# Split clique graph complexity

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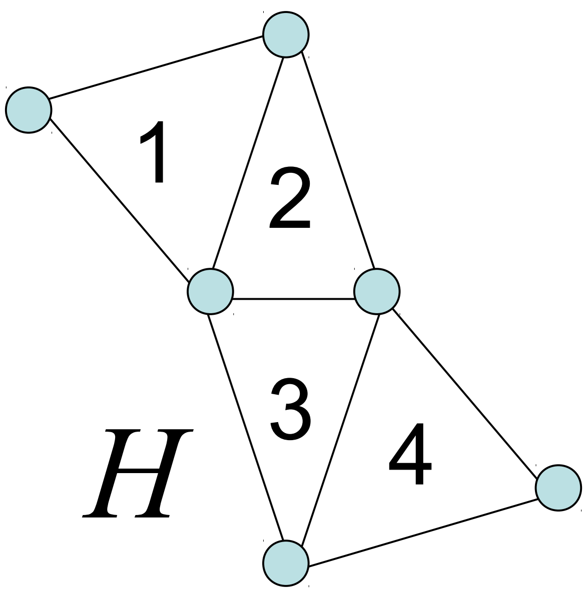
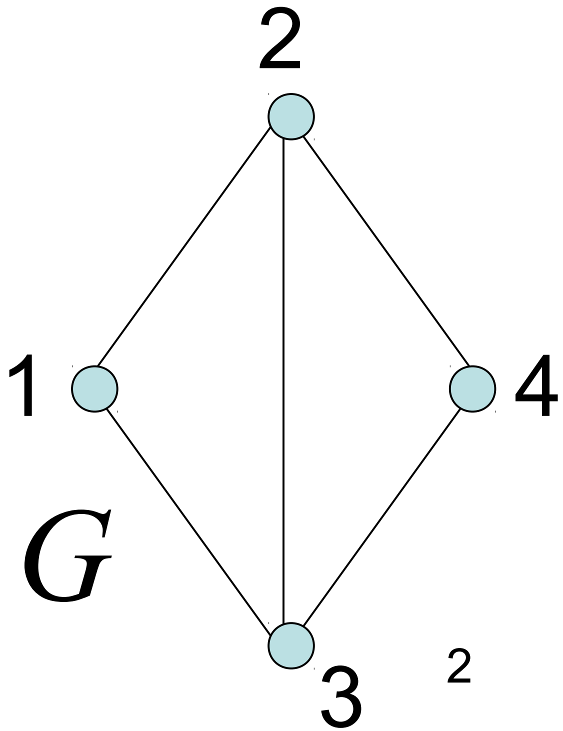
# Clique graph

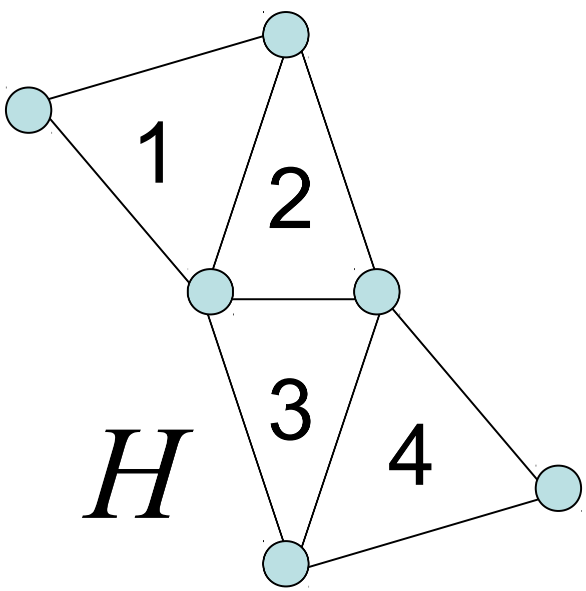
- A graph $G$ is the *clique graph* of a graph $H$ if $G$ is the graph of intersection in vertices of the maximal cliques of $H$.

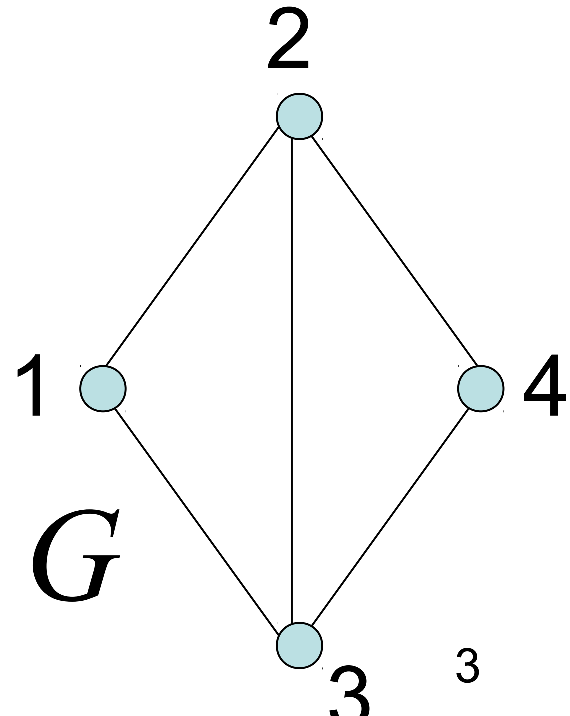


# Our Problem

CLIQUE GRAPH
INSTANCE: A graph $G = (V, E)$.
QUESTION: Is there a graph $H$ such that $G = K(H)$?

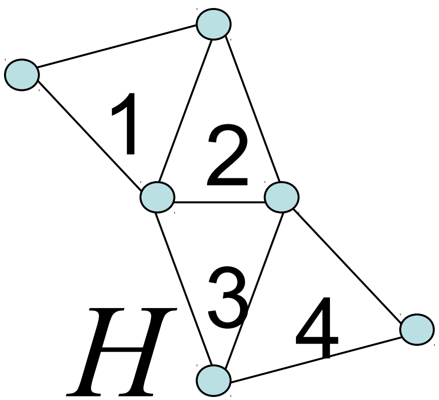


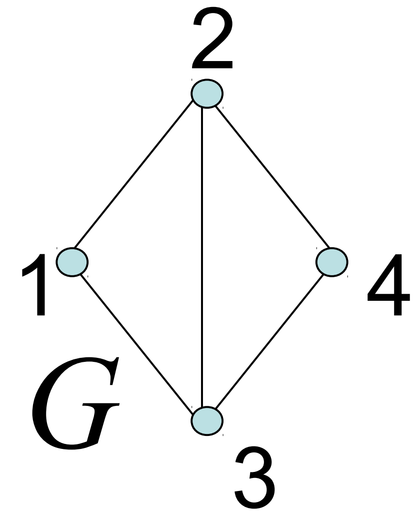
# Previous result

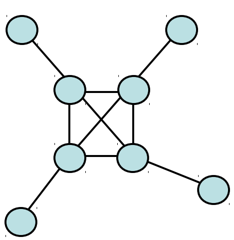
- WG'2006 – CLIQUE GRAPH is NPC for graphs with maximum degree 14 and maximum clique size 12

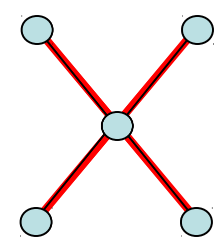
# Classes of graphs where CLIQUE GRAPH is polynomial




| CLASS $\mathcal{A}$ | $K(\mathcal{A})$ |
|---|---|
| *BLOCK* | *BLOCK* |
| *CLIQUE-HELLY* | *CLIQUE-HELLY* |
| *CHORDAL* | *DUALLY CHORDAL* |
| *CLOCKWORK* | *CLOCKWORK* |
| *DE* | *DUALLY DE* |
| *DIAMOND FREE* | *DIAMOND FREE* |
| *DISK-HELLY* | *DISK-HELLY* |
| *DISMANTABLE* | *DISMANTABLE* |
| *DUALLY CHORDAL* | *CHORDAL* $\cap$ *CLIQUE-HELLY* |
| *DUALLY DE* | *DE* |
| *DUALLY DV* | *DV* |
| *DUALLY RDV* | *RDV* |
| *DV* | *DUALLY DV* |
| $\mathcal{H}_1$ | $\mathcal{H}_1$ |
| *HELLY CIRCULAR ARC* | *CIRCULAR CLIQUE* |
| *HELLY HEREDITARY* | *HELLY HEREDITARY* |
| *INTERVAL* | *PROPER INTERVAL* |
| *MIN PROPER INTERVAL* | *PROPER INTERVAL* |
| *PROPER INTERVAL* | *PROPER INTERVAL* |
| *PTOLOMAIC* | *PTOLOMAIC* |
| *RDV* | *DUALLY RDV* |
| *SPLIT* | *STAR* |
| *STRONGLY CHORDAL* | *STRONGLY CHORDAL* |
| *TREE* | *BLOCK* |
| *UV* | *DUALLY CHORDAL* |





Survey of Jayme L. Szwarcfiter

**Theorem 1 (Roberts and Spencer '71)** *G is a clique graph if and only if there exists a complete edge cover of G satisfying the Helly property.*

# A quest for a non-trivial

Polynomial decidable class

# Chordal?

- Clique structure, simplicial elimination sequence, … ?

# Split?

- Same as chordal, one clique and one independent set … ?

