On collinear sets in straight-line drawings

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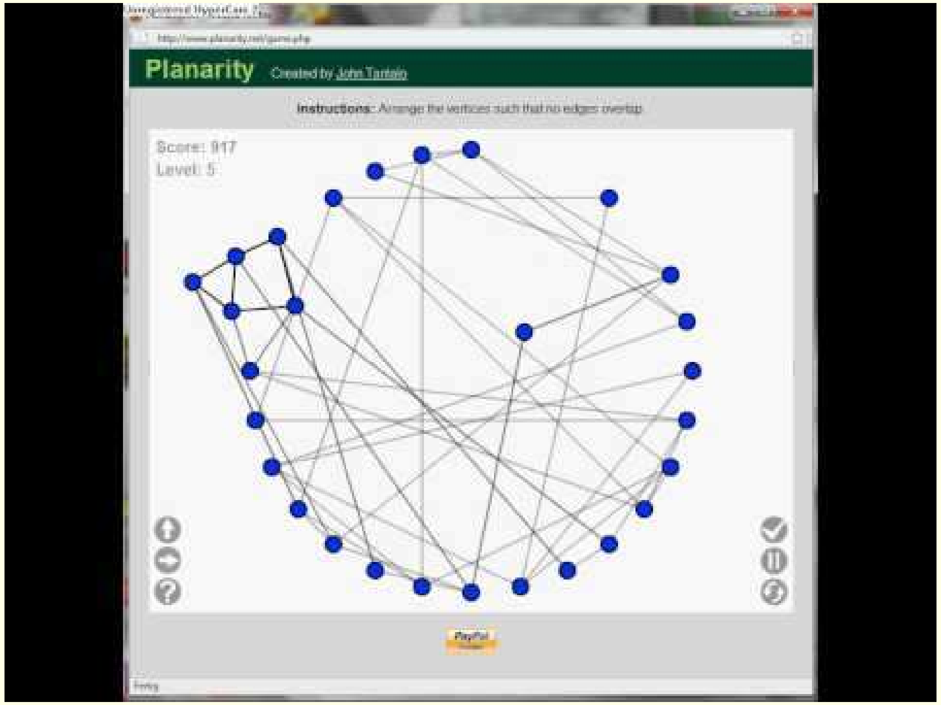
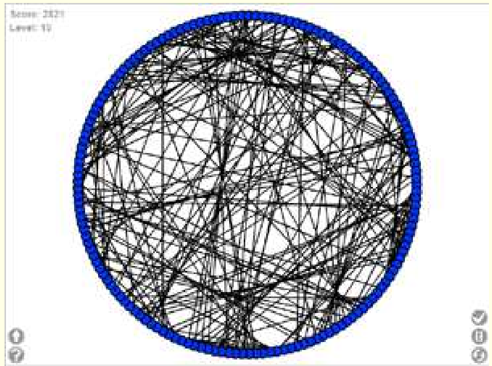
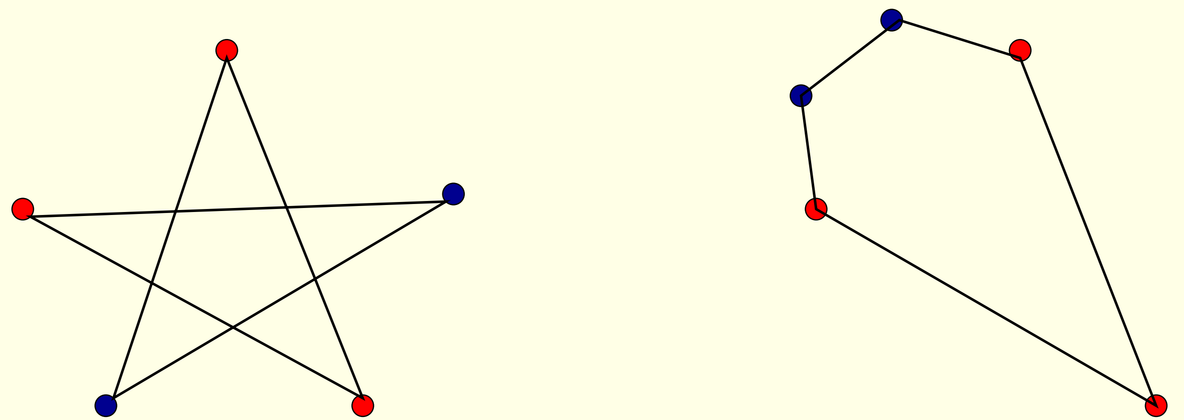
Planarity game





