An upper bound on the number of Steiner triple systems

Nathan Linial and Zur Luria
Presented by Josef Cibulka

Definitions.

- Steiner triple system (STS) on V *... collection of triples* $T \subseteq \binom{V}{3}$ *such that each pair of vertices is contained in exactly one triple from* T.
- $STS(n)$ *... number of STS's on* V *of size* n
- $F(n)$ *... number of* 1*-factorizations of* K_n
- $L(n)$ *... number of* $n \times n$ *Latin squares*

Observation. 1*-factorizations of* K_n *and STS's correspond to special subclasses of Latin squares*

Previous bounds.

$$\left(\frac{n}{e^2 3^{3/2}}\right)^{\frac{n^2}{6}} \le STS(n) \le \left(\frac{n}{e^{1/2}}\right)^{\frac{n^2}{6}}$$

$$\left((1+o(1))\frac{n}{4e^2}\right)^{\frac{n^2}{2}} \le F(n) \le \left((1+o(1))\frac{n}{e^2}\right)^{\frac{n^2}{2}}$$

$$L(n) = \left((1+o(1))\frac{n}{e^2}\right)^{n^2}.$$

Theorem 1.

$$STS(n) \le \left((1+o(1))\frac{n}{e^2}\right)^{\frac{n^2}{6}}.$$

Conjecture.

$$STS(n) = \left((1+o(1))\frac{n}{e^2}\right)^{\frac{n^2}{6}}$$

$$F(n) = \left((1+o(1))\frac{n}{e^2}\right)^{\frac{n^2}{2}}.$$

Proof

- Entropy of a uniformly distributed random $X \in STS(n)$

$$H(X) = \sum_{X \in STS(n)} -1/|STS(n)| \log(1/|STS(n)|) = \log(|STS(n)|)$$

- Fix an ordering $\ll$ of the vertices and an ordering $\prec$ of the edges such that if $i \ll j$, then $(i,k) \prec (j,l)$ for every k and l
- $X_{i,j}$ is the t such that $\{i,j,t\}$ is a triple in X
- *conditional entropy* $\ldots H(X|Y) = \sum_y \Pr(Y=y) H(X|Y=y)$
- *Chain rule:* $H(X) = \sum_{(i,j)} H(X_{i,j} | X_e : e \prec \{i,j\})$

Definitions.

- $\mathcal{A}_{i,j}$ *is the set of values unavailable for* $X_{i,j}$ *due to the values of* $X_{i',j'}$ *with* $i' \ll i$
- $\mathcal{B}_{i,j}$ *is the set of values unavailable for* $X_{i,j}$ *due to the values of* $X_{i',j'}$ *with* $(i',j') \prec (i,j)$
- $\mathcal{M}_{i,j} := (V \setminus \{i,j\}) \setminus \mathcal{A}_{i,j}$
- $\mathcal{N}_{i,j} := \mathcal{M}_{i,j} \setminus \mathcal{B}_{i,j}$
- $N_{i,j} := |\mathcal{N}_{i,j}|$ *and* $M_{i,j} := |\mathcal{M}_{i,j}|$

$$\log(STS(n)) \le \sum_{(i,j)} \mathbb{E}_X[\mathbb{E}_{\prec}[\log(N_{i,j})]]$$

- Fix X, i and j and estimate $\mathbb{E}_{\prec}[\log(N_{i,j})]$

Definitions.

- *Let predicate $F_{i,j}$ be true if and only if $i \ll j$, $i \ll X_{i,j}$ and $(i,j) \prec (i, X_{i,j})$*
- *Let predicate $F'_{i,j}$ be true if and only if $i \ll j$, $i \ll X_{i,j}$*
- *Let predicate $R_{i,p}$ be true if and only if the position of i in $\ll$ is p*

$$\mathbb{E}_{\prec}[\log(N_{i,j})] = \Pr(F_{i,j})\mathbb{E}_{\prec|F_{i,j}}[\log(N_{i,j})] = \frac{1}{6}\mathbb{E}_{\prec|F_{i,j}}[\log(N_{i,j})].$$

- First, consider $M_{i,j}$ instead of $N_{i,j}$

$$\mathbb{E}_{\prec|F_{i,j}}[\log(M_{i,j})] \le \mathbb{E}_{\ll|F'_{i,j}}[\log(M_{i,j})] \le \mathbb{E}_p[\log(\mathbb{E}_{\ll|R_{i,p},F'_{i,j}}[M_{i,j}])].$$

Lemma 1.

$$\Pr_{\ll|F'_{i,j}}(R_{i,p}) = \frac{\binom{n-p}{2}}{\binom{n}{3}}$$

Lemma 2.

$$\mathbb{E}_{\ll|R_{i,p},F'_{i,j}}[M_{i,j}] = 1 + (n-p-2)\frac{\binom{n-p-3}{2}}{\binom{n-4}{2}}$$

- By combining these lemmas and some calculations

$$\mathbb{E}_{\prec|F_{i,j}}[\log(N_{i,j}) \le \log(n) - 1 + o(1)$$

- Now we will estimate $\mathbb{E}_{\prec|F_{i,j},M_{i,j}=l}[\log(N_{i,j})]$ for every $1 \le l \le n$.

Definition.

- *Predicate $Q_{i,j,q}$ is true if and only if (i,j) is q'th among the edges $(i,\star)$ under $\prec$*

Lemma 3.

$$\Pr_{\prec|F_{i,j}}(Q_{i,j,p}) = \frac{n-q}{\binom{n}{2}}$$

Lemma 4.

$$\mathbb{E}_{\prec|Q_{i,j,p},F_{i,j},M_{i,j}=l}[N_{i,j}] = 1 + (l-1)\frac{\binom{n-q-1}{2}}{\binom{n-2}{2}}$$

- By combining these lemmas and some calculations

$$\mathbb{E}_{\prec|F_{i,j},M_{i,j}=l}[\log(N_{i,j})] \le \log(l) - 1 + o(1)$$

- Thus

$$\mathbb{E}_{\prec|F_{i,j}}[\log(N_{i,j})] \le \sum_{l=1}^{n} \Pr_{\prec|F_{i,j}}(M_{i,j}=l)(\log(l) - 1 + o(1)) =$$
$$\mathbb{E}_{\prec|F_{i,j}}[\log(M_{i,j})] - 1 + o(1) \le \log n - 2 + o(1)$$

- And finally

$$\log(STS(n)) \le \frac{1}{6}\sum_{(i,j)}(\log n - 2 + o(1)) = \frac{\binom{n}{2}}{3}(\log n - 2 + o(1))$$