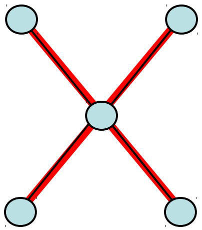
# Our class: Split

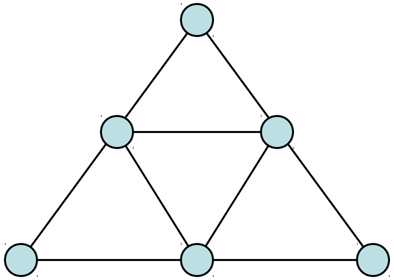
- G=(V,E) is a split graph if V=(K,S), where K is a complete set and S is an independent set

# Main used statements

**Theorem 1 (Roberts and Spencer '71)** *$G$ is a clique graph if and only if there exists a complete edge cover of $G$ satisfying the Helly property.*

**Lemma 2 (Alcón and Gutierrez '04)** *Let $\mathcal{F}$ be a complete edge cover of $G$. The following conditions are equivalent:*
*i) $\mathcal{F}$ has the Helly property.*
*ii) For every $T \in T(G)$, the subfamily $\mathcal{F}_T$ has the Helly property.*
*iii) For every $T \in T(G)$, the subfamily $\mathcal{F}_T$ has nonempty intersection, this means $\cap \mathcal{F}_T \neq \emptyset$.*



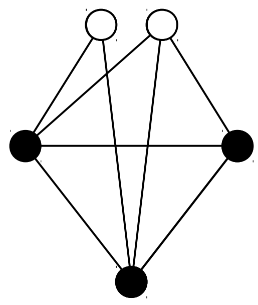


# ksplit$_p$

- Given a pair of integers $k \geq p$, G is ksplit$_p$ if G is a split (K,S) graph and for every vertex s of S, $p \leq d(s) \leq k$.

Example

of <u>3split$_2$</u>



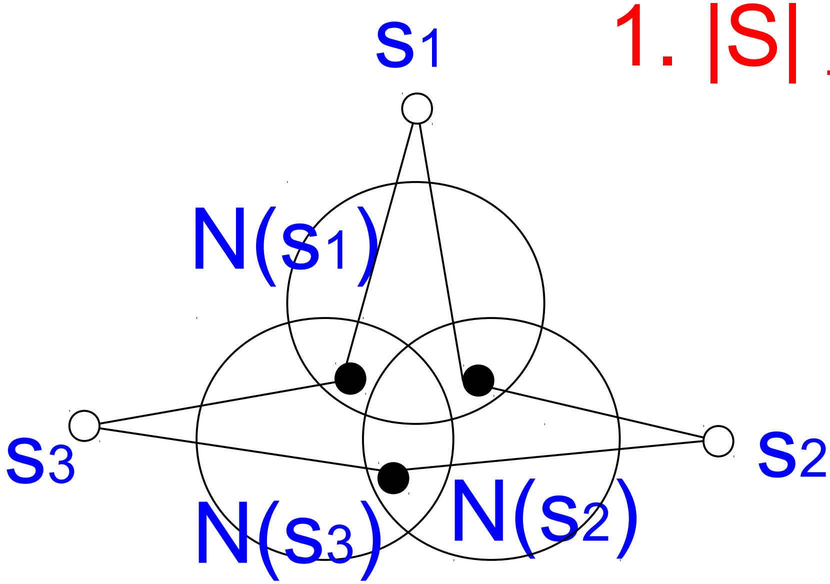
# In this talk

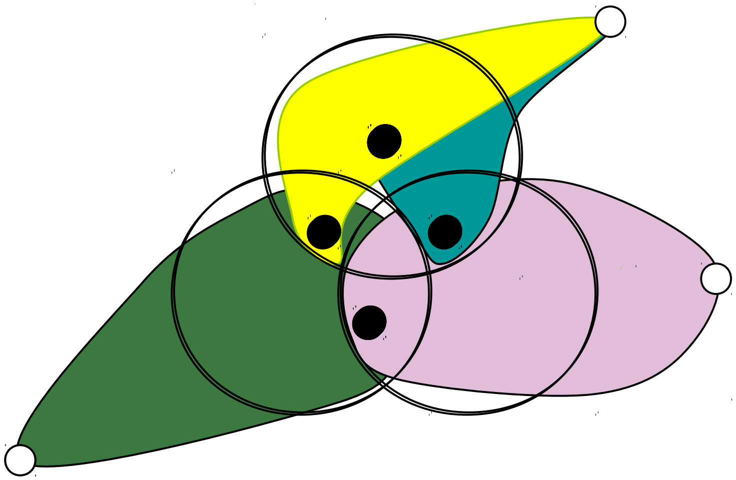
- Establish 3 subclasses of split (K,S) graphs in P:

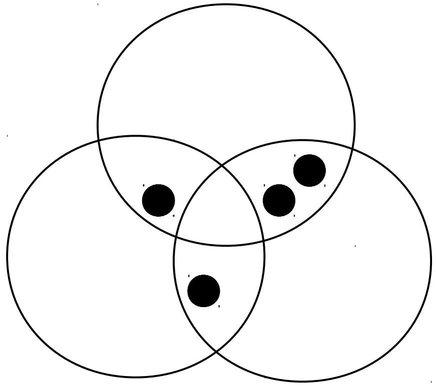
1. $|S| \leq 3$ 2. $|K| \leq 4$ 3. s has a private neighbour

- CLIQUE GRAPH is NP-complete for $3split_2$

1. $|S| \leq 3$



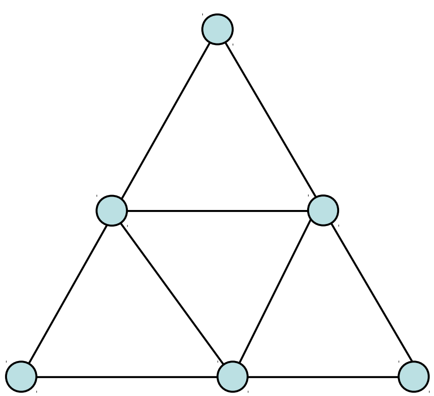





G is clique graph

iff

G is not

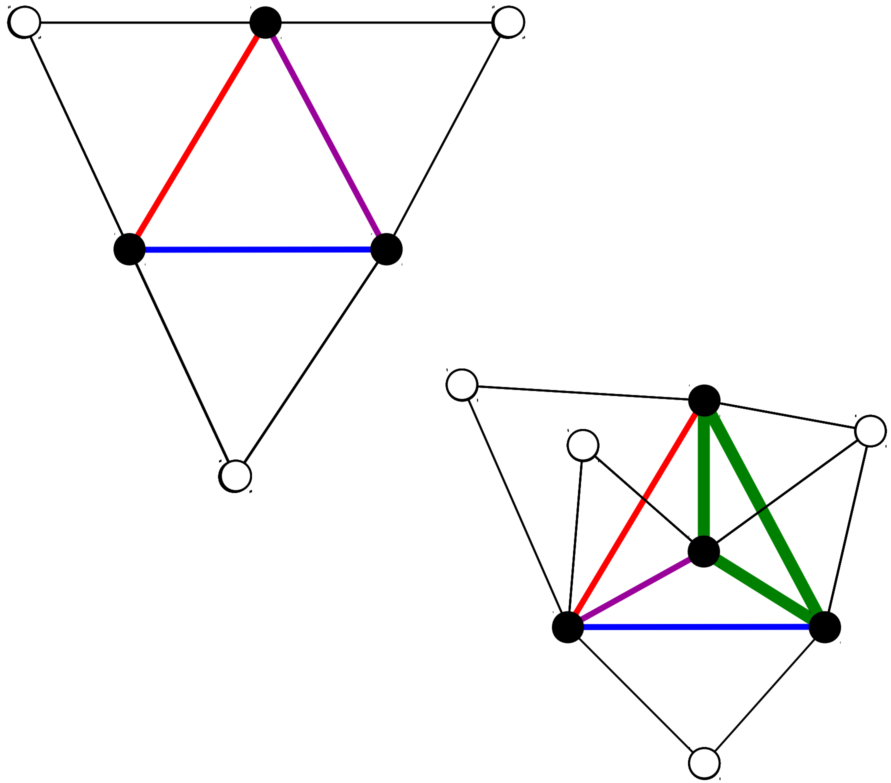


# 2. |K| $\leq$ 4

G is clique graph

iff

G has no bases set with



# 3. s has a private neighbour

- G is a clique graph

# Our NPC problem

**NPC**

## CLIQUE GRAPH $3split_2$

INSTANCE: A 3 $3split_2$ graph G=(V,E) with partition (K,S)

QUESTION: Is there a graph H such that G=K(H)?

## $3SAT_3$

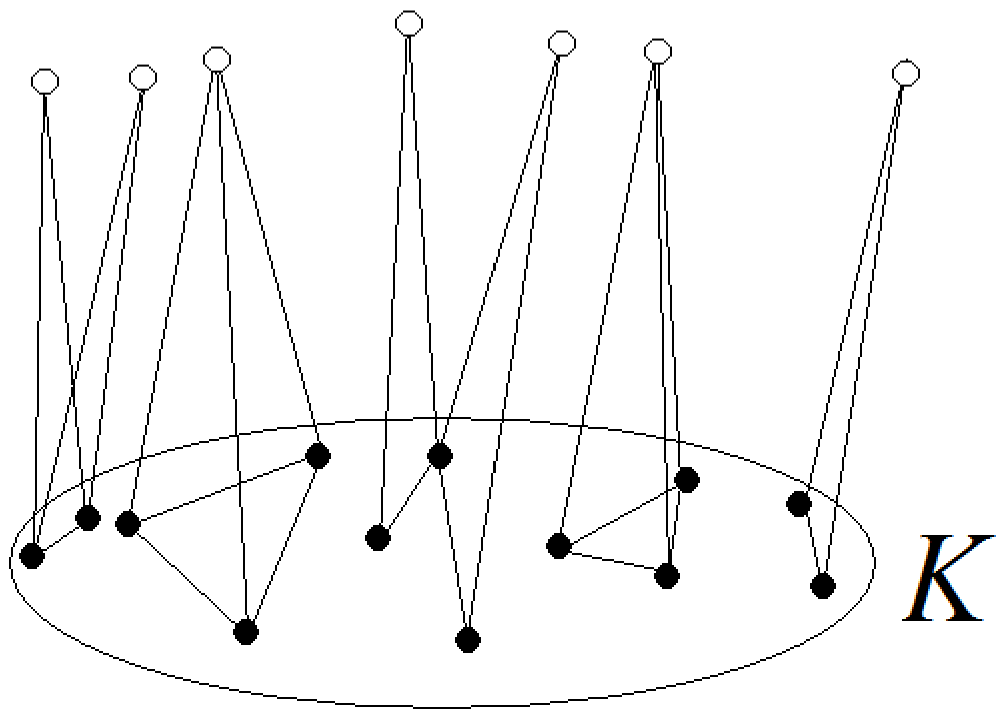
INSTANCE: A set of variables U, a collection of clauses C, s.t. if c of C, then |c|=2 or |c|=3, each positive literal u occurs once, each negative literal occurs once or twice.

