Asymptotically optimal frugal colouring

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1 Introduction and motivation

- A proper vertex coloring is β-frugal if no vertex has more than β members of any colour class in its neighbourhood.

Theorem 1 *There exists a constant Δ_0 such that every graph with maximum degree $\Delta \geq \Delta_0$ has a $(50 \log \Delta / \log \log \Delta)$-frugal $(\Delta + 1)$-colouring.*

- Motivation: total colouring
- History: Proved for $\beta = O(\log^8 \Delta)$ and later for $\beta = O(\log^2 \Delta / \log \log \Delta)$.
- Alon: class of examples that do not have a $(\log \Delta / \log \log \Delta)$-frugal $(\Delta + 1)$-colouring.
- Proving the existence of frugal colouring is quite easy (using Lovasz Local Lemma) if we do not require proper colouring.

2 Dense decomposition

Assume G is Δ-regular. There exists a dense decomposition of G into dense sets $D_1, \ldots, D_\ell$ and a collection S of sparse vertices. Let $\varepsilon = 10^{-6}$. The dense decomposition has the following properties. For every D_i

1. $\Delta - 5\varepsilon\Delta < |D_i| < \Delta + 2\varepsilon\Delta$
2. there are at most $4\varepsilon\Delta^2$ edges from D_i to $G - D_i$
3. every $v \in S$ has at least $\varepsilon\binom{\Delta}{2}$ pairs of non-adjacent vertices in its neighbourhood
4. each vertex is in D_i iff it has at least $\frac{3}{4}\Delta$ neighbours in D_i

Let $D = \bigcup_{i=1}^{\ell} D_i$. We colour each D_i by partitioning it into a set of classes C_i each of size 1 or 2. The properties are

1. $\Delta - 15\varepsilon\Delta \leq |C_i| \leq \Delta + 1$
2. each class in C_i has at most $(1/4 + 4\sqrt{\varepsilon})\Delta < \Delta/3$ external neighbours

More definitions:

- for $v \in D_i$, Out_v is the set of neighbours of v that are not in D_i (called external neighbours of v)
- D_i is ornery if $|C_i| > \Delta - \log^4 \Delta$

For ornery D_i we define

- kernel K_i, the set of vertices in D_i with at most $\log^6 \Delta$ external neighbours
- Big_i is the set of vertices outside D_i which have at least $\Delta^{7/8}$ neighbours in D_i
- $Notbig(i, x)$ is the set of vertices in D_i which do not have any external neighbours in $G - Big_i$ with colour x.
- u, v are big-neighbours if they are both in Big_i for some i

More properties:

1. Every vertex in G has at most $\Delta^{1/4} \log^7 \Delta$ big-neighbours.

3 Random colouring procedure

First of all, create the dense decomposition of G.

3.1 Phase 1 (Initial colouring)

All random choices are made independently.

1. Assign a uniformly random colour from $\{1, \dots, \Delta + 1\}$ to each $v \in S$.
2. For each D_i use $|C_i|$ colours uniformly from $\{1, \dots, \Delta + 1\}$ and then assign a random permutation of those colours to C_i.
3. Let $\{(x_1, y_1), \dots, (x_\ell, y_\ell)\}$ be the set of all pairs of neighbours or big-neighbours that are assigned the same colour. For each pair in that set choose one member, uniformly at random, to correct. To correct $v \in S$, uncolour v. To correct $v \in D$, label v as being temporarily coloured.

More definitions:

- $U \subseteq S$ is the vertices of S uncoloured in Step 3
- $Temp_i$ is the set of vertices of D_i that are labelled as temporarily coloured
- $Temp_i^*$ is the set of $u \in D_i$ such that $\{u, v\} \in C_i$ for some $v \in Temp_i$
- $Temp_i^+ \subseteq Temp_i^*$ is the set of vertices $u \in D_i$ such that $\{u, v\} \in C_i$ for some $v \in Temp_i$ with $|Out_v| < |Out_u|$
- $Temp_i(a)$ is the set $v \in Temp_i$ with $|Out_v| \leq a$
- $Temp = \bigcup Temp_i$

All vertices in $Temp$ will be recoloured during Phase 2 and 3. Vertices of $Temp^*$ might be also be recoloured during Phase 2.

For each ornery D_i we will recolour the vertices in $Temp_i \cap K_i$ during Phase 2 by swapping their colours with other vertices in D_i. To do this, we need one more step

4 For each ornery D_i we select uniformly at random a set F_i of $\frac{9}{10}\Delta$ of the vertices of K_i that are classes of size one in C_i.

F is the union over all ornery D_i of F_i.