Untangling problem



Given:	a straight-line drawing of a planar graph G
A move:	shifting a vertex to a new position (the incident edges stay straight)
Goal:	eliminate all edge crossings
Score:	the number of vertices left fixed
$\mathit{fix}(G) =$	the score that can always be gained (whatever drawing of G is given)

Example

Theorem (Pach and Tardos 02, Cibulka 10).

$$2^{-5/3}n^{2/3} - O(n^{1/3}) \leq \mathit{fix}(C_n) \leq O((n \log n)^{2/3})$$

Bounds

A general lower bound

$\mathit{fix}(G) \geq (n/3)^{1/4}$ for all G (Bose et al. 09)

Upper bounds

$\mathit{fix}(G) = O((n \log n)^{2/3})$ for all 3-connected G
$\mathit{fix}(G) = O(\sqrt{n}(\log n)^{3/2})$ for all G with logarithmic maximum degree and diameter (Cibulka 10)

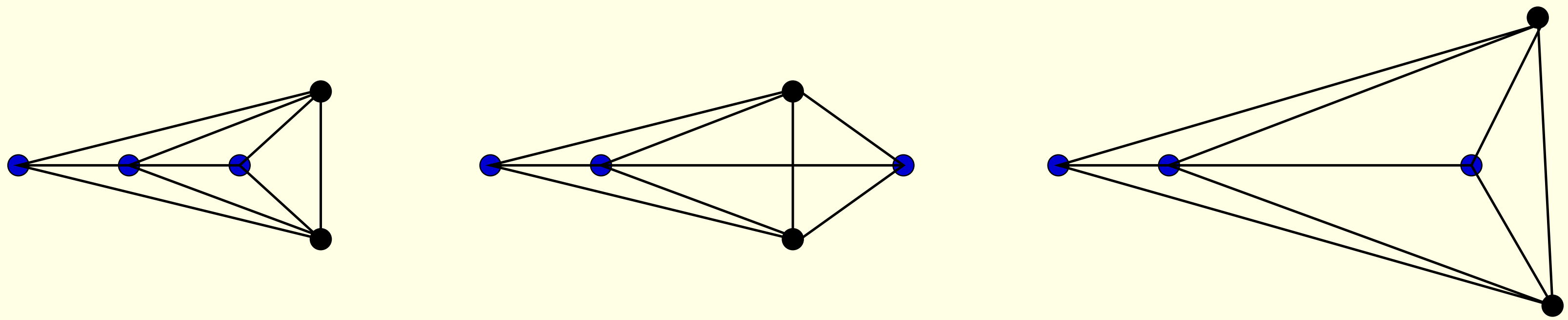
$\mathit{fix}(G) = O(\sqrt{n})$ for some G, even acyclic
(Bose et al. 09, Goaoc-Kratochvíl-et al. 09, Kang et al. 11)

Lower bounds for classes of graphs

$\mathit{fix}(G) \geq \sqrt{n/2}$ for all trees
(Bose et al. 09, Goaoc-Kratochvíl-et al. 09)
and, more generally, for all outerplanar graphs
(Goaoc-Kratochvíl-et al. 09)

$\mathit{fix}(G) \geq \sqrt{n/30}$ for all G of tree-width at most 2 (this talk)

A working tool: Free collinear sets



Definition. Let $G = (V, E)$ be a planar graph. Let $\pi : V \to \mathbb{R}^2$ be a crossing-free drawing of G and ℓ be a line. A set $S \subset \pi(V) \cap \ell$ is called free if, whenever we displace the vertices in S along ℓ without violating their order (thereby introducing edge crossings), we are able to untangle the modified drawing by only moving the vertices in $\pi(V) \setminus S$. By $\tilde{v}(G)$ we denote the largest size of a free collinear set maximized over all drawings of G.

