Sorting under Partial Information (without the Ellipsoid Algorithm)

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BASIC DEFINITIONS:

Number of linear extensions:
Let P be a partially ordered set (poset). Denote $e(P)$ the number of linear extensions of P.

Problem definition:
Let $V = \{v_1, v_2, \ldots, v_n\}$ be a set equipped with an unknown linear order $\leq$. Given a subset of the relations $v_i \leq v_j$ determine the complete linear order by queries of the form: "*is* $v_i \leq v_j$?"

Entropy definition:
Let G be a grah. Then define

$$STAB(G) := conv\{\chi^S \in \mathbb{R}^{V(G)} : S \text{ stable set in } G\}$$

where χ^S is characteristic vector of the subset S. The entropy of G is defined as:

$$H(G) := \min_{x \in STAB(G)} -\frac{1}{n} \sum_{v \in V(G)} \log x_v$$

Definition of (in)comparability graph:
Let P be a poset. Then define comparability graph $G(P)$ as a graph with vertex set equal to ground set of P and two distinct vertices v and w are adjacent in $G(P)$ whenever they are comparable in P. The incomparability graph $\bar{G}(P)$ is defined as complement of $G(P)$. We denote by $H(P)$ the entropy of $G(P)$ and by $H(\bar{P})$ the entropy of $\bar{G}(P)$.

MAIN RESULTS:

Algorithm	Global complexity	Number of comparisons
K&K	$O(n \log n \cdot EA(n))$	$\leq 9.82 \cdot \log e(P)$
ALGORITHM 1	$O(n^2)$	$O(\log n \cdot \log e(P))$
ALGORITHM 2	$O(n^{2.5})$	$\leq (1+\varepsilon) \log e(P) + O_\varepsilon(n)$
ALGORITHM 3	$O(n^{2.5})$	$\leq 15.09 \log e(P)$

ALGORITHMS AND NECESARY LEMMAS AND THEOREMS:

Lemma 1 *For any poset P of order n,*

$$\log e(P) \leq nH(\bar{P}) \leq \min\{\log e(P) + \log e \cdot n, c_1 \log e(P)\}$$

where $c_1 = (1 + 7\log e) \simeq 11.1$.

Theorem 1 *For any poset P of order n,*

$$nH(\bar{P}) \leq 2\log e(P)$$

Lemma 2 *Assume G is a perfect graph of order n, then*

$$H(G) + H(\bar{G}) = \log n$$

Lemma 3 *In any poset P of order n that is not a chain there are a, b incomparable such that*

$$\max\{nH(\overline{P(a < b)}), nH(\overline{P(b < a)})\} \leq nH(\bar{P}) - c_2$$

where $c_2 = \log(1 + 17/112) \simeq 0.2$.

Algorithm 1:
phase 1: find a maximum chain $C \subset P$
phase 2: while $P - C \neq \emptyset$, remove an element of $P - C$ and insert it in C with a binary search
phase 3: return C

Lemma 4 *Let P be a poset of order n and let C be a maximum chain in P. Then $|C| \geq 2^{-H(P)}n$.*

Lemma 5 *For all $x \in \mathbb{R}$, $1 - 2^{-x} \leq \ln 2 \cdot x$.*

Theorem 2 *Let G be a perfect graph on n vertces and denote by $\widetilde{g}$ the entropy of an arbitrary greedy point in $STAB(G)$. Then for any $\varepsilon > 0$,*

$$\widetilde{g} \leq (1+\varepsilon)H(G) + (1+\varepsilon)\log(1 + \frac{1}{\varepsilon})$$

Algorithm 2:
phase 1: find a greedy chain decomposition $C_1, \ldots, C_k$ of P; $C \leftarrow \{C_1, \ldots, C_k\}$
phase 2: while $|C| > 1$
 pick the two smallest chains A and B in C
 merge A and B into a chain D, linearly
 $C \leftarrow C \setminus \{A, B\} \cup \{D\}$
phase 3: return the chain in C

Theorem 3 *The query complexity of Algorithm 2 is for every $\varepsilon > 0$ at most*

$$(\widetilde{g} + 1)n \leq (1+\varepsilon)\log e(P) + O_\varepsilon(n)$$