A Center Transversal Theorem for Hyperplanes and Applications to Graph Drawing

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Combinatorial input

Theorem 1 (Erdös-Szekeres) For all integers r, s, any sequence of $n > (r-1)(s-1)$ numbers contains either a non-increasing subsequence of lenght r of increasing sequence of lenght s.

Theorem 2 (bichomatic Ramsey for hypergraphs) For all $p, a, b \in \mathbb{N}$, there is a natural number $R = R_p(a, b)$ such that for any set S of size R and any 2-coloring $c : \binom{S}{p} \to \{1, 2\}$ of all subsets of S of size p, there is either a set A of size a such that all p-tuples in $\binom{A}{p}$ have colour 1 or a set B of size b such that all p-tuples in $\binom{B}{p}$ have colour 2.

Theorem 3 (Erdös-Szekeres happy ending theorem) For any k there is a number $ES(k)$ such that in any set $S \subset \mathbb{R}^2$ of $ES(k)$ points in general position there is k points in a convex position.

Center transversal theorem

Center transversal theorem is generalization of both ham-sandwich cut theorem and centerpoint theorem.

Let $\mathcal{H}$ be the set of all open halfspaces in $\mathbb{R}^d$ and $\mathcal{G}$ be a family of subsets of $\mathbb{R}^d$ containing $\mathcal{H}$ and closed under union. A *charge* μ is a finite set function that is defined for all set $X \in \mathcal{G}$, and is *monotone* ($X \subseteq Y$ implies $\mu(X) \leq \mu(Y)$) and *subadditive*, that is $\mu(X \cup Y) \leq \mu(X) + \mu(Y)$. A charge μ is *concentrated on set* X, if for all halfspaces $h \in \mathcal{H}$ such that $h \cap X = \emptyset$ we have $\mu(h) = 0$.

Theorem 4 (Doľnikov, 1992) For arbitrary k charges $\mu_i, i = 1, \ldots, k$, defined on $\mathcal{G}$ and concentrated on bounded sets, there is a $(k-1)$-flat π such that

$$\mu_i(h) \geq \frac{\mu_i(\mathbb{R}^d)}{d-k+2}, \quad i = 1, \ldots, k, \tag{1}$$

for all open halfspaces $h \in \mathcal{H}$ containing π.

We say that charge μ is $\mathcal{H}$-subaditive if for any finite set $H \subseteq \mathcal{H}$ we have

$$\mu(\cup_{h \in H} h) \leq \sum_{h \in H} \mu(h).$$

Totally ordered unital magma $(M, \oplus, \leq, e)$ is a totally ordered set M endowed with binary operation $\oplus$ such that M is closed under $\oplus$, operation $\oplus$ has neutral element e, and is monotone, that is, $a \oplus c \leq b \oplus c$ and $c \oplus a \leq c \oplus b$ holds whenever $a \leq b$. Further, for all $x \in M$ and $c \in \mathbb{N}$, define $cx = \oplus^c x := x \oplus (x \oplus (\ldots \oplus x) \ldots)$, where the operation $\oplus$ was used c-times.

Theorem 5 Let $\mu_i, i = 1, \ldots, k$, be k set functions defined on $\mathcal{G}$ and taking values in a totally ordered unital magma $(M, \oplus, \leq, e)$. Let $\delta_i \in M$ be such that $(d-k+2)\delta_i \leq \mu_i(\mathbb{R}^d)$.

If the functions μ_i are monotone, $\mathcal{H}$-subaditive and concentrated on bounded sets, there is a $(k-1)$-flat π such that

$$\mu_i(h) \geq \delta_i, \quad i = 1, \ldots, k, \tag{2}$$

for all open halfspaces $h \in \mathcal{H}$ containing π.

Center transversal theorem for arrangements

Let A be an arrangement of n hyperplanes in $\mathbb{R}^d$. Denote by $V(A)$ the set of all vertices of A and by $CH(A) = CH(V(A))$ the convex hull of those points.

We say that the arrangements $A_1, A_2, \ldots, A_k$ are *disjoint* if their convex hulls do not intersect. They are *separable* if they are disjoint and no hyperplane intersects $d+1$ of them simultaneously.

Let $\mathcal{H}$ be the set of all open halfspaces in $\mathbb{R}^d$ and $\mathcal{G}$ be an additive family of subsets of $\mathbb{R}^d$ that contains $\mathcal{H}$. For any set $S \in \mathcal{G}$, let $\mu_A(S)$ be the maximum number of hyperplanes that have all their vertices inside S, that is,

$$\mu_A(S) = \max_{B \subseteq A, V(B) \subseteq S} |B|.$$

In particular, $\mu_A(\mathbb{R}^d) = \mu_A(CH(A)) = n$ and $\mu_A(\emptyset) = d-1$.

Case of lines in $\mathbb{R}^2$

Lemma 6 For any two sets $S_1 \in \mathcal{H}$ and $S_2 \in \mathcal{G}$ we have

$$\mu_A(S_1 \cup S_2) \leq \mu_A(S_1)\mu_A(S_2).$$

Corollary 7 The set function μ_A, which takes values in the totally ordered unital magma $(\mathbb{R}, \cdot, \leq, 1)$, is monotone and $\mathcal{H}$-subaditive.