QUESTION: Is there a truth assignment for U satisfying each clause of C?

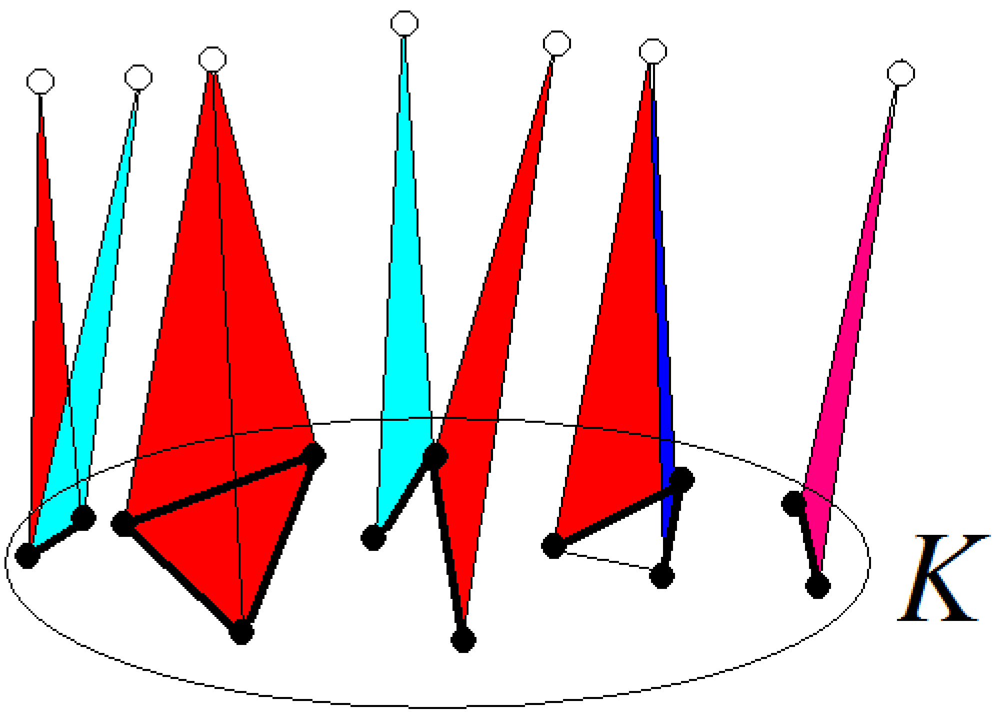
# Some preliminaries

- Black vertices in K
- White vertices in S
- Theorem – K is assumed in every 3split$_2$ RS-family

