List coloring in the absence of a linear forest

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Introduction

$G = (V, E)$ is finite, undirected graph, no loops, no multiple edges.

A coloring of G is a mapping $c : V \to \{1, 2, \dots\}$ such that

$$c(u) \neq c(v) \text{ whenever } uv \in E.$$

A coloring c of G is a k-coloring if $c(u) \in \{1, \dots, k\}$ for all $u \in V$.

COLORING
Instance: a graph G and an integer k.
Question: does G have a k-coloring?

k-COLORING
Instance: a graph G.
Question: does G have a k-coloring?

Motivation

We are interested in special graph classes because of the following well-known result.

Theorem (Karp, 1972)

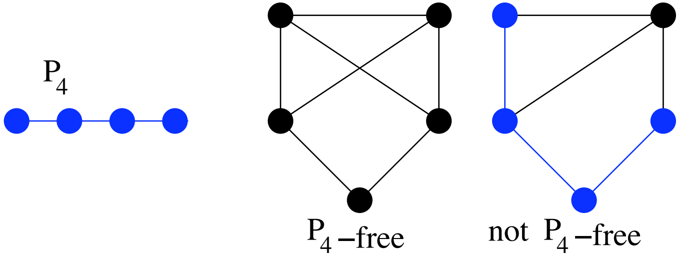
k-Coloring can be solved in polynomial time for $k \leq 2$, and it is NP-complete for $k \geq 3$.

H-free graphs

Let G and H be two graphs.

G is H-free if G contains no *induced* subgraph isomorphic to H.

Example. Let P_ℓ denote the path on ℓ vertices.



Let $F + G$ denote the disjoint union of graphs F and G.

Theorem (Král', Kratochvíl, Tuza & Woeginger, 2001)

Let H be a fixed graph.

If H is a (not necessarily proper) induced subgraph of P_4 or of $P_1 + P_3$ then Coloring *can be solved in polynomial time for H-free graphs.*

If not then Coloring *is* NP*-complete for H-free graphs.*

This theorem can be extended by forbidding more than one induced subgraph, e.g., the triangle and some other graph.

Then the classification is still open, but partial results have been obtained, see e.g. recent work of Dabrowski, Lozin, Raman and Ries [2010].

Overview

In contrast to COLORING, the complexity classification of

k-COLORING for H-free graphs

is still wide open, although partial results are known.

1. We state known results.
2. We discuss new results, in particular for LIST k-COLORING.
3. We mention some open borderline cases.

The girth of a graph G is the number of vertices in a shortest induced cycle of G.

Theorem (Kamiński & Lozin, 2007)

*For any fixed $k \geq 3$ and $p \geq 3$, k-*Coloring *is* NP*-complete for graphs of girth at least p.*

Let H be a graph with girth q that is not a forest. Note that $q \geq 3$ and that q is finite.

Then any graph of girth at least $q + 1 \geq 4$ is H-free.

By Kamiński & Lozin's theorem, k-Coloring is NP-complete for graphs of girth at least $q + 1$.

Corollary

*For any fixed $k \geq 3$, k-*Coloring *is* NP*-complete for H-free graphs if H has a cycle.*

Hence, we only need to consider the cases when H is a forest.

A claw is a 4-vertex star.

Theorem (Holyer, 1981)

3-Coloring *is* NP*-complete for claw-free graphs.*

Theorem (Leven & Galil, 1983)

For any fixed $k \geq 4$*,* k-Coloring *is* NP*-complete for claw-free graphs.*

If H is a forest with a vertex of degree at least 3, then any claw-free graph is H-free.

Corollary

Fro any fixed $k \geq 3$*,* k-Coloring *is* NP*-complete for* H*-free graphs if* H *is a forest with a vertex of degree at least 3.*

Hence, we only need to consider the cases when H is a linear forest, i.e., the disjoint union of one or more paths.

Linear Forests

- We first consider linear forests that are connected: the paths.
- We then consider linear forests that are not connected.