Theorem 8 For any arrangemets A_1 and A_2 of lines in $\mathbb{R}^2$, there exists a line ℓ bounding closed halfplanes ℓ^+ and ℓ^- and sets $A_i^\delta, i \in \{1,2\}, \delta \in \{+,-\}$ such that $A_i^\delta \subseteq A_i, |A_i^\delta| \geq |A_i|^{1/2}$ and $V(A_i^\delta) \in \ell^\delta$.

The bound in the above theorem is tight. Applying generalized center transversal theorem with $k = 1$ gives:

Theorem 9 For any arrangemets A of lines in $\mathbb{R}^2$, there exists a point q such that for every halfplane h containing q there is a set $A' \subseteq A, |A'| \geq |A|^{1/3}$, such that $V(A') \in h$.

This bound is not tight as proves following theorem. However, we do not know, whether the following bound is tight.

Theorem 10 For any arrangemets A of lines in $\mathbb{R}^2$, there exists a point q such that for every halfplane h containing q there is a set $A' \subseteq A, |A'| \geq (|A|/3)^{1/2}$, such that $V(A') \in h$.

Case of hyperplanes in $\mathbb{R}^d$

Lemma 11 For any two sets $S_1 \in \mathcal{H}$ and $S_2 \in \mathcal{G}$ we have

$$\mu_A(S_1 \cup S_2) \leq R_d(\mu_A(S_1) + 1, \mu_A(S_2) + 1) - 1.$$

Define the operator $\oplus$ as $a \oplus b = R_d(a+1, b+1) - 1$. The operator $\oplus$ is increasing, closed on the set $\mathbb{N}_{\geq d-1}$. Since $R_d(d, x) = x$ for all $x \in \mathbb{N}_{\geq d-1}$, d is a neutral element. Therefore $(\mathbb{N}_{\geq d-1}, \oplus, \leq, d-1)$ is a totally ordered unital magma.

Corollary 12 The set function μ_A takes values in the totally ordered unital magma $(\mathbb{N}_{\geq d-1}, \oplus, \leq, d-1)$, it is monotone and $\mathcal{H}$-subaditive.

Define $Q(x, c) := \max\{y \in \mathbb{N} : \oplus^c y \leq x\}$. Apply generalized center transversal theorem to obtain:

Theorem 13 Let $A_1, \ldots, A_k$ be k sets of hyperplanes in $\mathbb{R}^d$. There exist a $(k-1)$-flat π such that for every open halfspace h that contains π,

$$\mu_{A_i}(h) \geq Q(|A_i|, d-k+2). \tag{3}$$

Corollary 14 Let $A_1, \ldots, A_d$ be d sets of hyperplanes in $\mathbb{R}^d$. There exists a hyperplane π bounding the two closed halfspaces π^+ and π^- and sets $A_i^\delta \subseteq A_i, i \in [d], \delta \in \{+, -\}$ such that $V(A_i^\delta) \in \pi^\delta$ and $|A_i^\delta| \oplus |A_i^\delta| \geq |A_i|$.

Corollary 15 Let $A_1, \ldots, A_d$ be d sets of hyperplanes in $\mathbb{R}^d$, no $r+1$ of which intersect in a common point. There exists a hyperplane π bounding the two open halfspaces π^+ and π^- and sets $A_i^\delta \subseteq A_i, i \in [d], \delta \in \{+, -\}$ such that $V(A_i^\delta) \in \pi^\delta$ and $(|A_i^\delta| \oplus |A_i^\delta|) \oplus r \geq |A_i|$.

Same-type lemma for hyperplane arrangements

A *transversal* of a collection of sets $X_1, \ldots, X_m$ is a m-tuple $(x_1, \ldots, x_m)$ where $x_i \in X_i$. The sets $X_1, \ldots, X_{d+1}$ in $\mathbb{R}^d$ have *same-type transversals* if they are well separated, that is, for all disjoint sets of indices $I, J \subseteq [d+1]$ there is a hyperplane separating the sets $X_i, i \in I$ from the sets $X_j, j \in J$. The sets $X_1, \ldots, X_m$ in $\mathbb{R}^d$, $m > d+1$, have same-type transversals if any $d+1$ of them have the same-type transversals.

Lemma 16 For any integers d, m, r, there is a growing function $f = f_{m,d,r}$ such that for any collection of m hyperplane arrangements $A_1, \ldots, A_m$ in $\mathbb{R}^d$, where no $r+1$ hyperplanes intersect in a common point, there are sets $A_i' \subseteq A_i, i \in [m]$, such that $|A_i'| \geq f(|A_i|)$ and the sets $CH(A_1'), \ldots, CH(A_m')$ have same-type transversals.

Theorem 17 For every integers k, r, c, there is an integer N such that any arrangement A of N lines in $\mathbb{R}^2$, such that no $r+1$ lines go through a common point, contains disjoint subsets $A_1, \ldots, A_k$ with $|A_i| \geq c$ and such that every transversal of $CH(A_1), \ldots, CH(A_k)$ is in convex position.

Applications in graph drawing

A set of n lines in the plane labelled from 1 to n *supports G with vertex labelling π* if there exist straight-line crossing-free drawing of G where for each $i \in [n]$, vertex labelled i in G is mapped to a point on a line i. A set L of n lines labelled from 1 to n *supports an n-vertex graph G* if for every vertex labelling π of G, L supports G with labelling π.

Theorem 18 For some absolute constant c' and every $n \geq c'$, there exist no set of n lines in the plane that support all labelled n-vertex planar graphs.

Theorem 19 Given a set L of n lines in the plane, every planar graph has a straight-line crossing-free drawing where each vertex of G is placed on the distinct line of L.