Theorem. $\mathit{fix}(G) \geq \sqrt{\tilde{v}(G)}$.

Proof (sketch)

Let $\lambda : V \to \mathbb{R}^2$ be crossing-free, $S \subset V$, $(k-1)^2 < |S| \leq k^2$, and $\lambda(S)$ be free. Given an arbitrary $\pi : V \to \mathbb{R}^2$, we can untangle it with k vertices fixed.

Step 1. Make π 1-dimensional: choose a coordinate system (x, y) so that $\pi(V)$ is between $y = 0$ and $y = 1$, and projection p_x is injective on $\pi(V)$. By Erdős-Szekeres, there is a subset $S' \subset S$ of k points appearing in $p_x\pi$ and λ in the same order.

Step 2. Turn back to 2D: Consider a (crossing-free) modification λ' of λ such that $\lambda(v) = p_x\pi(v)$ for all $v \in S'$ (possible because $\lambda(S)$ is free).

Step 3. Make a small perturbation of λ' (still crossing-free):

$$\lambda''(v) = \begin{cases} (p_x\pi(v), \epsilon p_y\pi(v)) & \text{if } v \in S', \\ \lambda'(v) & \text{otherwise.} \end{cases}$$

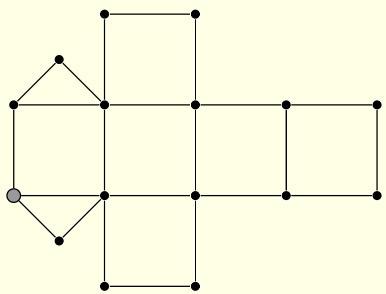
Step 4. Apply linear transformation $a(x, y) = (x, \epsilon^{-1}y)$. Then $a\lambda''$ is a crossing-free drawing of G such that $a\lambda''(v) = \pi(v)$ for all $v \in S'$.

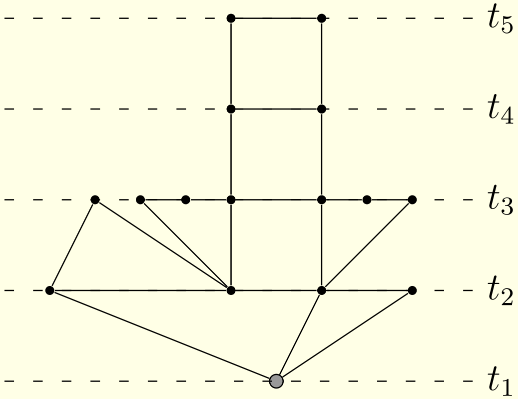
Application to outerplanar graphs

If G is outerplanar, then $\mathit{fix}(G) \geq \sqrt{n/2}$ because $\tilde{v}(G) \geq n/2$.

In a track drawing every vertex lies on one of parallel lines, called tracks, and every edge either lies on one of the tracks or connects vertices lying on two consecutive tracks.

Lemma (Felsner, Liotta, and Wismath 03).
Outerplanar graphs are track drawable.