Existing results for paths

Theorem (Hoàng, Kamiński, Lozin, Sawada & Shu, 2008)

For any fixed $k \geq 1$, the k-COLORING problem can be solved in polynomial time for P_5-free graphs.

The proof for $k = 3$ goes as follows. Let G be a P_5-free graph.

A vertex subset D of a graph G is dominating if every vertex of G is either in D or adjacent to a vertex of D.

A clique is a set of mutually adjacent vertices.

Lemma (Bacsó & Tuza, 1990)

G has a dominating P_3 or a dominating clique.

If G contains a clique of size 4, then G is not 3-colorable. Hence:
If G is 3-colorable then G has a dominating set D of size $|D| \leq 3$.

Then we do as follows for a given P_5-free graph G on n vertices.

Step 1. Check if G has a dominating set of size at most 3.

If not, then output `No`. Suppose that we find a dominating set D of size $|D| \leq 3$.

Step 2. Guess a 3-coloring of D.

Afterwards there are only 2 possible colors left for every $u \in V \setminus D$, because such a vertex u has a neighbor in D.

Step 3. Translate the problem to 2-SATISFIABILITY and solve it.

If a solution is found, then output `Yes`.

Otherwise, go to step 2 until all 3-colorings of D are considered.

Running Time

- *Step 1 takes $O(n^3)$ time by brute force.*
- *D has at most $3^{|D|} \leq 3^3$ different 3-colorings.*
- *2-SATISFIABILITY can be solved in polynomial time.*

Hence, the algorithm runs in polynomial time.

More results for paths

Theorem (Randerath & Schiermeyer, 2004)

3-COLORING *can be solved in polynomial time for* P_6*-free graphs.*

Theorem (Broersma, Fomin, Golovach & P., 2009)

6-COLORING *is* NP*-complete for* P_7*-free graphs.*

Theorem (Broersma, Golovach, P. & Song, 2010)

4-COLORING *is* NP*-complete for* P_8*-free graphs.*

k-Coloring for P_ℓ-free graphs

	$k \rightarrow$						
P_ℓ-free	3	4	5	6	7	8	...
$\ell = 1$	P	P	P	P	P	P	...
$\ell = 2$	P	P	P	P	P	P	...
$\ell = 3$	P	P	P	P	P	P	...
$\ell = 4$	P	P	P	P	P	P	...
$\ell = 5$	P	P	P	P	P	P	...
$\ell = 6$	P	?	?	?	?	?	...
$\ell = 7$	?	?	?	NP-c	NP-c	NP-c	...
$\ell = 8$	?	NP-c	NP-c	NP-c	NP-c	NP-c	...
$\ell = 9$	?	NP-c	NP-c	NP-c	NP-c	NP-c	...
$\ell = 10$	?	NP-c	NP-c	NP-c	NP-c	NP-c	...
$\ell = 11$	?	NP-c	NP-c	NP-c	NP-c	NP-c	...
$\ell = 12$	?	NP-c	NP-c	NP-c	NP-c	NP-c	...
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

ℓ-Coloring for P_k-free graphs

	$k \rightarrow$			
P_ℓ-free	3	4	5	≥ 6
$\ell \leq 5$	P	P	P	P
$\ell = 6$	P	?	?	?
$\ell = 7$	?	?	?	NP-c
$\ell \geq 8$	?	NP-c	NP-c	NP-c

1. Is 3-Coloring polynomial-time solvable for P_7-free graphs?
2. Is 4-Coloring polynomial-time solvable for P_6-free graphs?
3. Is 5-Coloring NP-complete for P_7-free graphs?
4. Is 3-Coloring NP-complete for P_ℓ-free graphs for some ℓ?
5. Is k-Coloring NP-complete for P_6-free graphs for some k?

Disconnected Linear forests

The following results also include disconnected linear forests.