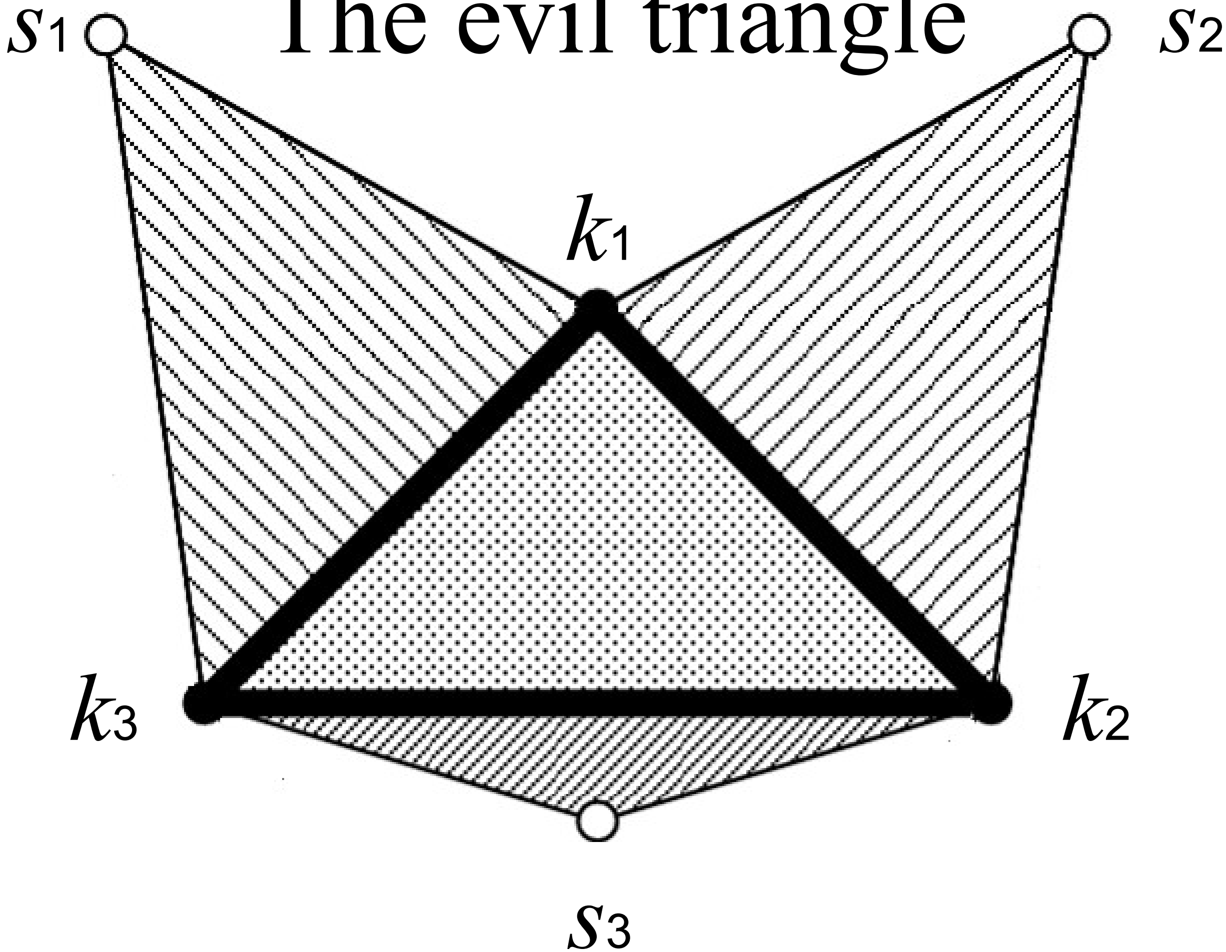
# 3split$_2$



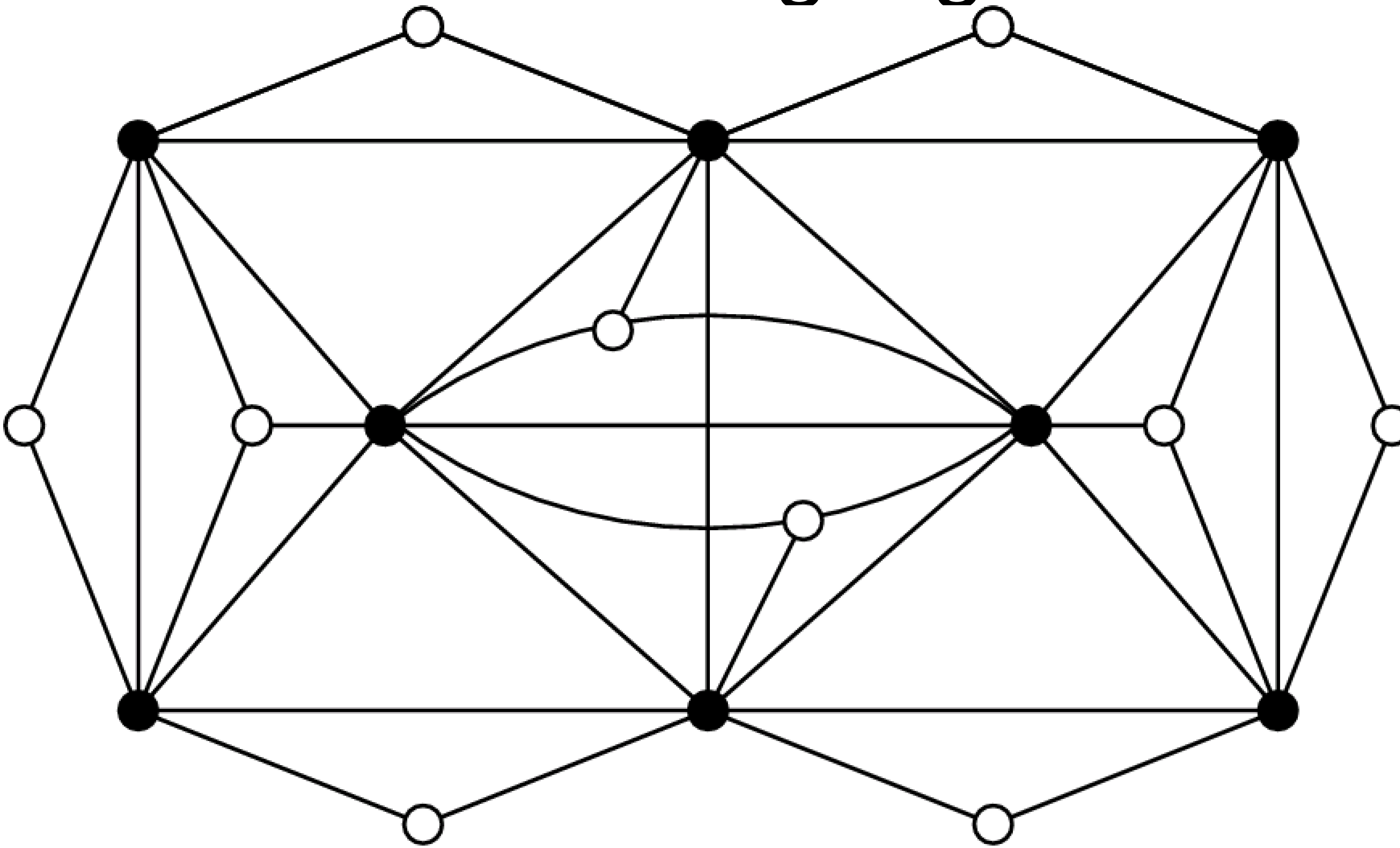
# 3split$_2$



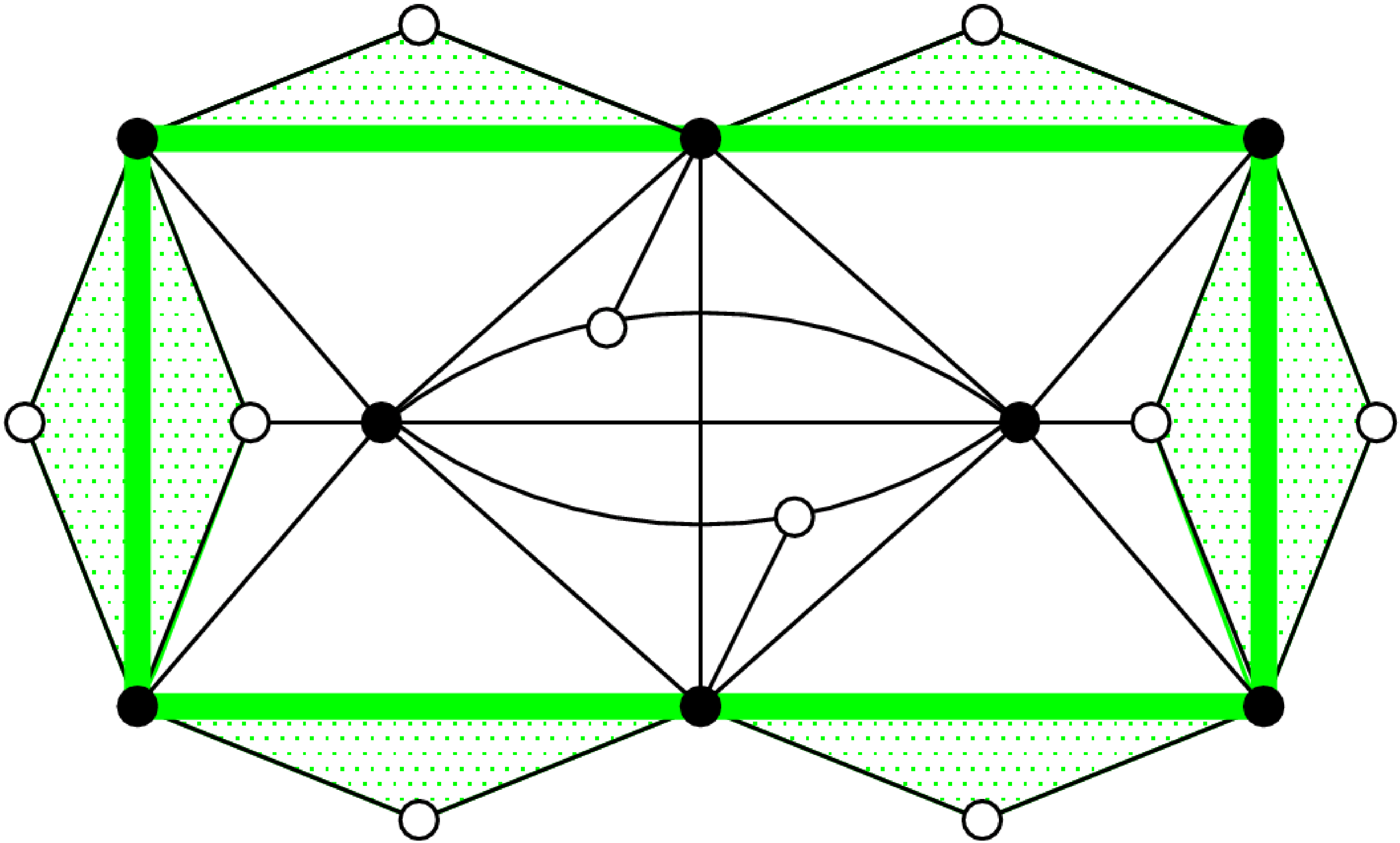
# The evil triangle

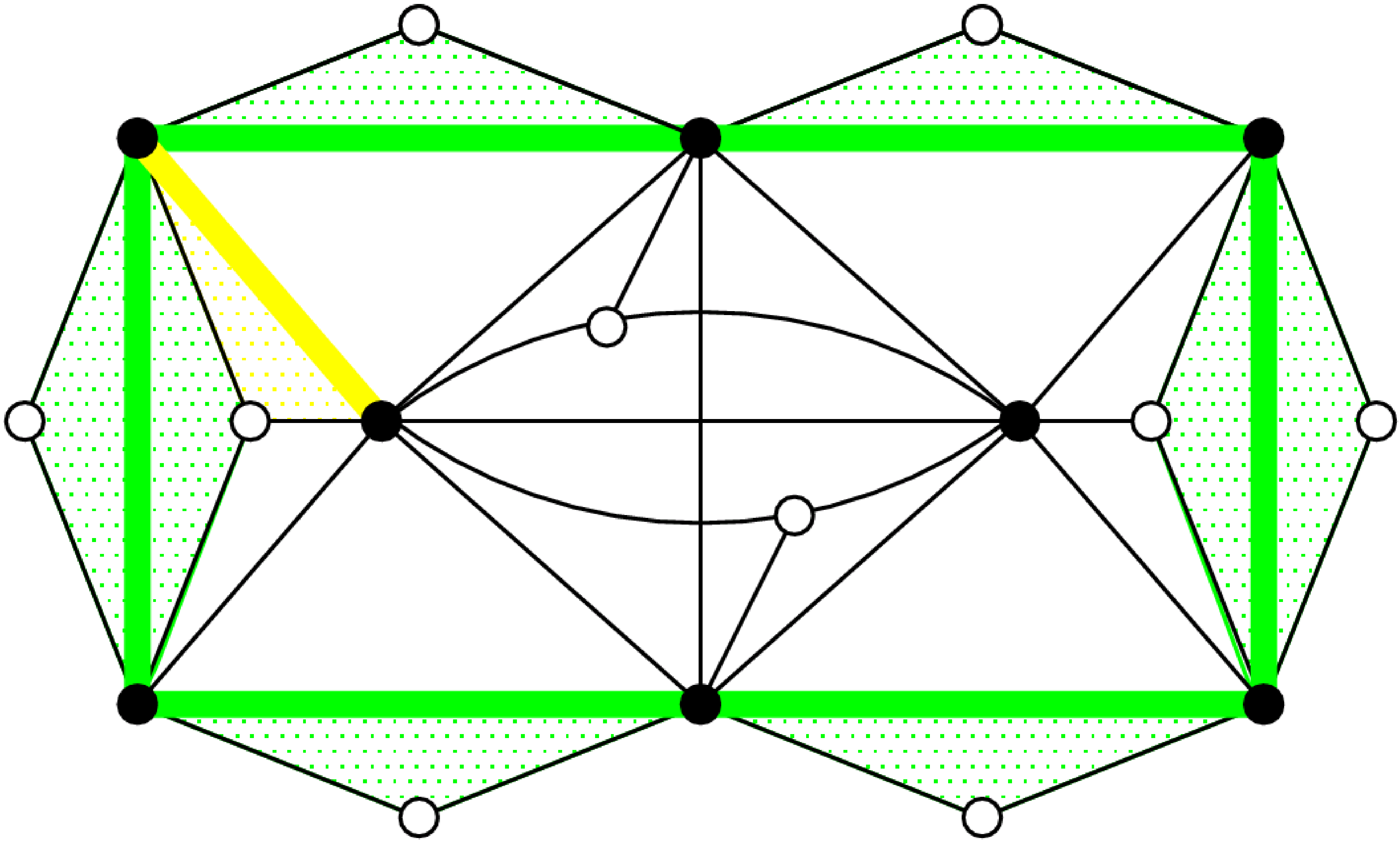


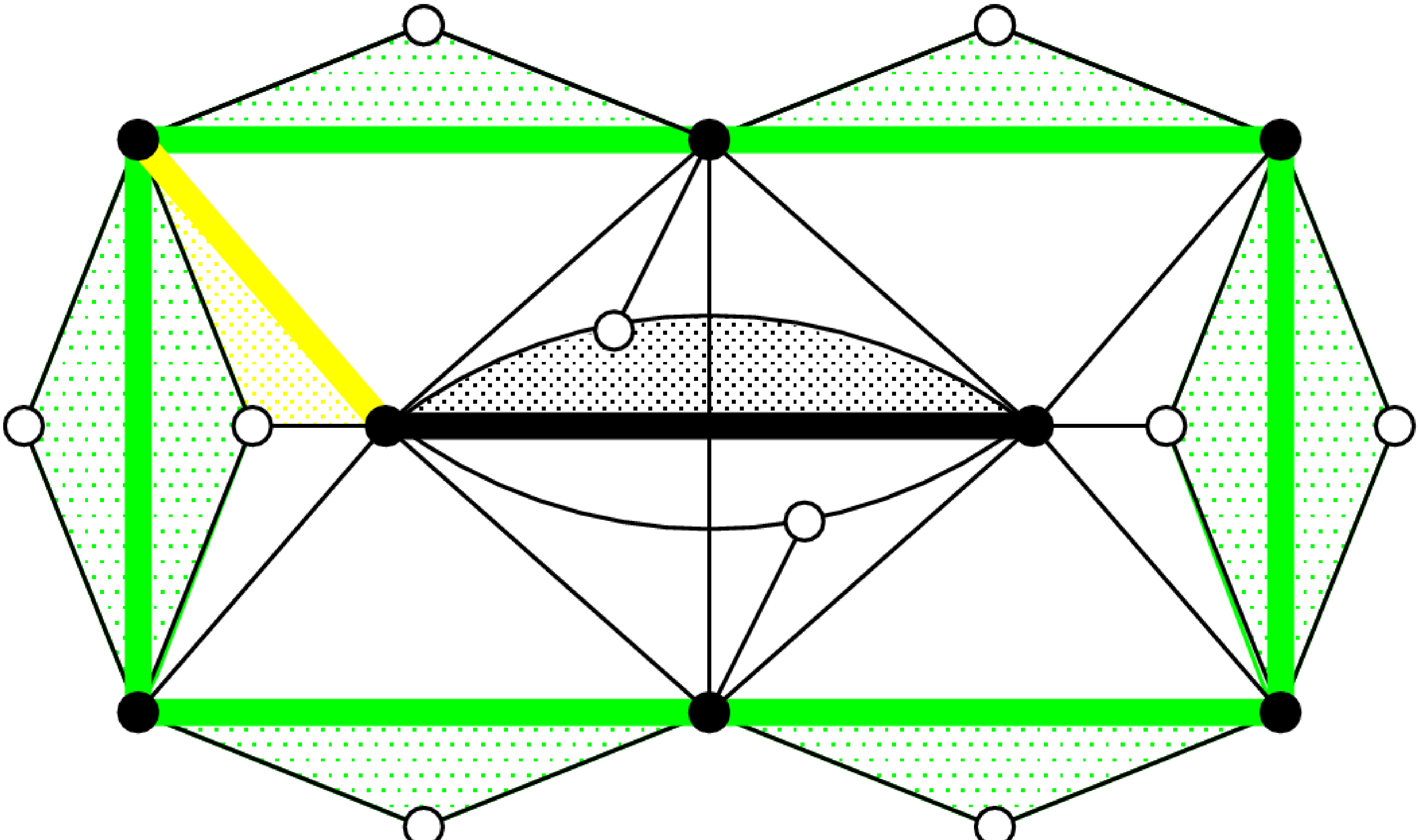
