Point configurations that are asymmetric yet balanced

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• Finite set $\mathcal{C} \subset S^{n-1}$ of points on the unit sphere in $\mathbb{R}^n$

• Given $x \in \mathcal{C}$ and $u \in \mathbb{R}$, let $S_u(x)$ be the set of $y \in \mathcal{C}$ such that $\langle x, y \rangle = u$.

• $\mathcal{C}$ is *balanced* iff it is in equilibrium under any force law
 that is, if $\forall x \in \mathcal{C}\ \forall u \in \mathbb{R}\ \exists c \in \mathbb{R} : \sum_{y \in S_u(x)} y = cx$

• *Isometry group of* $\mathcal{C}$ ...Elements are bijection $f : \mathbb{R}^n \to \mathbb{R}^n$ that preserve distances and map $\mathcal{C}$ on $\mathcal{C}$. The operation is composition.

• $\mathcal{C}$ is *group-balanced* iff $\forall x \in \mathcal{C}$: the stabilizer of x in the isometry group fixes only multiples of x

Observation. *Every group-balanced configuration is balanced.*

Theorem 1. *(Leech 1957) Every balanced configuration in* $\mathbb{R}^3$ *is group-balanced.*

Theorem 2. *(Main theorem) There exists a configuration in* $\mathbb{R}^7$ *that is balanced, but not group-balanced.*

Conjecture. *Every balanced configuration in* $\mathbb{R}^4$ *is group-balanced.*

• $\mathcal{C} \subset S^{n-1}$ is a *spherical t-design* iff for every polynomial $p : \mathbb{R}^n \to \mathbb{R}$ of total degree at most t the average of p over $\mathcal{C}$ is the same as over S^{n-1}.

Theorem 3. *If for each x in a spherical t-design $\mathcal{C}$*

$$|\{\langle x, y \rangle : y \in \mathcal{C}, y \neq \pm x\}| \leq t,$$

then $\mathcal{C}$ is balanced.

Proof. Let $\{u_1, \dots u_k\} = \{\langle x, y \rangle : y \in \mathcal{C}, y \neq \pm x\}$.
For each $x \in \mathcal{C}$ and $i \in [k]$ fix $y \in S^{n-1}$ orthogonal to x. Take polynomial

$$p(z) = \langle y, z \rangle \prod_{j \in [k] \setminus \{i\}} (\langle x, z \rangle - u_j).$$

• $p(z)$ is an odd function on cross-sections where $\langle x, z \rangle$ is constant $\Rightarrow$ average over S^{n-1} is 0

• On $\mathcal{C}$, $p(z)$ is nonzero only when $\langle x, z \rangle = u_i \Rightarrow$ sum of such z's is orthogonal to x

□

Lemma 1. $\{x_1, \dots, x_k\} \subset S^{n-1}$ *is a spherical* t*-design iff it is a spherical* $(t-1)$*-design and there exists* $c \in \mathbb{R}$ *such that*

$$\forall v \in S^{n-1} : \sum_{i=1}^{k} \langle x_i, v \rangle^t = c.$$

Proof of the Main Theorem.

• simplex is a spherical 2-design

• $\mathcal{C}_n \subset S^{n-1}$ ... midpoints of edges of the n-dim. simplex (properly scaled)

• $\mathcal{C}_n$ is a spherical 2-design

• $\mathcal{C}'_7$ ... $\mathcal{C}_7$ with the midpoints of 4 disjoint edges replaced by their antipodes (the *special points*)

• $\mathcal{C}_7$ and $\mathcal{C}'_7$ are 2-distance sets

• $\mathcal{C}'_7$ is a spherical 2-design

$\Rightarrow \mathcal{C}'_7$ is balanced

• There are $2^4 4!$ symmetries of $\mathcal{C}'_7$ - we can only permute the 4 pairs of vertices corresponding to the special points and swap the vertices of each pair.

• The group of symmetries of $\mathcal{C}'_7$ has two orbits: the special points and the remaining ones.

• Every stabilizer of every point in the larger orbit fixes at least one more point $\Rightarrow \mathcal{C}'_7$ is not group-balanced. $\square$