Holographic algorithms

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1 Definitions

With a graph G we associate the perfect matching polynomial PerfMatch(G) over $n(n-1)/2$ variables $\{x_{i,j} | 1 \leq i < j \leq n\}$ as follows:

$$\mathrm{PerfMatch}(G) = \sum_{E'} \prod_{(i,j)\in E'} x_{i,j}$$

where the summation is over all perfect matchings E' of G.

A planar matchgate Γ is a triple (G, X, Y) where G is a planar embedding of a planar graph (V, E, W) where $X \subseteq V$ is a set of input nodes, $Y \subseteq V$ is a set of output nodes, and where X, Y are disjoint.

Further, as one proceeds anticlockwise around the outer face starting from one point one encounters first the input nodes labeled $1, 2, \ldots, |X|$ and then the output nodes $|Y|, \ldots, 2, 1$ in that order. The arity of the matchgate is $|X| + |Y|$.

For $Z \subseteq X \cup Y$ we define the standard signature of Γ with respect to Z to be PerfMatch($G-Z$), where $G - Z$ is the graph obtained by removing from G the node set Z and all edges that are incident to Z.

Further we define the standard signature of Γ to be the $2^{|X|} \times 2^{|Y|}$ matrix $u(\Gamma)$ whose elements are the standard signatures of Γ with respect to Z for the $2^{|X|}2^{|Y|}$ choices of Z. The labeling of the matrix is as follows: Suppose that X and Y have the labeling described, i.e., the nodes are labeled $1, 2, \ldots, |X|$ and $|Y|, \ldots, 2, 1$ in anti-clockwise order. Then each choice of Z corresponds to a subset from each of these labeled sets. If each node present in Z is regarded as a 1, and each node absent as a 0, then we have two binary strings of length $|X|, |Y|$, respectively, where the nodes labeled 1 correspond to the leftmost binary bit. Suppose that i, j are the numbers represented by these strings in binary. Then the entry corresponding to Z will be the one in row i and column j in the signature matrix $u(\Gamma)$.

A basis of size k is a set of distinct nonzero vectors each of length 2^k with entries from a field F . Often we will have just two basis vectors that represent 0 and 1, respectively, and in that case we shall call them n and p. In this paper all bases will be of size $k = 1$, so that $n = (n_0, n_1)$ and $p = (p_0, p_1)$. The basis $b_0 = [n, p] = [(1, 0), (0, 1)]$ we call the standard basis. In general, the vectors in a basis do not need to be independent

$$b_1 = [n, p] = [(-1, 1), (1, 0)]$$

If we have two vectors q, r, of length l, m, respectively, then we shall denote the tensor product $s = q \otimes r$ to be the vector s of length lm in which $s_{im+j} = q_i r_j$ for $0 \leq i < l$.

We say that a matchgate is a generator if it has zero input nodes and nonzero output nodes, and a recognizer if it has zero output nodes and nonzero input nodes.

Let $b = \{n, p\}$ be a basis and let G be a generator of arity k and let

$$u(G) = \sum_{x = x_1 \otimes \cdots \otimes x_k \in \{n,p\}^k} \alpha_x x,$$

then the signature of generator G with respect to basis b is the vector

$$(\alpha_x)_{x = x_1 \otimes \cdots \otimes x_k \in \{n,p\}^k}.$$

Let $x \in \{n, p\}^k$ we denote by $\mathrm{valG}(G, x)$ the signature element α_x.

Let R be a recognizer and let x be a vector then $\mathrm{valR}(G, x) := u(g)x$ (scalar product of standard signature of G and x).

We define a matchgrid Ω over a basis b to be a weighted undirected planar graph G that consists of the disjoint union of a set of g generator matchgates $B_1, \ldots, B_g$, r recognizer matchgates $A_1, \ldots, A_r$, and f connecting edges $C_1, \ldots, C_f$ where each C_i edge has weight one and joins an output node in a generator matchgate with an input node of a recognizer matchgate, such that every input and output node in every constituent matchgate has exactly one such incident connecting edge.

$$\mathrm{Holant}(\Omega) = \sum_{x \in b^f} \left[\prod_{1 \le j \le g} \mathrm{valG}(B_j, x_j) \right] \left[\prod_{1 \le i \le g} \mathrm{valR}(A_i, x_i) \right]$$

Theorem 1.1. *For any matchgrid Ω over any basis b if Ω has weighted graph G then*

$$\mathit{Holant}(\Omega) = \mathit{PerfMatch}(G).$$