# The main gadget

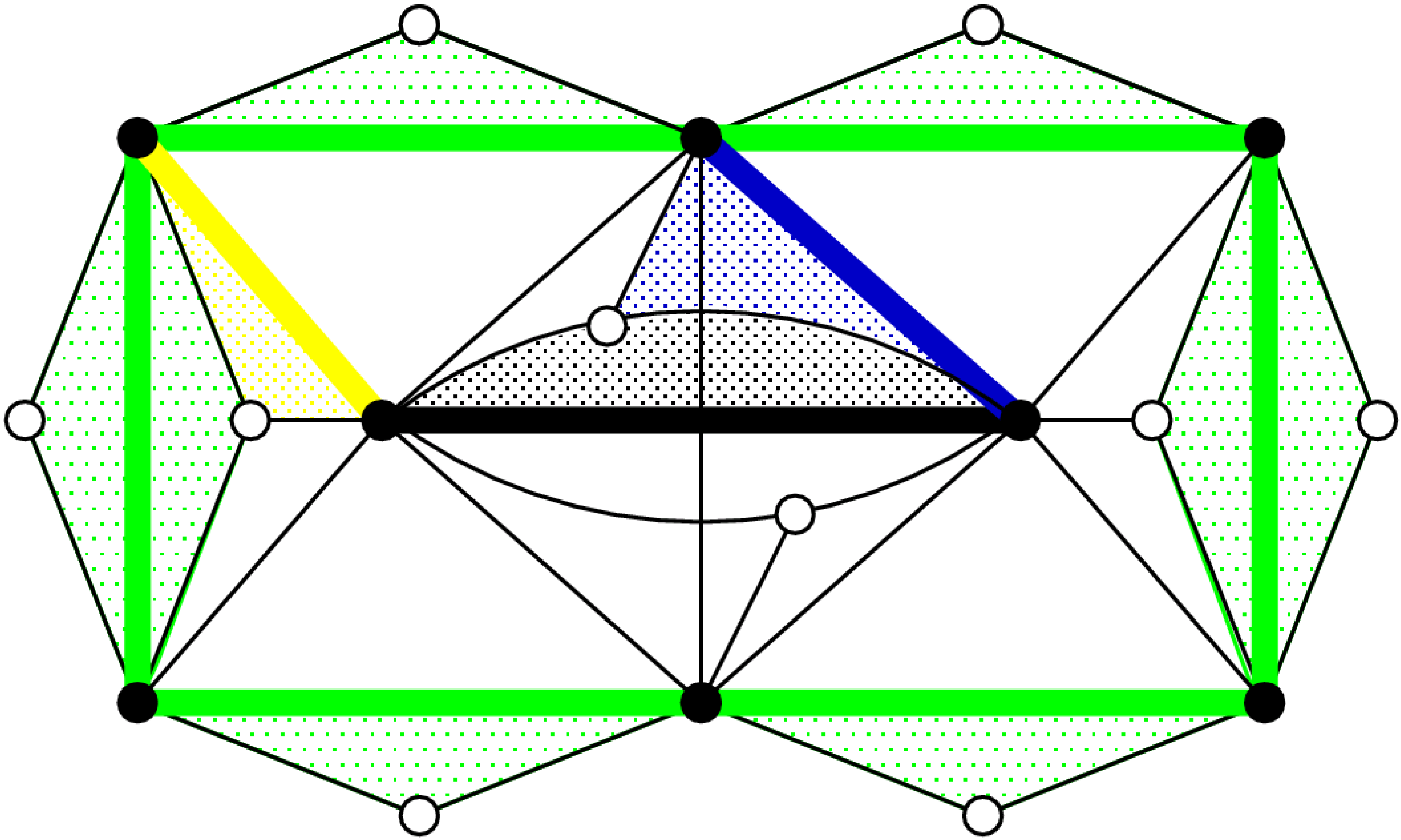


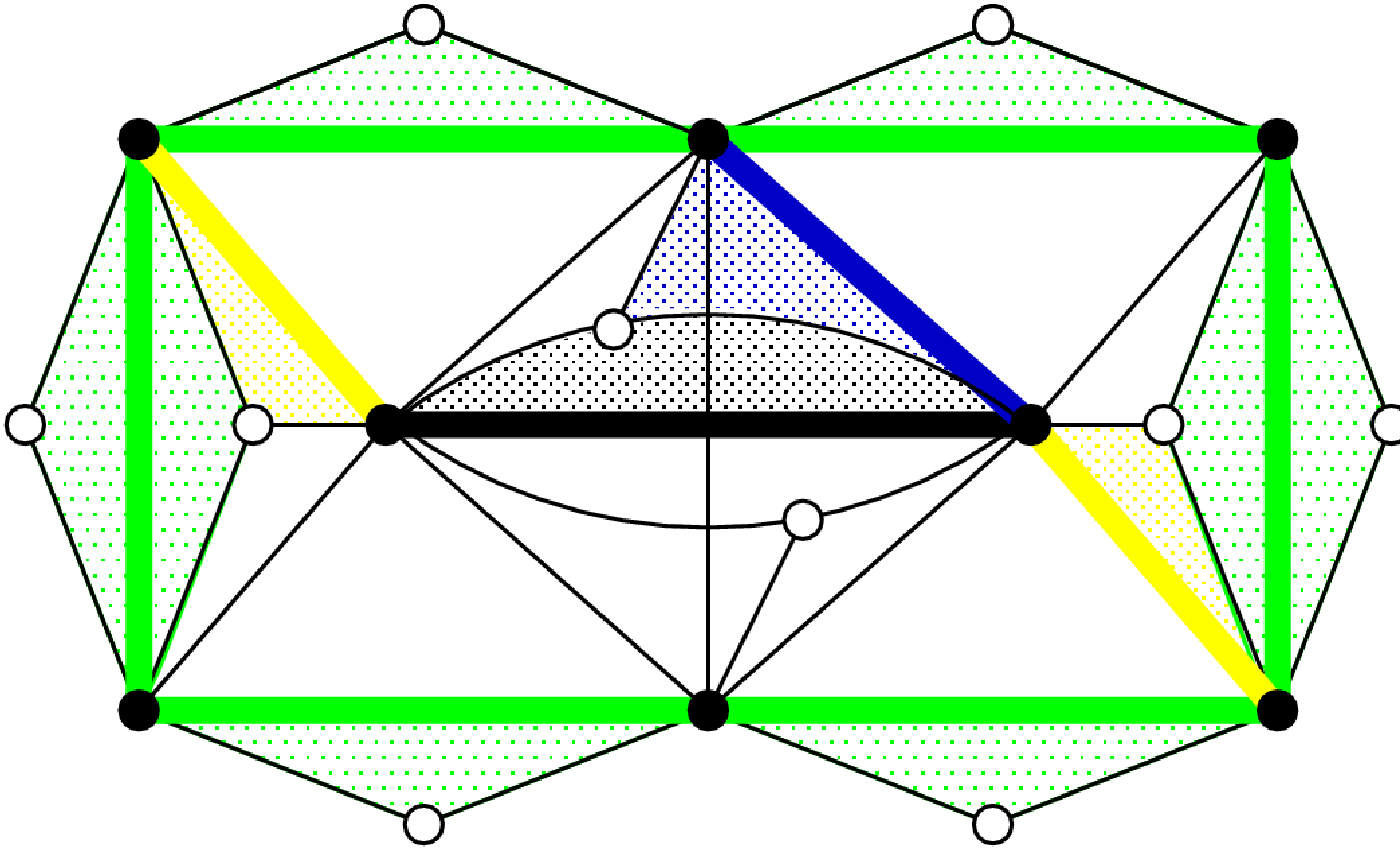
<u>Variable</u>

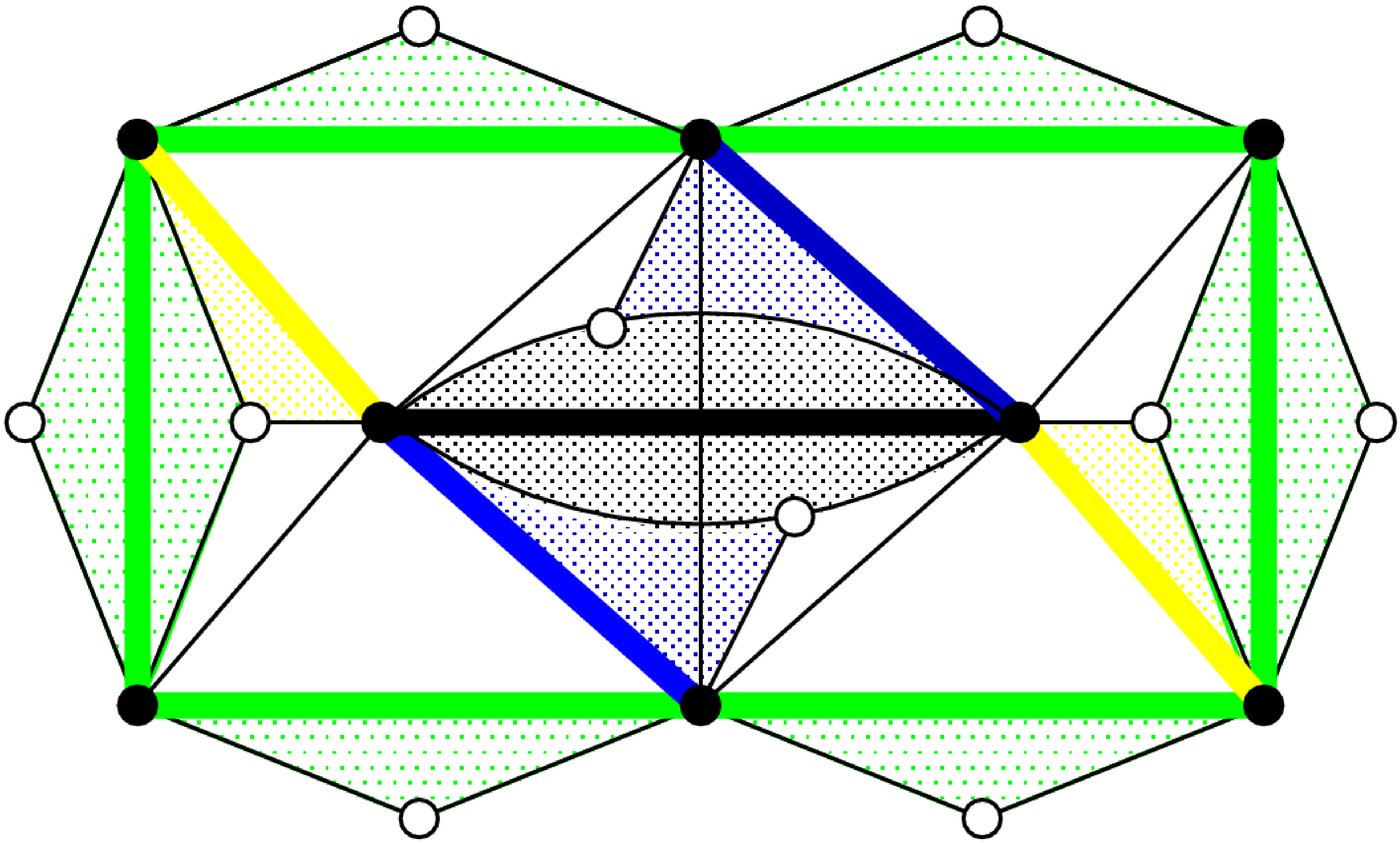




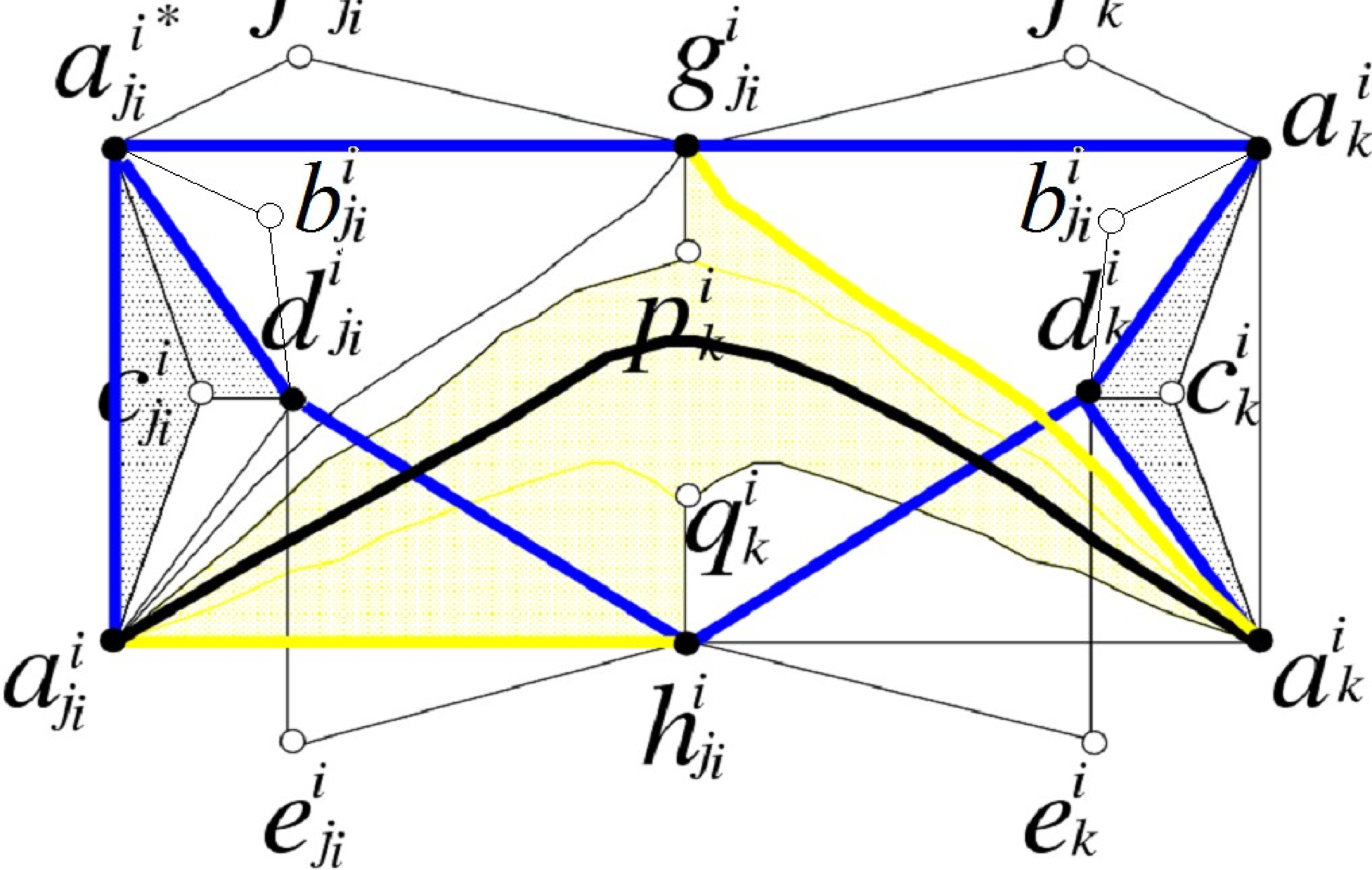




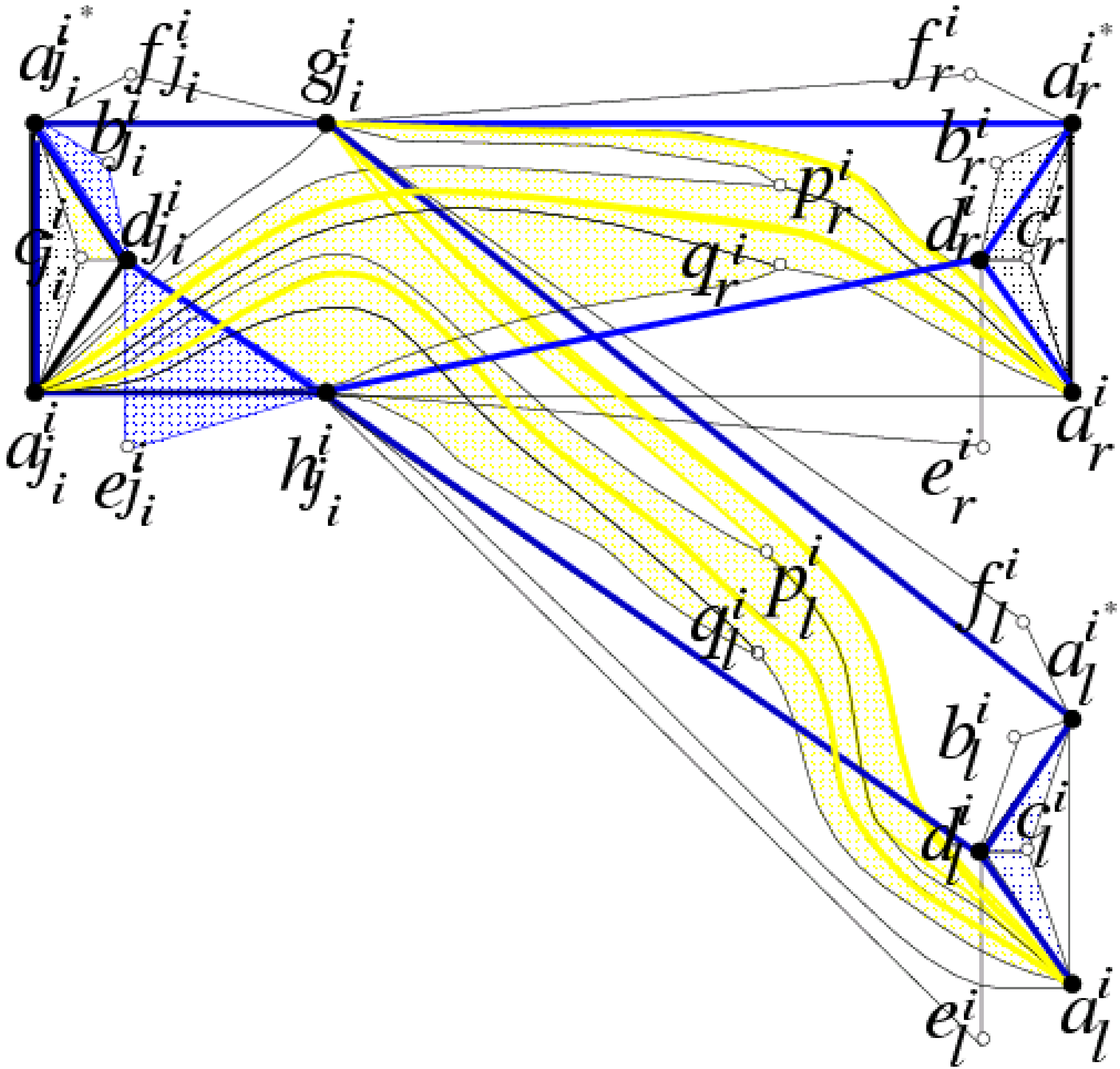




