Define 1 ((r,t)-RS)**.** *Graph G is an (r,t)-Ruzsa-Szemerédi graph if its set of edges consists of t pairwise disjoint induced matchings, each of size r.*

Theorem 2 (Frankl and Füredi)**.** *For any fixed r there are (r,t)-RS graphs on N vertices with $rt = (1-o(1))\binom{N}{2}$.*

Construction 1. *Graph $G=(V,E)$. Set $V=[C]^n$ for some constant C. Let $N=C^n$ be the number of vertices. Each $x \in V$ is interpreted as integer vector in n dimensions with coordinates in $[C]=\{1,2,\ldots,C\}$.*

Let $\mu = E_{x,y}[||x-y||_2^2]$ where x and y are sampled uniformly at random from V.

A pair of vertices x and y are adjacent iff $\left| ||x-y||_2^2 - \mu \right| \le n$.

Lemma 3 (Number of edges)**.** $\binom{N}{2} - |E| \le \binom{N}{2} 2e^{-n/2C^4}$

$||x-y||_2^2 = \sum_{i=1}^n (x_i-y_i)^2 \implies$ sum of independent random variables, each bounded in $[0, C^2]$.

Hence we apply Hoeffding's inequality:

$$Pr[|\ ||x-y||_2^2 - \mu| > n] \le 2e^{-n/2C^4}$$

Define 4. *For each $z \in V$ let $G_z = G|V_z$ be induced subgraph.*

$$V_z = \{x \in V : |\ ||x-z||_2^2 - \mu/4| \le 3n/4\}$$

Lemma 5. *Let a be a vector in which the absolute value of each entry is at most C. Then there is a vector w where each is $\pm 1/2$ such that $|(a,w)| = |\sum_{i=1}^n a_i w_i| \le C/2 \le n/4$.*

Lemma 6. *For all $(x,y) \in E$, there is a z such that $x,y \in V_z$.*

Lemma 7. *For all $z \in V$, the maximum degree of G_z is at most $(10.5)^n$*

Antipodal point to x in V_z is $x' = 2z - x$. Parallelogram Law: The sum of the squares of the four side lengths equals to the sum of the squares of the lengths of the two diagonals.

$$||x-y||_2^2 + ||x+y-2z||_2^2 = 2||x-z||_2^2 + 2||y-z||_2^2 = ||x-y||_2^2 + ||y-x'||_2^2$$

Hence $||y-x'||_2^2 = 2||x-z||_2^2 + 2||y-z||_2^2 - ||x-y||_2^2$.

$x,y \in V_z$ and x,y are adjacent $\implies ||y-x'||_2^2 \le 4n$

Thus we can bound degree of x in G_z by the number of lattice points in a ball of radius $2\sqrt{n}$. The volume of those points does not exceed ball of radius $2.5\sqrt{n}$.

$$\frac{\pi^{n/2}(2.5\sqrt{n})^n}{(n/2)!} < (2\pi e)^{n/2}\frac{(2.5\sqrt{n})^n}{n^{n/2}} = (2.5\sqrt{2\pi e})^n < 10.5^n$$

Lemma 8. *Let H be a graph with maximum degree d. Then H can be covered by $O(d^2)$ induced matchings.*

Two edges are in conflict e_1, e_2 of H if they share a common end or if there is an edge in H connecting an endpoint of e_1 and an endpoint of e_2. There is at most $2d-2+(2d-2)(d-1) < 2d^2$ such edges in H. Thus partition edges of H such that in each part are no two edges in conflict.

