Transitive Sets in Euclidean Ramsey Theory

Imre Leader, Paul A. Russell and Mark Walters

Definition. A finite set X in some Euclidean space $\mathbb{R}^n$ is *Ramsey* if for every positive integer k there exists a positive integer d such that whenever $\mathbb{R}^d$ is k-colored it must contain a monochromatic subset congruent to X.

Examples

- (the vertex set of an) n-dimensional regular simplex
- any (non-degenerate) triangle [Frankl, Rödl, 1986]
- any (non-degenerate) simplex [Frankl, Rödl, 1990]
- any trapezoid [Křížž, 1991]
- if $X \subset \mathbb{R}^n$, $Y \subset \mathbb{R}^m$ are Ramsey, then $X \times Y$ is Ramsey [Erdős, Graham, Montgomery, Rothschild, Spencer, Straus, 1973]
- $\{0, 1, 2\}$ is not Ramsey

Theorem. [Erdős et al., 1973] Every Ramsey set must be *spherical* (a subset of a sphere in some $\mathbb{R}^m$).

Conjecture. [Graham, 1994] ($1000) Every finite spherical set is Ramsey.

Theorem. [Křížž, 1991] Every finite set with transitive solvable group of isometries is Ramsey. (It is enough if the set is transitive and has a solvable group of isometries with at most two orbits).

Definition. A set is *transitive* if its symmetry group acts transitively ("every point looks the same").

Observation. Every transitive set is spherical.

(Meta)Observation. All known proofs that some set X is Ramsey use an embedding of X into some (higher-dimensional) transitive set.

Main conjecture

Conjecture A. A finite set $X \subset \mathbb{R}^n$ is Ramsey if and only if it is (congruent to) a subset of a finite transitive set.

Theorem. [Leader, Russell, Walters, 2010] Let x, y, z and w be four distinct points lying on a circle such that

$$w = z + \alpha(x - z) + \beta(y - z),$$

where $\alpha \neq 1$ and β is transcendental over $\mathbb{Q}(\alpha)$. Then $xyzw$ does not embed into a transitive set.

Corollary. [Leader, Russell, Walters, 2010] The cyclic quadrilateral with vertices $(-1, 0), (1, 0), (a, \sqrt{1 - a^2}), (a, -\sqrt{1 - a^2})$, where a is transcendental, does not embed into any transitive set.

Five equivalent conjectures

Definition. $Y \subset \mathbb{R}^d$ is *k-Ramsey for X* if any k-coloring of Y yields a monochromatic subset congruent to X.

Conjecture B. ("stronger") Let $X \subset \mathbb{R}^m$ be a finite transitive set. Then, for any k, there exists an n such that some *scaling* λX^n of $X^n \subset \mathbb{R}^{mn}$ is k-Ramsey for X.

Definition. Let G be a group, n a positive integer and $I \subset [n]$. Suppose $\overline{g} = (g_1, g_2, \ldots, g_n) \in G^n$ and $h \in G$. We write $\overline{g} \times_I h$ for the word $(k_1, k_2, \ldots, k_n) \in G^n$ where $k_i = g_i$ if $i \notin I$ and $k_i = g_i h$ if $i \in I$.

Conjecture C. ("removing geometry") Let G be a finite group. Then for any positive integer k there exist positive integers n and d such that whenever G^n is k-colored there exists a word $\overline{g} \in G^n$ and a set $I \subset [n]$ with $|I| = d$ such that the set $\{\overline{g} \times_I h; h \in G\}$ is monochromatic.

Definition. A *block permutation set* in $[m]^n$ is a set B formed in the following way. First, select pairwise disjoint subsets $I_1, \ldots, I_m \subset [n]$ and elements $a_i \in [m]$ for each $i \notin \bigcup_{j=1}^m I_j$. For each $\pi \in S_m$, define $a^\pi \in [m]^n$ by $(a^\pi)_i = \pi(j)$ if $i \in I_j$ and $(a^\pi)_i = a_i$ otherwise (if $i \notin \bigcup_{j=1}^m I_j$). Now set $B = \{a^\pi; \pi \in S_m\} \subset [m]^n$.

If $\sum_{j=1}^m |I_j| = d$ then we say that B is of degree d. We refer to the sets I_j as *blocks*.

Conjecture D. ("removing groups") Let m and k be positive integers. Then there exist positive integers n and d such that whenever $[m]^n$ is k-colored it contains a block permutation set of degree d.

Definition. A *template* over $[m]$ is a non-decreasing word $\tau \in [m]^\ell$ for some ℓ. Let S be a set of all *rearrangements* of τ. A *block set with template τ* in $[m]^n$ is a set B formed in the following way. First, select pairwise disjoint subsets $I_1, \ldots, I_\ell \subset [n]$ and elements $a_i \in [m]$ for each $i \notin \bigcup_{j=1}^\ell I_j$. For each $\pi \in S$, define $a^\pi \in [m]^n$ by $(a^\pi)_i = \pi(j)$ if $i \in I_j$ and $(a^\pi)_i = a_i$ otherwise (if $i \notin \bigcup_{j=1}^\ell I_j$). Now set $B = \{a^\pi; \pi \in S\} \subset [m]^n$.

As before, if $\sum_{j=1}^\ell |I_j| = d$ then we say that B is of degree d and we refer to the sets I_j as *blocks*. The block set is *uniform* if all blocks have the same size.

Conjecture E. ("arbitrary patterns") Let m and k be positive integers and let τ be a template over $[m]$. Then there exist positive integers n and d such that whenever $[m]^n$ is k-colored it contains a monochromatic block set of degree d with template τ.

Conjecture F. ("uniform patterns") Let m and k be positive integers and let τ be a template over $[m]$. Then there exist positive integers n and d such that whenever $[m]^n$ is k-colored it contains a monochromatic *uniform* block set of degree d with template τ.

Theorem. Conjecture E is true for $m = 3$ and templates of the form $1 \ldots 12 \ldots 23$.

Corollary. For any distinct reals α, β and γ, the set $X \subset \mathbb{R}^{r+s+1}$ consisting of all those points x having r coordinates α, s coordinates β and one coordinate γ is Ramsey.