Lemma 1 *(Properties of Phase 1 colouring) With positive probability*

1. *every $v \in S$ has at least $\frac{\varepsilon}{10^9}\Delta$ colours that appear twice in $N(v) - (U \cup Temp \cup Temp^* \cup F)$.*

2. *every $v \in D$ with $|Out_v| \geq \log^3 \Delta$ has at least $\frac{\varepsilon}{10^9}|Out_v|$ colours that appear twice in $N(v) - (U \cup Temp \cup Temp^* \cup F)$.*

3. *for each D_i and $a \in \{\lceil \log^3 \Delta \rceil, \ldots, \Delta\}$, we have $|Temp_i(a)| \leq 2a$*

4. *for each $v \in G$, no colour is assigned to more that $20 \log \Delta / \log \log \Delta$ vertices in $N(v)$*

5. *for each colour x and ornery D_i, $|Notbig(i, x)| \leq \Delta^{19/20}$*

6. *for each vertex $v \in G$,*

$$\sum_{u \in N(v) \cap (Temp \cup Temp^+)} \frac{1}{\max(|Out_u|, \log^3 \Delta)} \leq 299999.$$

3.2 Phase 2 (Kernels of the ornery dense sets)

Here we colour all vertices in kernels, that have the same colour as a neighbour. Consider any ornery D_i and any $v \in Temp_i(\log^6 \Delta)$. We will recolour v by swapping its colour with a suitable vertex in F_i. There are several conditions on vertices that are swapped with – we omit all details, but the conditions are that a single swap will not create a conflict. From these Swappable vertices we select 20 members uniformly at random – call them candidates. However, only some of these will not create conflicts by making multiple swaps.

We omit all details.

Lemma 2 *(Properties of Phase 2 colouring) With positive probability*

1. *for each ornery D_i, every vertex in $Temp_i(\log^6 \Delta)$ has a good candidate*

2. *for each vertex $v \in G$ and each colour x, at most $20 \log \Delta / \log \log \Delta$ neighbours of v have a candidate with colour x or are a candidate of a vertex with colour x.*

3.3 Phase 3 (Completing the colouring)

$L(u)$ denotes the set of colours that do not appear on neighbours of u.

1. Uncolour every vertex in $Temp$.

2. Let $v_1, \ldots, v_\ell$ be and ordering of uncoloured vertices such that the vertices of $Temp$ appear in non-decreasing order of $|Out_v|$.

3. For $i = 1$ to ℓ, assign to v_i a colour chosen uniformly at random from $L(v_i)$.

We define

- for $v \in U$, $Q(v) = \frac{\varepsilon}{10^9}\Delta$
- for $v \in Temp$, $Q(v) = \frac{\varepsilon}{10^9} \max\{|Out_v|, \log^3 \Delta\}$

Lemma 3 1. *when we colour $v \in U \cup Temp$, we have $|L(v)| \geq Q(v)$*

2. *for each vertex $v \in G$*

$$\sum_{u \in N(v) \cap (U \cup Temp)} \frac{1}{Q(u)} \leq \frac{3 \cdot 10^{14}}{\varepsilon}.$$

Lemma 4 *(Properties of Phase 3 colouring) With positive probability, for each $v \in G$ and each colour x, at most $4 \log \Delta / \log \log \Delta$ neighbours of v are assigned x during Phase 3.*

The tool used is

Lemma 5 *(Lopsided Local Lemma) Let $A = \{A_1, \ldots, A_n\}$ be a set of random events. Suppose that for each A_i we have a subset $B_i \subseteq A$ such that*

1. *for any subset $B \subset A - B_i$,*
$$P\left[A_i | \bigcap_{A_j \in B} \overline{A_j}\right] \leq p$$
2. $|B_i| \leq d$
3. $pd \leq 1/4$

Then $P[\overline{A_1} \cap \ldots \cap \overline{A_n}] > 0$.

Lemma 6 *For every $u \in (U \cup Temp) - N(v)$, choose any colour $c(u) \in L_0(v)$ such that for every adjacent u_1, u_2 we have $c(u_1) \neq c(u_2)$. Conditioning on the event that each such u is assigned $c(u)$ during Phase 3, the conditional probability of $A(v)$ is at most $\Delta^{-2}/4$.*

Lemma 7 *Consider any set of vertices $w_1, \ldots, w_t$ and any colour x. For every $u \in U \cup Temp - \{w_1, \ldots, w_t\}$, choose any colour $c(u) \in L_0(u)$ such that for every adjacent u_1, u_2 we have $c(u_1) \neq c(u_2)$. Conditioning on the event that each such u is assigned $c(u)$ during Phase 3, the conditional probability that $w_1, \ldots, w_t$ are all assigned x is at most $e^{6 \cdot 10^{14} t/\varepsilon} \cdot \prod_{i=1}^{t} 1/Q(w_i)$.*