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KERNELCLUSTERING

Input: $A = (a_{ij}) \in \mathbb{R}^{n \times n}$, $B = (b_{lp}) \in \mathbb{R}^{k \times k}$,
 A, B are PSD

Output: $\sigma : [n] \rightarrow [k]$

Goal: maximize $\sum_{i,j=1}^n a_{ij} b_{\sigma(i), \sigma(j)}$

Theorem. Let $\sigma : [n] \rightarrow [k]$ be given by the Algorithm- s on matrices A and B where A is centered, i.e. $\sum_{i,j=1}^n a_{ij} = 0$. Then it holds true

$$\mathbf{E} \left[\sum_{i,j} a_{ij} b_{\sigma(i)\sigma(j)} \right] \geq \frac{R_1^s \sum_{l \in S} \|v_l - v\|_2^2}{(s-1)R_B^2} OPT,$$

where $R_1^{(s)}$ is a positive constant and $OPT = \text{Clust}(A|B)$ is the optimum of the KERNELCLUSTERING problem. In the case of Algorithm-2 we have $R_1^{(2)} = 1/\pi$ and we have

$$\text{Clust}(A|B) \leq \pi(1 - 1/k) \mathbf{E} \left[\sum_{i,j=1}^n a_{ij} b_{\sigma(i)\sigma(j)} \right].$$

(It can be further proved that the best bounds from the Theorem are obtained for $s = 2$ or $s = 3$!)

Outline:

1. $\text{SDP} \geq \text{Clust}(A|B)$
2. WLOG $\|x_i^*\|_2^2 = 1$ for every $i = 1, \dots, n$
3. $\text{SDP} = R_B^2 \sum_{i,j=1}^n a_{ij} \langle x_i^*, x_j^* \rangle$
4. To prove Theorem, it suffices to show
5. Similarly to MAX-CUT, we evaluate expected contribution for every pair of indices i, j separately. We want to show that

$$\mathbf{E}[b_{\sigma(i)\sigma(j)}] = \|v\|_2^2 + \frac{\sum_{l \in S} \|v_l - v\|_2^2}{s-1} \sum_{m=1}^{\infty} R_m^{(s)} \sum_{i,j=1}^n \langle x_i^*, x_j^* \rangle^m.$$

6. To get 5. we start by using the symmetry of g_l 's (independent and identically distributed):

$$\mathbf{E}[b_{\sigma(i)\sigma(j)}] = \Pr[\sigma(i) = \sigma(j)] \sum_{l \in S} b_{ll}/s + (1 - \Pr[\sigma(i) = \sigma(j)]) \sum_{l \neq p} b_{lp}/(s(s-1))$$

and proceed with terrible computations.

Algorithm- s , $s = 2, 3, \dots$

1. Compute $v_1, \dots, v_k \in \mathbb{R}^k$ s.t. $\langle v_l, v_p \rangle = b_{lp}$ (Cholesky decomposition)
2. Compute the smallest ball $B = (w_B, R_B)$ containing v_1, \dots, v_k
3. Solve: $\text{SDP} := \max \{ \sum_{i,j=1}^n a_{ij} \langle u + R_B x_i, u + R_B x_j \rangle \mid u, x_1, \dots, x_n \in \mathbb{R}^{n+1}, \|u\|_2^2 = \|w_B\|_2^2, \forall i : \|x_i\|_2^2 \leq 1 \}$
4. Find $S \subseteq [k], |S| = s$ maximizing $\sum_{l \in S} \|v_l - v\|_2^2$ where $v = \sum_{p \in S} v_p / s$
5. Choose random indep. Gaussian $g_l \in \mathbb{R}^{n+1}$ for $l \in S$
6. Let $\sigma(i) = l$ whenever $\langle x_i^*, g_l \rangle = \max_{p \in S} \langle x_i^*, g_p \rangle$
7. Output σ