# Variable



$f^i_{ji}$
$f^i_k$
$a^{i*}_{ji}$
$g^i_{ji}$
$a^i_k$
$b^i_{ji}$
$b^i_{ji}$
$d^i_{ji}$
$p^i_k$
$d^i_k$
$c^i_{ji}$
$c^i_k$
$q^i_k$
$a^i_{ji}$
$h^i_{ji}$
$a^i_k$
$e^i_{ji}$
$e^i_k$



$a_{j_i}^{i*}$
$f_{j_i}^i$
$g_{j_i}^i$
$f_r^i$
$a_r^{i*}$
$b_{j_i}^i$
$b_r^i$
$p_r^i$
$c_i^i$
$d_{j_i}^i$
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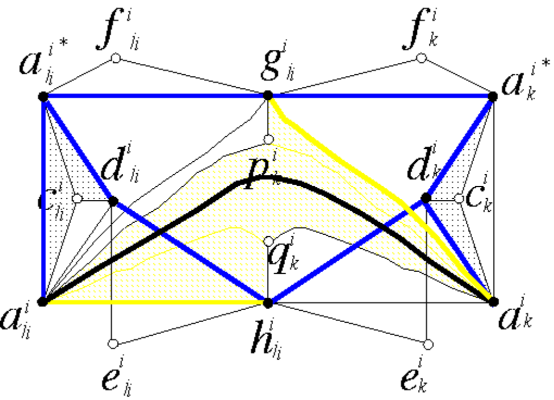
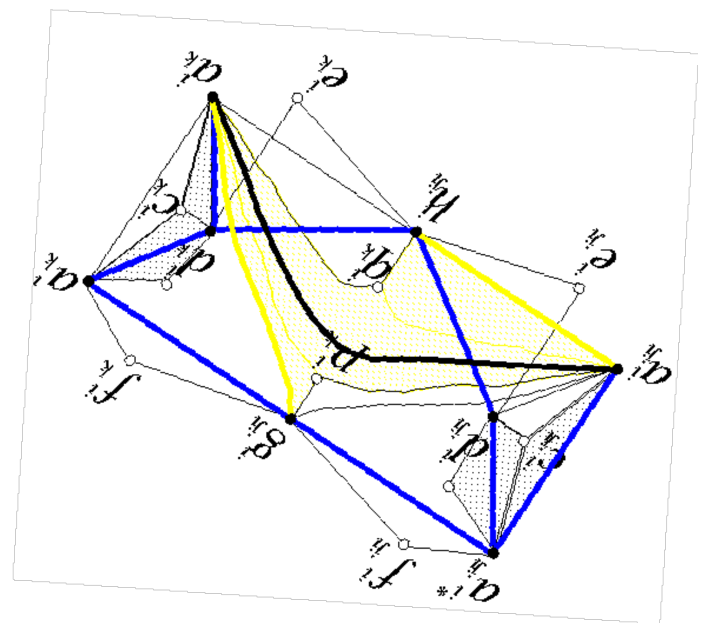
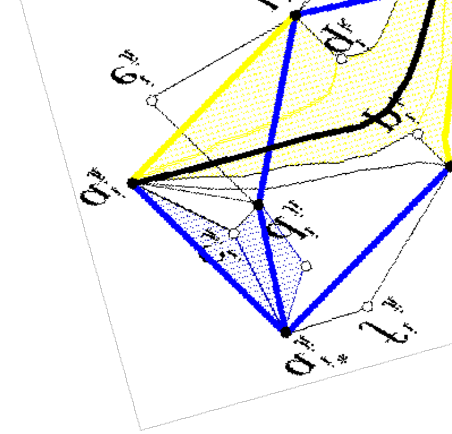
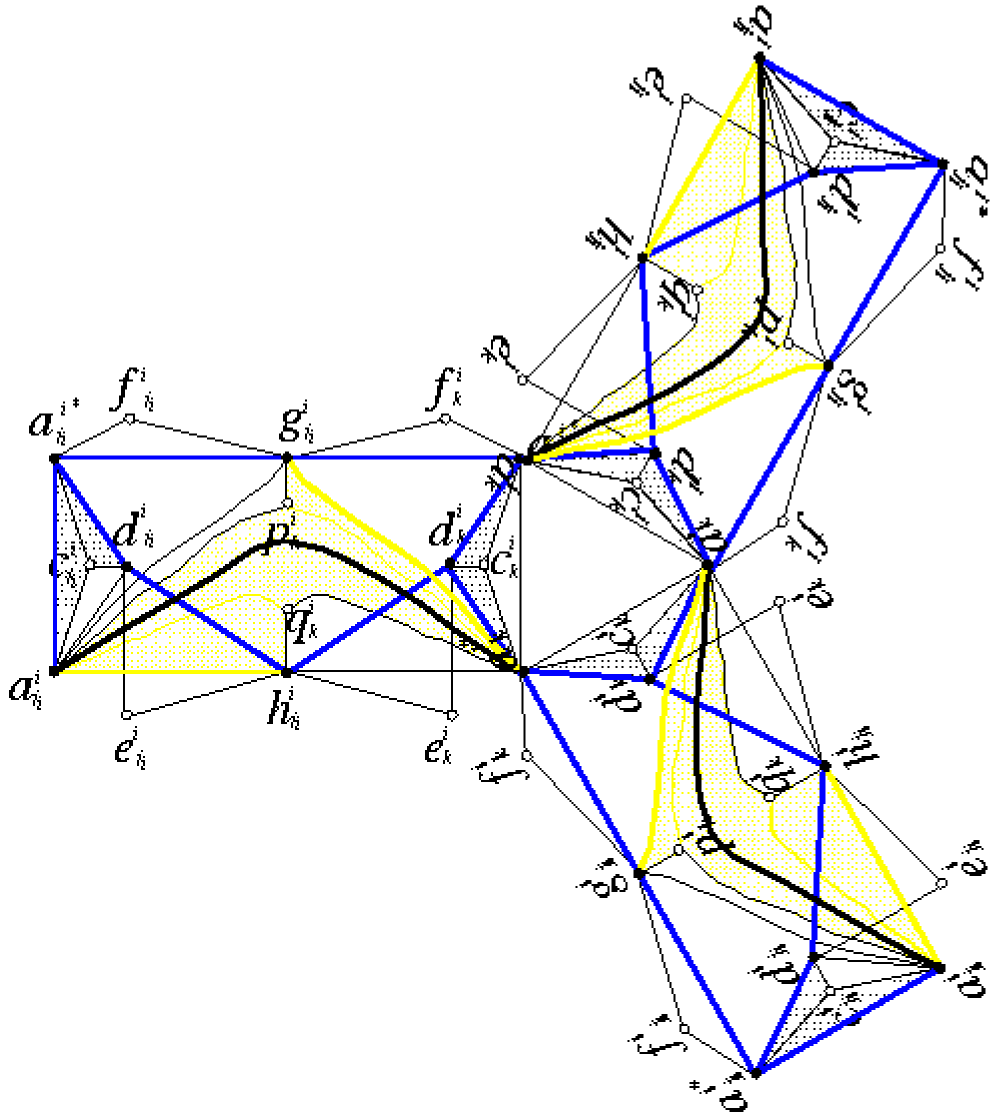


