Small Complete Minors Above the Extremal Edge Density
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Definition. G graph, subgraph H of G is a *minor* of G if it can be obtained from G by a sequence of edge and vertex deletions and contractions.
K_t*-minor* of a graph G: connected subgraphs $S_1, \ldots S_t$ of G, such that there exist internally disjoint paths $P_{i,j}$ joining each S_i, S_j.

Definition. $c(t) = min\{c : d(t) \geq c \implies G has\ a\ K_t minor\}$

THE RESULT:

Theorem. 1. *For all* $\epsilon > 0$, *integer* t, $\exists n_0(\epsilon, t)$ *such that each graph* G *fulfilling* $|V(G)| \geq n_0$ *and* $d(G) \geq c(t)+\epsilon$ *has a* K_t*-minor of order at most* $O(\frac{c(t)t^2}{\epsilon} log(n)loglog(n))$.

WE WILL PROVE:

Lemma. 3.1. $\forall \epsilon > 0$, $t \geq 3$, $n \geq n_1(\epsilon, t)$
each graph G of order n and with $d(G) \geq c(t) = \epsilon$ has a K_t-minor of order at most $O(\frac{c(t)t^2}{\epsilon} log(n)(loglog(n))^3)$.

Definition. **δ-expander** Graph $H, |V(H)| = m$ is a δ-expander if $\forall d : 0 \leq d \leq loglog(m) - 1$, $\forall S : S \subseteq V(H), |S| \leq \frac{m}{2^{2^d}} \implies$

$$|N(S)| \geq \frac{\delta 2^d}{log(m)loglog(m)^2}|S|.$$

Lemma. KEY LEMMA 1.2. Graph G, $d(G) = c$, then $\forall \delta$: $0 < \delta < \frac{1}{256}$ $\exists H$ subgraph of G such that: $d(H) \geq (1 - \delta)c$ and H is a δ-expander.

Claim. 2.1. Graph G on n vertices, $d(G) = c$, subset $S \subset V(G)$ such that $|N(S)| < \gamma|S| \implies$ one of the following is fulfilled:

1. $d(G[V \setminus S])| \geq c$
2. $d(G[S \cup N(S)])| \geq (1 - \gamma)c$.

Proof. (KEY LEMMA)

- sequence of graphs: $G = G_0, \ldots$
- end when $|G_t| \leq 256$ or G_t is a γ-expander
- use claim 2.1. and define G_{t+1}:
- *case1* $G_{t+1} = G[V(G_t) \setminus S_t]$
- *case2* $G_{t+1} = G[S_t \cup N(S_t)]$
- find the maximal edge loss when t passing through intervals $[2^{2^{k-1}}, 2^{2^k}]$

□

Lemma. 3.2. $\forall \delta > 0$, t, $m \geq m_0(\delta, t)$
each graph H is a δ-expander of order m then H has a K_t-minor of order at most $O(\frac{t^2}{\delta} log(n)(loglog(n))^3)$.

Definition. Ball with radius k and center v is a set $B_k(v)$ of all vertices whose distance from v is $\leq k$.

Claim. 3.3. Let U, V be subsets of $V(G)$ fulfilling:
$\forall d : 0 \leq d \leq loglog(m) - 1$, whenever $|B_k(U)| \leq \frac{m}{2^{2^d}}$ then

$$|N(B_{k+1}(U))| \geq \frac{\delta 2^d}{10log(m)(loglog(m))^2}|B_k(U)|,$$

and the same holds for the set V.
Then there exists a path from U to V of length at most $\frac{20}{\delta}log(m)(loglog(m))^3$.

Proof. (claim 3.3)

- sufficient: $|B_k(U)| > \frac{m}{2}$ for some $k \geq \frac{20}{\delta}log(m)(loglog(m))^3$

□

Definition. **δ-expanding ball** Ball $B_k(v)$ is δ-expanding if $\forall i : 1 \leq i \leq k-1$: $|B_{i+1}(v)| \geq |B_i(v)|(1+\gamma)$.

Claim. 3.5. $\forall \delta > 0$, t, $m \geq m_0(\delta, t)$ each δ-expander G of order m fulfills one of the following:

1. G has t vertices of order $> t$
2. G contains t disjoint sets $S_1, \ldots S_t$ such that $G[S_i]$ is γ-expanding, and $m^{\frac{1}{5}} \leq |S_i| \leq m^{\frac{1}{4}}$ $\forall i$ and $\forall v \in G[S_i]$ $deg(v) \leq log^4(m)$, where $\gamma = \frac{\delta}{5(loglog(m))^2}$.

Proof. (claim 3.5.)

- if not ad 1. then $T = \{v, deg(v) > log^4(m)\} < t$
- "nice" set S: $G[S]$ is a γ-expanding ball and $\frac{1}{5} < |S| < \frac{1}{4}$
- each $W \supseteq T$, $|W| \leq m^{\frac{1}{3}}$ contains a "nice" subset - one can iteratively pick S_i from $G \setminus W_i$, where $W_i = (\bigcup_{j<i} S_j) \cup$
- proof goes on by contradiction: suppose there is some W of the given properties, that does not contain a "nice" subset
- set $G_0 = G \setminus W$, take k_i the smallest that violates the expander condition for some fixed v_1
- set $T_1 = B_{k_1}(v_1)$, then $G_1 = G_0 \setminus T_1, \ldots$ until the first time $\bigcup_i T_i > \frac{\sqrt(m)}{2}$
- yield a contradiction

□

Claim. 3.6. $\forall \delta > 0$, t, $m \geq m_0(\delta, t)$ and G a δ-expander of order m , which has t vertices fulfilling ad 1. from Claim 3.5.
$\implies$ G has a K_t-minor of order at most $O(\frac{t^2}{\delta}log(n)(loglog(n))^3)$.

Claim. 3.6. $\forall \delta > 0$, t, $m \geq m_0(\delta, t)$ and G a δ-expander of order m , which has t subsets fulfilling ad 2. from Claim 3.5.
$\implies$ G has a K_t-minor of order at most $O(\frac{t^2}{\delta}log(n)(loglog(n))^3)$.