Triangle Detection Versus Matrix Multiplication: A Study of Truly Subcubic Reducibility

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Definitions: $(\mathcal{S}, \oplus, \odot)$ is a $(\oplus, \odot)$-*semiring*, if $(\mathcal{S}, \oplus)$ is commutative monoid with identity element 0, $(\mathcal{S}, \odot)$ is a monoid with identity element 1, the $\odot$ distributes over $\oplus$ and 0 is an annihilator with respect to $\odot$. A *Boolean semiring* is the $(\mathcal{B}, \vee, \wedge)$ semiring, where $\mathcal{B} = \{False, True\}$. Apart from semirings, we also consider $(\min, \odot)$ *structure* over a set $\mathcal{R}$, with $\odot : \mathcal{R} \times \mathcal{R} \to \mathbb{Z} \cup \{-\infty, \infty\}$. Such a structure is called *extended* if there exists $r_\infty \in \mathcal{R}$, such that $r_\infty \odot a = a \odot r_\infty = \infty$ for any $a \in \mathcal{R}$.

Definition: We generalize matrix multiplication over a $(\min, \odot)$ structure in a natural way – we use classical definition and replace plus and times operators with min and $\odot$, respectively.

Remark: There is an algorithm by Coppersmith and Winograd which multiplies two $n \times n$ matrices over a ring in $\mathcal{O}(n^{2.376})$ time. That is currently the best exponent (denoted usually as ω), although the optimal algorithm is conjectured to be $\tilde{\mathcal{O}}(n^2)$.

Definition: We say an algorithm on $n \times n$ matrices is *truly subcubic* if its time complexity is $\mathcal{O}(n^{3-\delta} \log M)$ for $\delta > 0$, where M is the absolute value of the largest entry.

Negative triangle detection problem over $\mathcal{R}$ is defined on a weighted tripartite graph with parts I, J, K. Edge weights between I and J are from $\mathbb{Z}$ and all other weights are from $\mathcal{R}$. The problem is to detect whether there is a triangle $i \in I, j \in J, k \in K$ so that $\omega(i,k) \odot w(k,j) + w(i,j) < 0$. If we negate all weights between I and J, the condition becomes $w(i,k) \odot w(k,j) < w(i,j)$.

Negative triangle finding problem over $\mathcal{R}$ extends negative triangle detection problem by listing one or more negative triangles.

Lemma 3.1: Let $T(n) = \Omega(n)$ be a non-decreasing function. If there is a $T(n)$ time algorithm for negative triangle detection over $\mathcal{R}$ on a graph $G = (I \cup J \cup K, E)$, then there is an $\mathcal{O}(T(n))$ algorithm which returns a negative triangle in G if one exists.

Theorem E.1: Suppose there is a truly subcubic algorithm for negative triangle detection over $\mathcal{R}$. Then there is a truly subcubic algorithm which lists Δ negative triangles over $\mathcal{R}$ in a graph with at least Δ triangles, for any $\Delta = \mathcal{O}(n^{3-\delta})$, $\delta > 0$.

Corollary E.1: There is an algorithm that lists up to Δ triangles from a given graph G in time $\mathcal{O}(\Delta^{1-\omega/3} n^\omega) \leq \mathcal{O}(\Delta^{0.208} n^{2.376})$.

Matrix product verification problem over $\mathcal{R}$ is to verify whether for all $i, j \in [n]$ $\min_{k \in [n]}(A[i,k] \odot B[k,j]) = C[i,j]$, where A, B, C are given $n \times n$ matrices with entries from $\mathcal{R}, \mathcal{R}, \mathbb{Z}$, respectively.

Theorem 1.2: Suppose matrix product verification over $\mathcal{R}$ can be done in time $T(n)$. Then the negative triangle problem for graphs over $\mathcal{R}$ can be solved in $\mathcal{O}(T(n))$ time.