Theorem (Broersma, Golovach, P. & Song, 2010)

Let H be a linear forest on at most 6 vertices. Then 3-COLORING *can be solved in polynomial time for H-free graphs.*

Let sH denote the disjoint union of s copies of H.

Theorem (Broersma, Golovach, P. & Song, 2010)

3-COLORING *can be solved in polynomial time for sP_3-free graphs for any fixed $s \geq 1$.*

New Results

Recall the result for P_5-free graphs.

Theorem (Hoàng, Kamiński, Lozin, Sawada & Shu, 2008)

For any fixed $k \geq 1$, the k-COLORING problem can be solved in polynomial time for P_5-free graphs.

As a matter of fact they prove something stronger.

The LIST k-COLORING problem is to test whether a given graph G has a k-coloring in which every vertex u receives a color from some given list $L(u) \subseteq \{1, \ldots, k\}$.

Theorem (Hoàng, Kamiński, Lozin, Sawada & Shu, 2008)

For any fixed integer $k \geq 1$, the LIST k-COLORING problem can be solved in polynomial time for P_5-free graphs.

Theorem (Couturier, Golovach, Kratsch, P., 2011)

For any fixed $k \geq 1$ and $r \geq 0$, the LIST k-COLORING *problem can be solved in polynomial time for $(P_5 + rP_1)$-free graphs.*

The proof is based on the technique of Hoàng, Kamiński, Lozin, Sawada & Shu.

Step 1. Check if the graph has a dominating set $D = \{d_1, \ldots, d_q\}$ with $q \leq r + 4$. If not then output `No`.

Step 2. Determine sets $F_1, \ldots, F_q$ where F_i is the set of vertices in $V \setminus D$ adjacent to d_i but not to any d_h with $h \leq i - 1$.

Step 3. For each precoloring of the vertices in D, try to separate the sets $F_1, \ldots, F_q$, i.e., to modify lists of admissibe colors such that the lists of any two vertices from any two different sets F_i and F_j have an empty intersection.

Step 4. Solve the problem recursively on the graphs induced by the sets $F_1, \ldots, F_q$, respectively.

Tightness

Theorem (Couturier, Golovach, Kratsch, P., 2011)

Let H be a supergraph of $P_1 + P_5$ with at least 5 edges. Then LIST 5-COLORING *is* NP*-complete for H-free graphs.*

Parameterized Complexity

Theorem (Jansen and Scheffler, 1997)

The LIST *k*-COLORING *problem is in* FPT *for P_4-free graphs when parameterized by k.*

Theorem (Couturier, Golovach, Kratsch, P., 2011)

The LIST *k*-COLORING *problem is in* FPT *for $(rP_1 + P_2)$-free graphs when parameterized by k and r.*

*For any fixed $r \geq 2$, the k-*COLORING *problem has a kernel of size $k^2(r-1)$ for $(rP_1 + P_2)$-free graphs when parameterized by k.*

Theorem (Couturier, Golovach, Kratsch, P., 2011)

The LIST *k*-COLORING *problem is in* FPT *for $(P_1 + P_3)$-free graphs when parameterized by k.*

Some more open problems

1. Classify the complexity of k-Coloring for H-free graphs, where H is a graph with $|V_H| \leq 5$ and k is an integer.

   The only remaining case is $H = P_2 + P_3$; all other cases are polynomial-time solvable.

2. Classify the complexity of 4-Coloring for H-free graphs, where H is a graph with $|V_H| \leq 6$.

   There are many remaining linear forests, e.g., $H = P_6$.

3. Recall that Coloring can be solved in polynomial time if H is an induced subgraph of $P_1 + P_3$ or of P_4.

   Is k-Coloring fixed-parameter tractable in k for H-free graphs, where H is a linear forest with $|V_H| \leq 4$ that is not isomorphic to an induced subgraph of $P_1 + P_3$ or of P_4?

   The only remaining case is $H = 2P_2$; all other cases are in FPT.