Theorem 9. *For every n, C with $n \ge 2C$, n even, there is a graph G on $N = C^n$ vertices that is missing at most N^g edges for*

$$g = 2 - \frac{1}{2C^4 \ln C} + o(1)$$

and can be covered by N^f disjoint induced matchings, where

$$f = 1 + \frac{2\ln 10.5}{\ln C} + o(1)$$

Construction 2. *Let $V = [C]^n$ and $N = C^n$. Consider two vertices $a, b \in V$, where $a = (a_1, a_2, \dots, a_n)$ and $b = (b_1, b_2, \dots, b_n)$ for $a_i, b_i \in [C]$. There is an edge between a and b iff $d_H(a, b) = \sum_{i=1}^{n} 1_{a_i \neq b_i} > n - d$*

Define 10. *A $[n, k, d]$ linear code $\mathcal{C}$ is a subspace consisting of 2^k length n binary vectors such that for all $x, y \in \mathcal{C}$ and $x \neq y, d_H(x, y) \geq d$. We will call n encoding length, k the dimension, and d the distance of the code.*

Define 11. *Call a linear code $\mathcal{C}$ proper if the all ones vector is a codeword.*

Parity check matrix $(n - k) \times n$ called B:
For each of the first $n - 1$ columns choose uniformly a random vector.
Choose the last columns to be a parity of the preceding $n - 1$ columns.

Choose the code $\mathcal{C} = \{x \in \{0, 1\}^n | Bx = \vec{0}\}$

Lemma 12. *For any fixed set S of columns of B, the probability that the sum is the all zeros vector is exactly $2^{-(n-k)}$*

Lemma 13. *If $\sum_{i=0}^{d} \binom{n}{i} < 2^{n-k}$, then there is a proper $[n, k, d]$ code. Thus, there is such a code in which $k = (1 - H(d/n))n$, where $H(x) = -x \log_2 x - (1 - x) \log_2(1 - x)$ is the binary entropy function.*

Claim:

$$\binom{N}{2} - |E| \leq \frac{1}{2} C^n \sum_{i=d}^{n} \binom{n}{i} (C - 1)^{n-i}$$

Lemma 14. *If $\frac{d}{n} \geq \frac{2}{C-1}$ then*

$$\frac{1}{2} C^n \sum_{i=d}^{n} \binom{n}{i} (C - 1)^{n-i} \leq \binom{n}{d} C^n (C - 1)^{n-d}$$

Define 15. *We will define a pair (a, b) (a', b') iff $S = \{i | a_i = b_i\} = S' = \{i | a'_i = b'_i\}$, $|S| < d$ and furthermore there is an $x \in \mathcal{C}$ such that (a', b') is the x-flip of (a, b).*

Since each code $\mathcal{C}$ has dimension k, each equivalence class has size exactly 2^k.

Lemma 16. *Each equivalence class is an induced matching consisting of 2^{k-1} edges.*

The number of induced matchings to cover G is $\frac{|E|}{2^{k-1}} \leq \frac{N^2}{2^k}$.

Theorem 17. *For every n, d, C such that $\frac{d}{n} \geq \frac{2}{C-1}$, there is a graph G on $N = C^n$ vertices that is missing at most N^e edges, for*

$$e = 1 + \frac{H(d/n) + (1 - d/n) \log_2(C - 1)}{\log_2 C} + o(1)$$

and can be covered by N^f disjoint induced matchings, where

$$f = 2 - \frac{1 - H(d/n)}{\log_2 C} + o(1)$$

Theorem 18. *If there exists an (r, t)-RS graph on N vertices, then there exists a graph on $N + t$ vertices with at least $3rt/2$ edges, in which every edge is contained in exactly one triangle. Thus one has to delete at least $rt/2$ edges to destroy all triangles and yet the graph contains only $rt/2$ triangles.*

Lemma 19. *Any graph that can be covered by disjoint, induced matchings of size two or more must miss $N^{3/2}$ edges.*

Theorem 20. *Let $G = (V, E)$ be a graph on N vertices, that can be covered by disjoint induced matchings of size $r \geq 3$. Then the number of missing edges satisfies*

$$\binom{N}{2} - |E| \geq (\frac{1}{2\sqrt{2}} - o(1)) r^{1/2} N^{3/2}$$