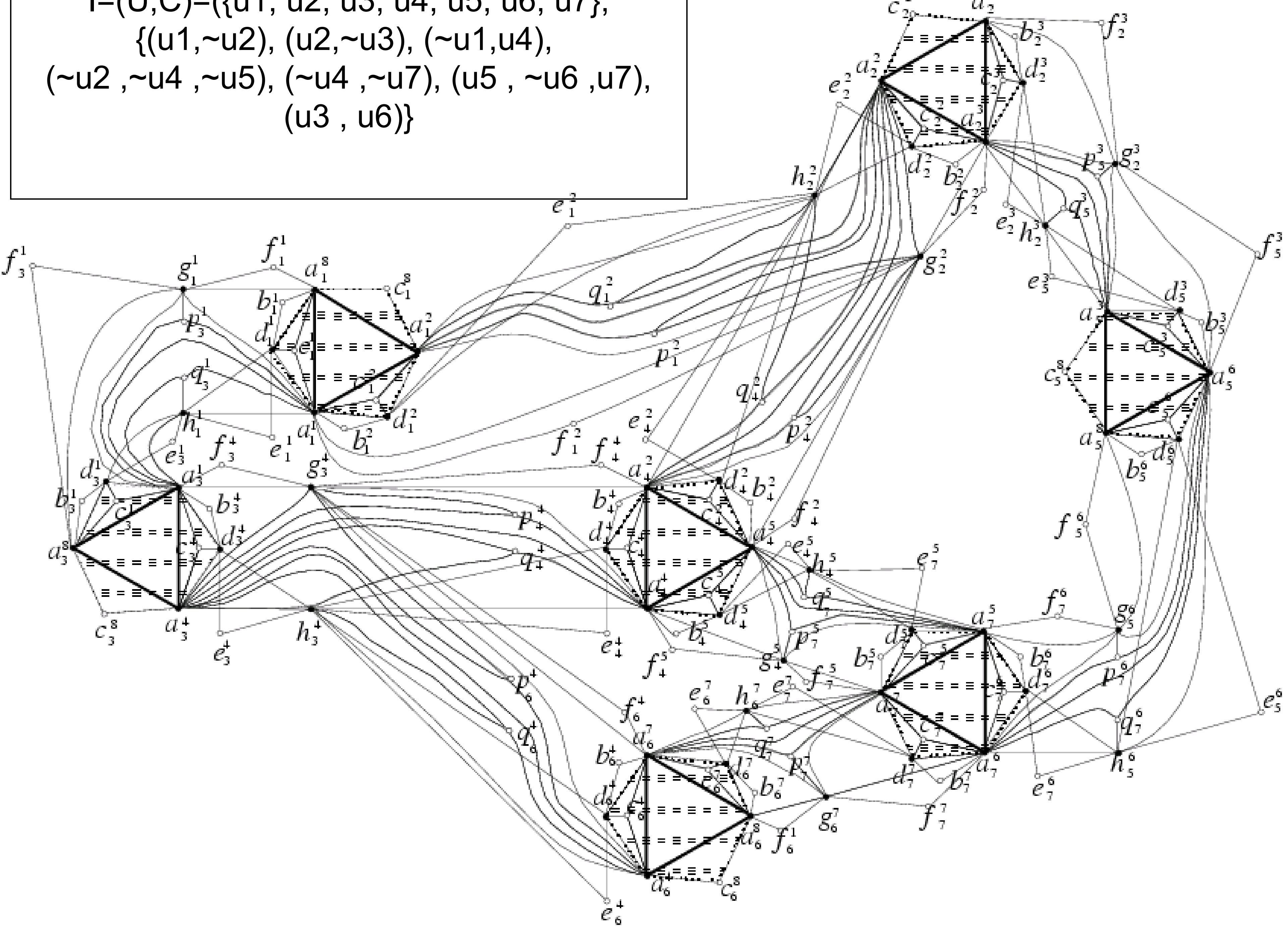




# Clause



I=(U,C)=({u1, u2, u3, u4, u5, u6, u7},
{(u1,~u2), (u2,~u3), (~u1,u4),
(~u2 ,~u4 ,~u5), (~u4 ,~u7), (u5 , ~u6 ,u7),
(u3 , u6)}



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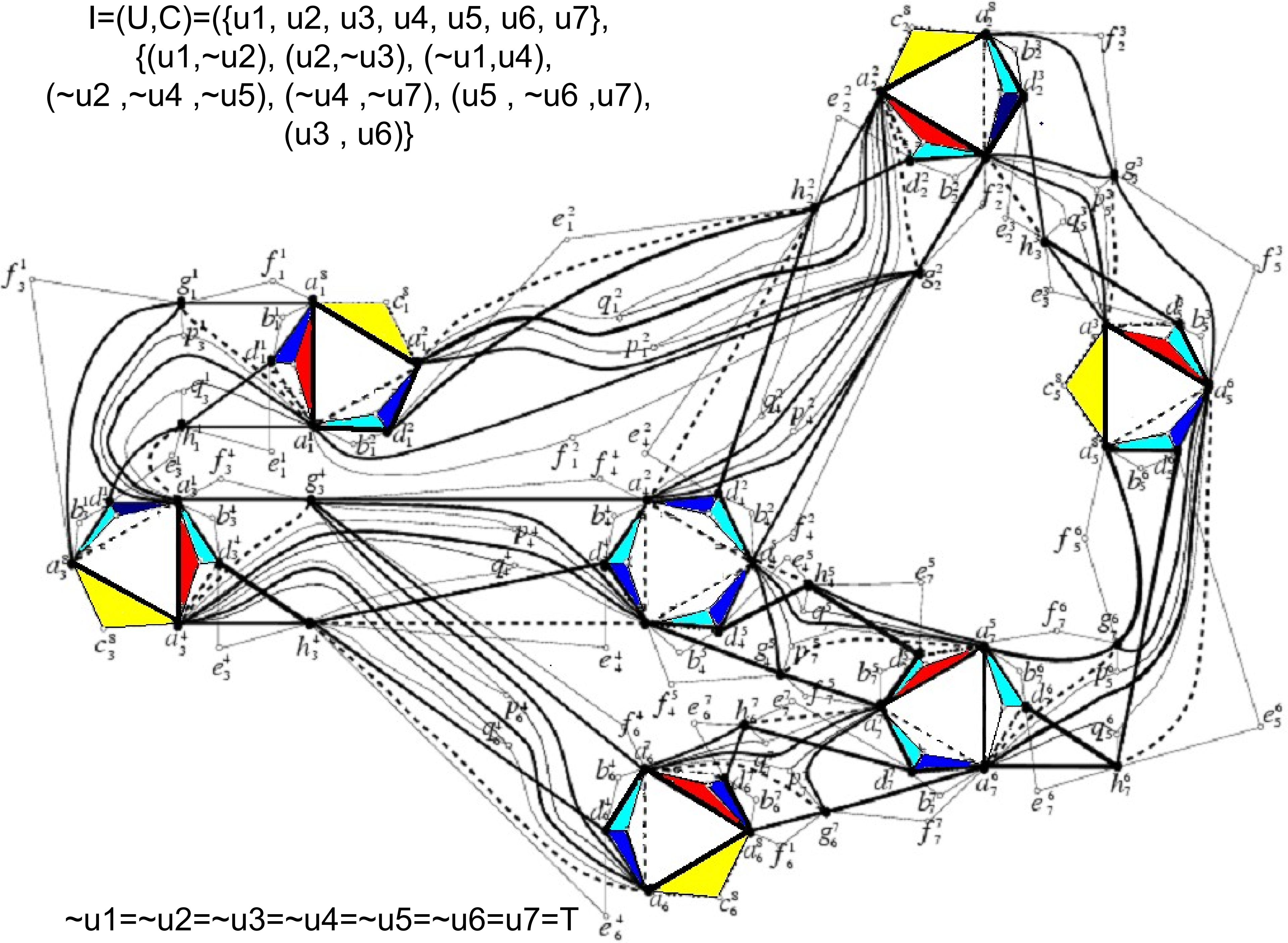

~u1=~u2=~u3=~u4=~u5=~u6=u7=T

# Problem of theory of the sets

Given a family of sets $\mathcal{F}$, decide whether there exists a family $\mathcal{F}'$, such that:

- $\mathcal{F}'$ is Helly,
- Each F' of $\mathcal{F}'$ has size |F'|>1,
- For each F of $\mathcal{F}$, $\bigcup_{F' \subset F} F' = F$

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# Our clique graph results

<table>
<tr><td></td><td>3split3</td><td>3split2</td><td>s of S has a private neighbor</td><td colspan="2">$|S|$ bounded</td><td colspan="2">$|K|$ bounded</td></tr>
<tr><td rowspan="2">Split graph<br>G = (V,E)<br>partition V=(K,S)</td><td rowspan="2">?</td><td rowspan="2">NPC</td><td rowspan="2">P</td><td>$|S| \leq 3$</td><td>general</td><td>$|K| \leq 4$</td><td>general</td></tr>
<tr><td>P</td><td>?</td><td>P</td><td>?</td></tr>
</table>

# Next step

- If G is a split planar graph => |K| $\leq$ 4 =>
  => split planar clique is in P.

- Is CLIQUE NP-complete for planar graphs?