Definition: Consider a tripartite graph with parts I, J, K. We say a set of triangles $T \subseteq I \times J \times K$ is IJ-*disjoint*, if for all $(i,j,k), (i',j',k') \in T$, $(i,j) \neq (i',j')$ holds.

Lemma 3.2: Let $T(n) = \Omega(n)$ be a non-decreasing function. Given a $T(n)$ algorithm for triangle detection, there is an algorithm L, which outputs a maximal set of IJ-disjoint triangles in a tripartite graph with distinguished parts (I, J, K) in $\mathcal{O}(T(n^{1/3})n^2)$ time.

Theorem 1.1: Let $T(n) = \Omega(n)$ be a non-decreasing function. Suppose the negative triangle problem over $\mathcal{R}$ in an n-node graph can be solved in $T(n)$ time. Then the product of two $n \times n$ matrices over $\mathcal{R}$ can be performed in $\mathcal{O}(n^2 T(n^{1/3}) \log W)$ time, where W is the absolute value of the largest integer in the output.

Corollary 1.1: Suppose the negative triangle problem over $\mathcal{R}$ is in truly subcubic time. Then the product of two $n \times n$ matrices over $\mathcal{R}$ can be computed in truly subcubic time.

Corollary 1.2: Let $T(n) = \Omega(n)$ be a non-decreasing function. Suppose matrix product verification problem over $\mathcal{R}$ is in time $T(n)$. Then matrix multiplication over $\mathcal{R}$ is in $\mathcal{O}(n^2 T(n^{1/3}) \log W)$ time, where W is the absolute value of the largest integer in the output, i.e., matrix product verification over $\mathcal{R}$ is truly subcubic iff matrix multiplication over $\mathcal{R}$ is truly subcubic.

Corollary 3.3: Suppose matrix distance product verification can be done in $\mathcal{O}(n^{3-\delta})$ time for some $\delta > 0$. Then negative triangle detection is in $\mathcal{O}(n^{3-\delta})$ time, the distance product of two matrices with entries in $\{-W, \ldots, W\}$ can be computed in $\mathcal{O}(n^{3-\delta/3} \log W)$ time, and all pairs shortest paths for n node graph with edge weights in $\{0, \ldots, W\}$ can be solved in $\mathcal{O}(n^{3-\delta/3} \log(nW))$ time.

Theorem 3.1: All-pairs shortest paths in undirected graphs is in $\tilde{\mathcal{O}}(n^{3-\delta} \log^c M)$ time iff all pairs shortest paths in directed graphs is in $\tilde{\mathcal{O}}(n^{3-\delta} \log^c M)$ time.

Theorem 3.2: Let $T(n, M)$ be non-decreasing. Then there is an $\mathcal{O}(n^2) + T(\mathcal{O}(n), \mathcal{O}(M))$ time algorithm for negative triangle problem in n node graphs with weights in $[-M, M]$ iff there is an $\mathcal{O}(n^2) + T(\mathcal{O}(n), \mathcal{O}(M))$ algorithm for the metricity problem on $[n]$ such that all distances are in $[-M, M]$.

Corollary: All following problems either all have truly subcubic algorithms, or none of them do:

- The all pairs shortest paths problem on directed graphs.
- The all pairs shortest paths problem on undirected graphs.
- Detecting if a weighted graph has a triangle of negative total edge weight.
- Reporting $n^{2.99}$ negative triangles in a graph.
- Checking whether a given matrix defines a metric.
- Matrix multiplication over the $(\min, +)$-semiring.
- Verifying the correctness of a matrix product over the $(\min, +)$-semiring.

Corollary: All following problems either all have truly subcubic *combinatorial* algorithms, or none of them do:

- Boolean matrix multiplication.
- Detecting if a graph has a triangle.
- Reporting $n^{2.99}$ triangles in a graph.
- Verifying the correctness of a matrix product over the Boolean semiring.

Without use of algebra, best known Boolean matrix multiplication algorithm runs in $\mathcal{O}(n^3 / \log^{2.25} n)$.