Long paths and cycles in random subgraphs of graphs with large minimum degree

M. Krivelevich, C. Lee, B. Sudakov

1 Introduction

Fix a sequence of graphs G_i, where $i \in \mathbb{N}$. For the purpose of this talk, we always assume that the minimum degrees of the graphs tend to infinity, and the minimum degree of G_i will be denoted by k_i. Let $(G_i)_{p_i}$ be a random subgraph obtained from the graph G_i by taking each edge of G_i independently with probability p_i. We say that the $(G_i)_{p_i}$ satisfies some property $\mathcal{P}$ asymptotically almost surely, if the probability that $(G_i)_{p_i}$ satisfies $\mathcal{P}$ tends to one as i goes to infinity. For the simplicity, when G and p are graphs parametrized by the minimum degree, we abuse a notation and consider G and p as sequences obtained by taking the minimum degree tending to infinity. We then say that G_p satisfies $\mathcal{P}$ a.a.s., if the underlying sequence does.

The main results of the paper shows that

- if $p = c/k$, then G_p a.a.s. contains a path of length $(1 - 2/\sqrt{c}) \cdot k$,
- if $p = \omega(1)/k$, then G_p a.a.s. contains a cycle of length $(1 - \varepsilon) \cdot k$, and
- if $p = (1 + \varepsilon) \log k/k$, then G_p a.a.s. contains a path of length k.

Note that if the graphs G_k are cliques on $k + 1$ vertices, we obtain the standard Erdős-Rényi model, and the results generalize various classical results about sparse random graphs. Specifically,

- the result of Ajtai, Komlós and Szemerédi about long paths in sparse random graphs, which was independently proven also by Fernandez de la Vega,
- the result of Bollobás, Fenner and Frieze about long cycles in sparse random graphs, and
- the result about Hamiltonicity threshold due to Bollobás, and Komlós and Szemerédi.

2 More formally

We present the following results about the case of paths.

Theorem 1.1. *Let G be a finite graph with minimum degree at least k, and let $p = c/k$ for some positive c satisfying $c = o(k)$ (c is not necessarily fixed). Then a.a.s. G_p contains a path of length $(1 - 2/\sqrt{c})k$.*

Theorem 1.2. *Let ε be a fixed positive real. For a finite graph G of minimum degree at least k and a real $p \geq (1 + \varepsilon) \log k/k$, G_p a.a.s. contains a path of length k.*

The key tools for proving Theorem 1.2 are the following two theorems:

Theorem 3.1. *Let $p = c/k$ for some $c = o(k)$, and let G be a graph of minimum degree at least k.*

(i) *G_p a.a.s. contains a path of length $(1 - 2/\sqrt{c})k$,*
(ii) *if G is bipartite, then G_p a.a.s. contains a path of length $(2 - 6/\sqrt{c})k$, and*
(iii) *if c tends to infinity with k, then for a fixed vertex v, there a.a.s. exists a path of length $(1 - 2/\sqrt{c})k$ in G_p which starts at vertex v.*

Theorem 3.2. *There exists a positive real ε_0 such that following holds for every fixed positive real $\varepsilon \leq \varepsilon_0$. Let G be a graph on n vertices of minimum degree at least $(1 - \varepsilon)k$, and assume that $n \leq (1 + \varepsilon)k$. For $p \geq \frac{(1+4\varepsilon)\log k}{k}$, a random subgraph G_p is Hamiltonian a.a.s.*