The Lonely Runner with Seven Runners

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4. března 2010

Problem:
Suppose $k+1$ runners having nonzero constant speeds run laps on a unit-length circular track starting at same time-space. A runner is said to be lonely if she is at distance at least $\frac{1}{k+1}$ along the track to every other runner. The lonely runner conjecture states that every runner gets lonely.

We denote by $(x)_n$ the residue class of x modulo n in $\{0, \ldots, N-1\}$and by $|x|_n$ the residue classs x or $-x$ modulo n in $\{0, \ldots, \lfloor N/2 \rfloor\}$.

The regular chromatic number $\chi_r(N, D)$ is define as

$$\chi_r(N, D) = \min\{k : \exists \lambda \in \mathbb{Z}_N \text{ such that } |\lambda d|_N \geq \frac{n}{k} \text{ for each } d \in D\},$$

if D contains no multiples of N and $\chi_r(N, D) = \infty$ otherwise.
We also define regular chromatic number of D as

$$\chi_r(D) = \liminf_{N \to \infty} \chi_r(N, D)$$

.

Conjecture 1. *For every set $D \subset \mathbb{Z}$ of positive integers with* $\gcd(D) = 1$*,*

$$\chi_r(D) \leq |D| + 1.$$

For positive integer x and a prime p, the p-adic valuation of x is

$$v_p(x) = \max\{k : x \equiv 0 \pmod{p^k}\}$$

and we also denote by $r_p(x) = (xp^{-v_p(x)})_p$ as p-ary expansion of x.

Notation D is set of positive integers, $m = \max v_p(D)$ and set $N = p^{m+1}$.

The p-levels of D are

$$D_p(i) = \{d \in D : v_p(d) = i\}.$$

Let $q = q_{p,m}$ be define as

$$q(x) = (\lfloor \frac{x}{p^m} \rfloor)_p.$$

Set of multipliers are define as follows

$$\Lambda_{j,p} = \{1 + p^{m-j}, 0 \leq k \leq p-1\}, j = 0, \dots, m-1$$

and

$$\Lambda_{m,p} = \{1, 2, \dots, p-1\}.$$

Lemma 2 (Prime Filtering). *Let p be a prime and let D be a set of positive integers. Set $m = \max v_p(D)$ and $N = p^{m+1}$. For each $d \in D$ let $F_d \subset \mathbb{Z}$. Suppose that*

$$\sum_{d \in D_p(j)} |F_d| \leq p-1, j = 0, 1, \dots, m-1$$

$$\sum_{d \in D_p(m)} |F_d| \leq p-2.$$

Then there is a multiplier λ such that, for each $d \in D$.

$$q(\lambda d) \notin F_d.$$

Corollary 3. *With the notation of Lemma 2, suppose that $|d|_N \geq N/p$ for each $d \in D_p(i)$ and each $i \geq i_0$ for some positive integer $i_0 \leq m$. If*

$$|D_p(j)| = \frac{p-1}{2}, j = 0, 1, \dots, i_0 - 1,$$

then

$$\chi_r(N, D) \leq p.$$