# Complexity measures of sign matrices

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**Definition 1.** Let $E_1$ be a norm on $\mathbb{C}^n$ and $E_2$ a norm on $\mathbb{C}^m$. We define a norm $\|\cdot\|_{E_1\to E_2}$ on the space of complex matrices $m\times n$ by

$$\|A\|_{E_1\to E_2} = \sup_{\|x\|_{E_1}=1} \|Ax\|_{E_2}.$$

We define a factorization constant for $\ell_2$ of $m\times n$ matrix

$$\gamma_2(A) = \min_{XY=A} \|X\|_{\ell_2\to\ell_\infty^m}\|Y\|_{\ell_1^n\to\ell_2}$$

$$\gamma_2(A) = \min_{XY=A}\max_{i,j} \|x_i\|_{\ell_2}\|y_j\|_{\ell_2},$$

where $x_i$ are rows of $X$ and $y_j$ are columns of $Y$.

**Definition 2.** We say that a sign matrix $A$ has *margin*

$$m(A) = \sup_{X,Y:sign(\langle x_i,y_j\rangle)=a_{ij}} \min_{i,j} \frac{|\langle x_i,y_j\rangle|}{\|x_i\|\|y_j\|}$$

and we denote $mc(A) = m(A)^{-1}$ *margin complexity* of $A$.

**Definition 3.** The *sign pattern* of a matrix $B$ is the sign matrix $(b_{ij})$. For a sign matrix $A$, let $SP(A)$ be a family of matrices $B$ satisfying $|b_{ij}|\geq 1$ and $sp(B)=A$.

**Lemma 1.** *For every $m\times n$ signed matrix $A$,*

$$mc(A) = \min_{XY\in SP(A)} \|X\|_{\ell_2\to\ell_\infty^m}\|Y\|_{\ell_1^n\to\ell_2}$$

**Corollary 2.** $mc(A)\leq\gamma_2(A)$.

**Fact 1.**

$$\|A^T\|_{\ell_\infty^m\to\ell_1^n} \leq \gamma_2^*(A^T) \leq K_G\|A^T\|_{\ell_\infty^m\to\ell_1^n}$$

**Corollary 3.** $mc(A) \geq \frac{mn}{\gamma_2^*(A^T)}$.

**Lemma 4.** *For every complex $m\times n$ signed matrix $A$,*

$$\gamma_2^2(A) \leq \|A\|_{\ell_1^n\to\ell_\infty^m} rank(A).$$

**Definition 4.** Let $A$ be a sign matrix $m\times n$. We denote $d(A)$ the smallest possible dimension $d$ such that there exist vectors $x_1\dots x_m$ and $y_1\dots y_n$ such that $sign(\langle x_i,y_j\rangle)=a_{ij}$ for every $i,j$.

For every sign matrix $d(A)\leq rank(A)$. Gap can be arbitrarily large: $2I_n - J_n$ has $rank=n$ and $d=2$.

**Lemma 5.** *For every $m\times n$ sign matrix $A$*

$$d(A) \geq \frac{\sqrt{mn}}{\gamma_2^*(A)}.$$

**Lemma 6.**

$$Pr(\|A\|_{\ell_\infty^n\to\ell_1^m} \leq 2mn^{1/2}) \geq 1-(e/2)^{-2m},$$

*where $m\geq n$, and the matrix $A$ is drawn uniformly at random from among the $m\times n$ sigh matrices.*

**Lemma 7.** *Let $m\geq n$ and let $A$ be a random $m\times n$ sign matrix. Denote $m_\gamma$ the median of $\gamma_2$, then*

$$Pr(|\gamma_2(A)-m_\gamma)| > c) \leq 4e^{-c^2/16}.$$