Ramsey games with giants

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1 Introduction and motivation

- $G_{n,m}$ is a random graph with m edges and n vertices, $G_{n,p}$ is a random graph on n vertices where each edge has a probability p of being included in the graph.

Theorem 1 *(Erdős, Rényi) If $m \leq (1-\varepsilon)\frac{n}{2}$ for a constant $\varepsilon > 0$, then w.h.p. $G_{n,m}$ has all of its connected components of order $O(\log n)$. If $m \geq (1+\varepsilon)\frac{n}{2}$ then w.h.p. $G_{n,m}$ has a unique connected component of linear size ("a giant").*

- controlled random graph processes, algorithms trying to achieve some goal on random graph, online and offline setting
- Achlioptas process: random edges arrive in pairs, online algoriths one puts into the graph and the second returns back. The goal is to delay the appearance of giant component as long as possible. Solution: $0.535n$ rounds.
- Our setting: algorithm colours incoming random edges by one of $r \geq 2$ colours.
- We study: creating a giant in every colour class, avoiding a giant in every colour class. Offline setting. Another variant studied in the paper is the online setting.

2 Tools

Theorem 2 *(Ajtai,Komlós,Szemerédi) For $m > (1+\varepsilon)\frac{n}{2}$ w.h.p. it is possible to colour the edges of $G_{n,m}$ with $r \geq 2$ colours, so that every colour class contains a component of size $\Omega(n)$.*

Fact 1 *("monotone properties for $G_{n,p}$ holds for $G_{n,m}$ and vice versa") Fix $\varepsilon > 0$, suppose that $m = m(n)$ tends to infinity, but $m = o(n^2)$. Then*

1. *$G_{n,m} \subseteq G_{n,p}$ w.h.p. for $p = (1+\varepsilon)\frac{2m}{n^2}$, and $G_{n,p} \subseteq G_{n,m}$ w.h.p. for $p = (1-\varepsilon)\frac{2m}{n^2}$.*
2. *The graph created by generating m random edges (possibly with repetition) is contained in $G_{n,m}$ and w.h.p. contains $G_{n,m'}$ with $m' = (1-\varepsilon)m$.*

Theorem 3 *(Cain, Sanders, Wormald; Fernholz, Ramachandran) For any integer $r \geq 2$, there is and explicit threshold ψ_r such that the following holds.*

1. *For $\varepsilon > 0$, if $m < (\psi_r - \varepsilon)n$, then $G_{n,m}$ is r-orientable w.h.p.*
2. *If $m > (\psi_r + \varepsilon)n$, then w.h.p. $G_{n,m}$ contains a subgraph with average degree at least $2r + c_\varepsilon$ ($c_\varepsilon > 0$).*
3. *The asymptotic dependence of ψ_r on r is $\psi_r = r - \frac{1}{2}\left(\frac{2}{e} + o(1)\right)^r$.*

- Parameter for studying the giant: susceptibility of graph $S(G) = \frac{1}{n}\sum_v C_v$, where C_v is the size of component containing v.
- $nS(G)$ equals the sum of squares of all component sizes.
- Spencer, Wormald: For $m < (1-\varepsilon)\frac{n}{2}$ is $S(G_{n,m})$ similar to the solution of the differential equation $\phi' = \frac{2}{n}\phi^2$ with $\phi(0) = 1$.

3 (Some) Results

Proposition 1 *("creating giant is easy") For $m < (1-\varepsilon)\frac{n}{2}$ it is w.h.p. impossible to create a giant in every colour class. For $m > (1+\varepsilon)\frac{n}{2}$ it is w.h.p. possible to colour edges of $G_{n,m}$ so that every class contains a giant.*

Theorem 4 *Given fixed r, let ψ_r be the threshold referenced above. For $\varepsilon > 0$, if $m < (\psi_r - \varepsilon)n$, then w.h.p. it is possible to colour the edges of $G_{n,m}$ with r colours such that every colour class contains components of size only $o(n)$. If $m > (\psi_r + \varepsilon)n$, then w.h.p. every edge r-colouring of $G_{n,m}$ has a colour class with a component of size at least $c_\varepsilon n$.*

To prove the previous theorem, we use the following lemmas.

Lemma 1 *The edges of any r-orientable graph G can be coloured by r colours such that for every distinct vertices u, v, there are at most 2 monochromatic paths in each colour connecting u and v.*

Lemma 2 *For any $\varepsilon > 0$ there is $c > 0$ such that the following holds. Let G be a graph on n vertices with $\Delta(G) \leq \log n$, where every distinct $u, v \in V(G)$ are connected by at most 2 distinct paths. Independently delete each edge of G with probability ε. Then, w.h.p. all components of the resulting graph have size at most $ne^{-c\frac{\log n}{\log\log n}} = o(n)$.*

Lemma 3 *For any $\lambda, \varepsilon > 0$, there is a constant $c > 0$ such that in $G_{n,p}$ with $p = \lambda/n$, w.h.p. every set of at most cn vertices induces a subgraph with average degree less than $2 + \varepsilon$.*

Corollary 1 *For any $\lambda, \varepsilon > 0$, there is a constant $c > 0$ such that in $G_{n,m}$ with $m = \lambda n$, w.h.p. every subgraph with average degree at least $2 + \varepsilon$ contains a component of order at least cn.*