Long Cycles in Subgraphs of (Pseudo)random Directed Graphs

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1 Definitions of the model

- Given a monotone increasing property P, the global resilience of a graph G with respect to P is the maximal integer R such that for every syubset $E_0 \subset E(G)$ of $|E_0| = R$ edges, the graph $G - E_0$ still possesses P. Analogously for a monotone decreasing property P.
- We consider directed graphs on n vertices, antiparallel edges are allowed.
- Graph (V, E) has edge density p if $|E| = pn^2$.
- The probability distribution $D(n, p)$: n vertices, for every distinct vertices x, y there is and edge from x to y with probability p, and independently from y to x with probability p.
- Directed graph G is (p, r)-pseudorandom if it has edge density p and for every disjoint $A, B \subseteq V(G)$, $|A| = |B|$, the number of edges from A to B (denoted by $e_G(A, B)$) satisfies

$$|e_G(A, B) - p|A||B|| \leq r|A|\sqrt{pn}.$$

Lemma 1 *For every constant $c > 0$ there is a constant $C > 0$ such that for $p \geq C/n$, a random directed graph $G \in D(n, p)$ is (p, c)-pseudorandom with high probability.*

2 Long cycles in graphs

Theorem 1 (Woodall) *Let $3 \leq \ell \leq n$. Every graph G on n vertices satisfying*

$$e(G) \geq \lceil \frac{n-1}{\ell-2} \rceil \cdot \binom{\ell-1}{2} + \binom{r+1}{2} + 1,$$

where $r = (n-1) \bmod (\ell - 2)$, has a cycle of length at least ℓ.

The bound is best possible.

For a given $0 \leq \alpha < 1$, define

$$w(\alpha) = 1 - (1-\alpha)\lfloor (1-\alpha)^{-1} \rfloor.$$

Theorem 2 (Dellamonica et al.) *Let $\alpha > 0$. For every $\beta > 0$ there is n_0 such that for every graph G on $n > n_0$ vertices satisfying*

$$|E(G)| \geq \binom{n}{2} \cdot (1 - (1 - w(\alpha))(\alpha + w(\alpha)) + \beta)$$

has a cycle of length at least $(1 - \alpha)n$.

Theorem 3 *Fix $0 < \gamma < 1/2$ and let $G = (V, E)$ be a (p, r)-pseudorandom directed graph on n vertices, where $r \leq \mu\sqrt{np}$ and $\mu(\gamma) > 0$ is a sufficiently small constant that depends only on γ and n is sufficiently large. Let G' be a subgraph of G with at least $(1/2 + \gamma)|E|$ edges. Then G' contains a directed cycle of length at least $(1 - \alpha - o(1))n$, where α satisfies*

$$2\gamma = 1 - (1 - w(\alpha))(\alpha + w(\alpha)).$$

Corollary 1 *For every $\gamma > 0$ there is a constant $c_1(\gamma) > 0$ such that the following holds. Let G be a (p,r)-pseudorandom graph on n vertices, $r \leq \mu\sqrt{np}$, where $\mu(\gamma) > 0$ is some sufficiently small constant that depends only on γ and n is sufficiently large. Let G' be a subgraph of G with at least $(1/2+\gamma)|E(G)|$ edges. Then G' contains a directed cycle of length at least $c_1 n$.*

Theorem 4 *Fix $0 < \gamma < 1/2$ and let G be a (p,r)-pseudorandom directed graph on n vertices, where $r = O(\sqrt{np})$ and $pn \to \infty$. There is a subgraph G' with $(1/2+\gamma)|E|$ edges that does not contain any directed cycle of length at least $(1-\alpha+o(1))n$, where α satisfies*

$$2\gamma = 1 - (1-w(\alpha))(\alpha + w(\alpha)).$$

3 The Regularity Lemma

- For a pair of disjoint sets of vertices U, W, let $E_G(U,W)$ be the set of edges directed from U to W, and let $e_G(U,W) = |E_G(U,W)|$.
- Graph G is (δ, D, p)-bounded if for any two disjoint sets U, W such that $|U|, |W| \geq \delta|V|$ we have
$$e_G(U,W) \leq Dp|U||W|.$$
- The edge density from a set U to W is defined by $\frac{e(U,W)}{|U||W|}$.
- Two sets U, W span a bipartite directed graph of bi-density p if it has edge density at least p in both directions.
- The directed p-density from U to W is
$$d_{G,p}(U,W) = \frac{e_G(U,W)}{p|U||W|}.$$
- For $0 < \delta \leq 1$, a pair (U,W) is (δ,p)-regular in a digraph G if for every $U' \subseteq U$ and $W' \subseteq W$ such that $|U'| \geq \delta|U|$ and $|W'| \geq \delta|W|$ we have both
$$|d_{G,p}(U,W) - d_{G,p}(U',W')| < \delta$$
and
$$|d_{G,p}(W,U) - d_{G,p}(W',U')| < \delta.$$
- A partition $\{V_0, V_1, \ldots, V_k\}$ of V is (δ, k, p)-regular if the following properties hold:
  1. $|V_0| \leq \delta|V|$.
  2. $|V_i| = |V_j|$ for all $1 \leq i < j \leq k$.
  3. At least $(1-\delta)\binom{k}{2}$ of the pairs (V_i, V_j), $1 \leq i < j \leq k$, are (δ,p)-regular.

Lemma 2 (Regularity Lemma) *For any real $\delta > 0$, any integer $k_0 \geq 1$ and any real $D > 1$, there exist constants $\eta = \eta(\delta, k_0, D)$ and $K = K(\delta, k_0, D) \geq k_0$ such that for any $0 < p(n) \leq 1$, any (η, D, p)-bounded directed graph G admits a (δ, k, p)-regular partition for some $k_0 \leq k \leq K$.*

4 Regular pair contains a long path

Lemma 3 *Let (U,W) be a (δ,p)-regular pair for $|U| = |W|$ with bi-density at least $2\delta p$, where $p > 0$. Then for every two sets $U' \subseteq U$ and $W' \subseteq W$ such that $|U'| \geq \delta|U|$ and $|W'| \geq \delta|W|$ there is a directed edge from U' to W'.*

Lemma 4 *Let $H = (V_1, V_2, E)$, where $|V_1| = |V_2| = t$, be a directed bipartite graph that satisfies the following property: for every two sets $A \subseteq V_1, B \subseteq V_2$ of size k, there is at least one edge from B to A. Then H contains a directed path of length $2t - 4k + 3$.*

Corollary 2 *Let (U,W) be a (δ,p)-regular pair with bi-density at least $2\delta p$ and $|U| = |W| = t$, $p > 0$. Then the bipartite directed graph between U and W contains a directed path of length $(1-2\delta)2t + 2$ that starts at U.*