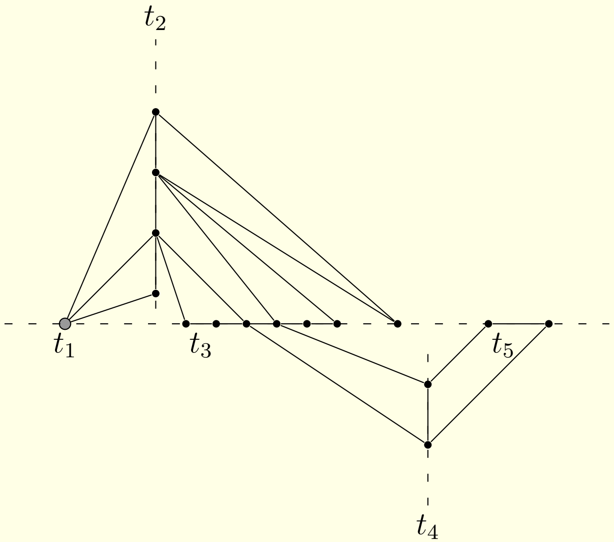


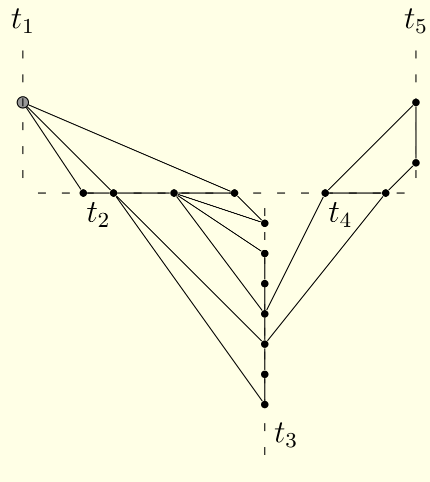
Application to outerplanar graphs

Key observation:

All odd tracks or all even tracks can be drawn on the same line.

The resulting collinear set is free!





Application to graphs of tree-width 2

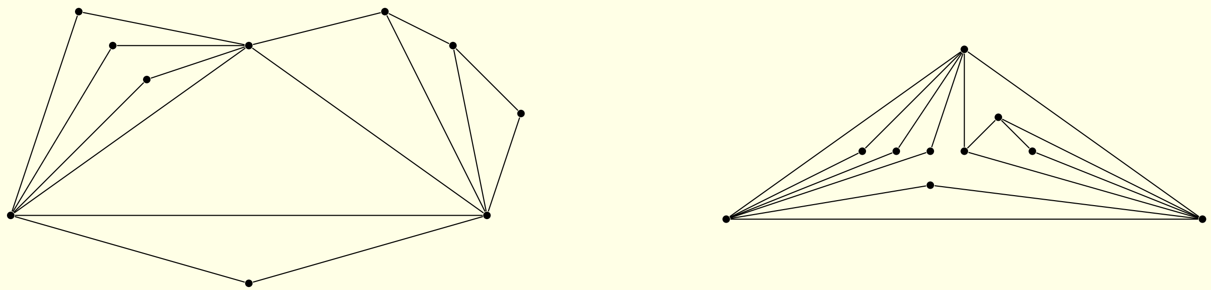
Definition. A triangle is a 2-tree. If we connect a new vertex to two adjacent vertices of a 2-tree, we obtain a 2-tree.

Known fact. Graphs of tree-width 2 are exactly subgraphs of 2-trees.

Therefore, it suffices to show that every 2-tree has a drawing with a large free set.

Application to graphs of tree-width 2

Call a drawing of a 2-tree folded if for any two triangles that share an edge, one contains the other.



Lemma 1. Every collinear set of vertices in a folded drawing is free.

Lemma 2. Every 2-tree has a folded drawing with at least $n/30$ collinear vertices.

Our second contribution

Let $\bar{v}(G)$ denote the number of collinear vertices maximized over all crossing-free drawings of G.

We show that for some planar graphs not only $\tilde{v}(G)$, but even $\bar{v}(G)$ is small.

Allocation problem

Given	a planar graph G on n vertices and an n-point set X in the plane
Draw	G without egde-crossings with as many vertices as possible in X

$\mathit{fit}^X(G) =$ the optimum

Gritzmann et al. 91: $\mathit{fit}^X(G) = n$ for all outerplanar G and all X in general position. But if G is not outerplanar, then $\mathit{fit}^X(G) < n$ for any X in convex position.

Gimenez, Flajolet, Noy: If X is in convex position, then $\mathit{fit}^X(G) < n$ for almost all planar G.

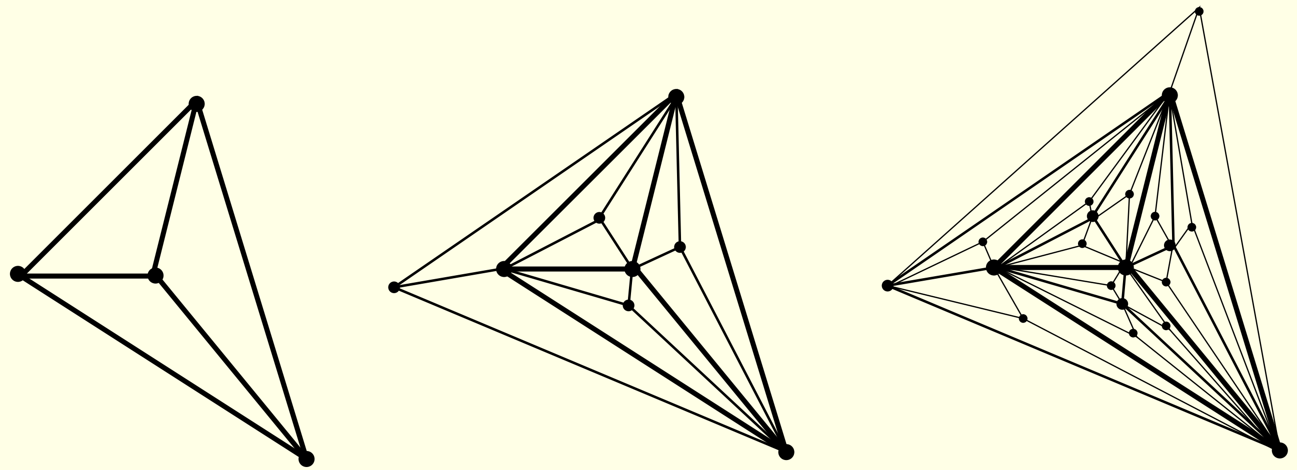
Let $\mathit{fit}(G) = \min_X \mathit{fit}^X(G)$. Note that

$$\sqrt{\tilde{v}(G)} \le \mathit{fix}(G) \le \mathit{fit}(G) \le \bar{v}(G).$$

An upper bound

Theorem. There are triangulations with $\bar{v}(G) = o(n^{0.99})$.

Given a triangulation G, define G^2 as follows: draw G and triangulate each face by a copy of G. Iterating, we obtain G^3, G^4, ...



Proof

Let $\bar{f}(G)$ be the maximum number of faces in some straight-line drawing of G whose interiors can be cut by a line.

Lemma 1. $\bar{v}(G^k) < \mathrm{const} \cdot v(G^k)^\alpha$ with $\alpha = \dfrac{\log(\bar{f}(G)-1)}{\log(f(G)-1)}$, where $f(G) = 2\,v(G) - 4$ is the number of faces of G.

Lemma 2. $\bar{f}(G) \le \mathrm{circ}(G^*)$, the maximum cycle length in the dual of G.

Lemma 3 (Grünbaum, Walther 73). There are triangulations G with $\mathrm{circ}(G^*)$ so small that this gives us α arbitrarily close to $\frac{\log 26}{\log 27} = 0.988$.

Further questions

1. How far or close are $\tilde{v}(G)$ and $\bar{v}(G)$?

2. We constructed examples of graphs with $\tilde{v}(G) \le \bar{v}(G) \le O(n^\sigma)$ for a graph-theoretic constant σ, for which it is known that $0.69 < \sigma < 0.99$. Are there graphs with $\bar{v}(G) = O(\sqrt{n})$? If so, this would be a qualitative strengthening of the bound $\mathit{fix}(G) = O(\sqrt{n})$. Are there graphs with, at least, $\tilde{v}(G) = O(\sqrt{n})$? If not, this would improve the bound $\mathit{fix}(G) = \Omega(n^{1/4})$.

3. We proved that $\tilde{v}(G) \ge n/30$ for any G with tree-width 2. For which other classes of planar graphs do we have $\tilde{v}(G) = \Omega(n)$ or, at least, $\bar{v}(G) = \Omega(n)$?

4. $\mathit{fit}(G) = \bar{v}(G)$?

5. Let $\mathit{fit}^\vee(G) = \min_X \mathit{fit}^X(G)$, where the minimization goes over all X in general position. Extend our upper bound $\mathit{fit}(G) = O(n^{0.99})$ to $\mathit{fit}^\vee(G) = o(n)$ (for infinitely many G).

Thank you!