On the Independence Number of Graphs with Maximum Degree 3

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Outline

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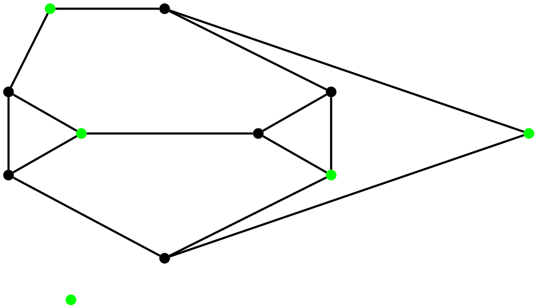
INDEPENDENT SET

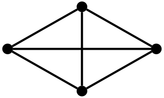
- An *independent set* in a graph is a set of vertices such that no two vertices in this set are adjacent
- A *maximum independent* set is an independent set of maximum cardinality/size
- $\alpha(G)$ denotes the cardinality of a maximum independent set of G
- We consider only graphs of maximum degree ≤ 3

INDEPENDENT SET



Combinatorial lower bounds

- Brook's Theorem (1960s) states that every K_4-free graph of maximum degree ≤ 3 is 3-colorable
- Therefore, every K_4-free graph G of maximum degree at most 3 has an independent set of size $\geq n(G)/3$, where $n(G)$ is the number of vertices in G



Combinatorial lower bounds

- Can this combinatorial lower bound be improved?
- Clearly triangles pose a direct obstacle
- If every vertex in the graph appears in a triangle then the lower bound is tight

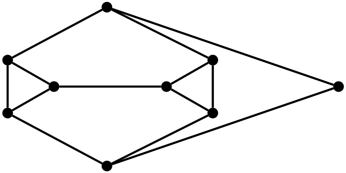
Combinatorial lower bounds

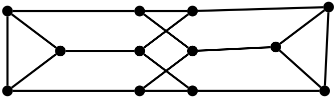
- Can we improve the $n(G)/3$ lower bound if G contains some nontriangle vertices?
- Let $nt(G)$ be the number of nontriangle vertices in G
- To benefit from the presence of nontriangle vertices, we need a lower bound of the form:

$$\alpha(G) \geq n(G)/3 + nt(G)/c \text{ for some constant } c > 1$$

Combinatorial lower bounds

- Such result is not possible, as illustrated by the following graphs:





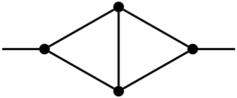
- Can we exclude certain subgraphs to make such a result possible?

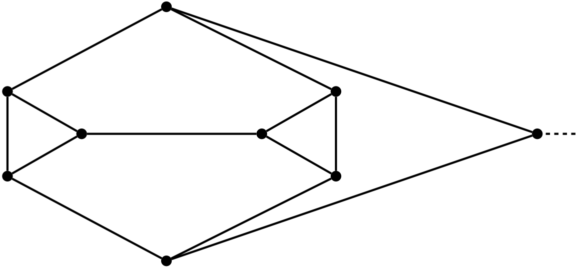
A Tight Combinatorial Result

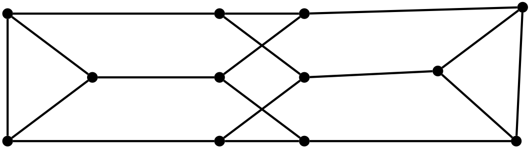
Theorem

Let G be a graph that excludes the following graphs as subgraphs then $\alpha(G) \geq n(G)/3 + nt(G)/42$

A Tight Combinatorial Result







A Tight Combinatorial Result

- This lower bound is tight because $5n(G)/14 = n(G)/3 + nt(G)/42$ is a tight bound on the independence number of triangle-free graphs of maximum degree ≤ 3
- We call the three graphs the *obstacle graphs*
- There are several combinatorial result of a similar nature on the independence number of graphs of maximum degree ≤ 3
- This result is orthogonal to them

The Proof

We need to prove the following theorem:

Theorem

Let G be a graph that excludes the 3 obstacle graphs as subgraphs then $\alpha(G) \geq n(G)/3 + nt(G)/42$

The Proof

- We use discharging coupled with amortized analysis
- The proof is broken into three phases
- In each phase we apply a sequence of operations to simplify the structure of the graph further
- We present each of the three phases next

Phase-I

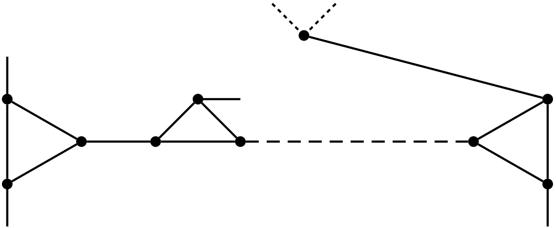
- We apply operations that remove some triangles from G to obtain a graph G_1 in which every triangle is contained in a special structure that we call a *steeple*
- None of these operations decreases the number of nontriangle vertices in the graph or introduces triangles
- Each of these operation guarantees that one vertex from every removed triangle can be added to the independent set of G
- That is, the independence number of the graph to which the operation is applied is at least as large as that of the resulting graph, plus one third the number of vertices removed by the operation

Phase-I

- The operations in this phase remove all paths/cycles of triangles
- Below is an example of an operation that removes a path of triangle in certain situations

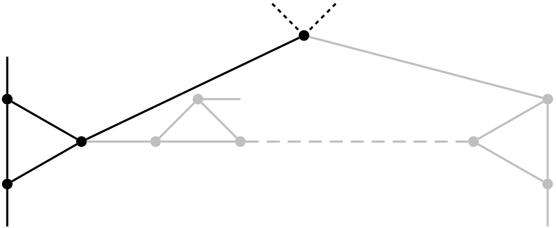
Phase-I

Consider the following path of triangles:



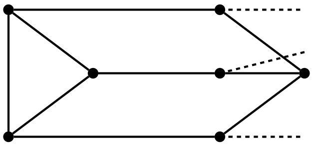
Phase-I

We apply the following operation:

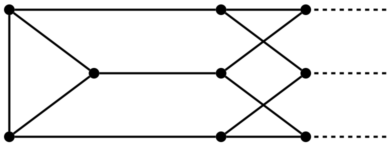


Phase-I

At the end of Phase-I every triangle is contained in one of the following two subgraphs called *steeples*:



Type-I Steeple

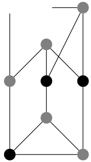


Type-II Steeple

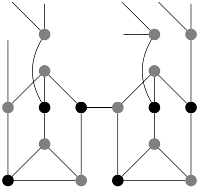
Phase-II

- We apply more operations to G_1 to simplify its structure further
- We make the steeples in the resulting graph G_2, and hence the triangles, farther apart
- Each of these operations removes a subgraph H from G_1 satisfying the local-ratio property: An independent set S_H of H of size at least $n(H)/3 + nt(H)/42$ can be added to any independent set of the resulting graph

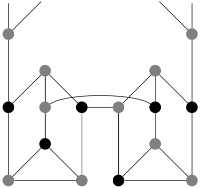
Examples:



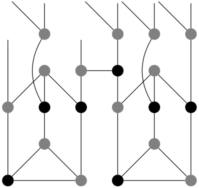
Op.: $n(H) = 8$,
$nt(H) = 5, |S_H| = 3$.



Op.: $n(H) = 17$,
$nt(H) = 11, |S_H| = 6$.



Op.: $n(H) = 16$,
$nt(H) = 10, |S_H| = 6$.



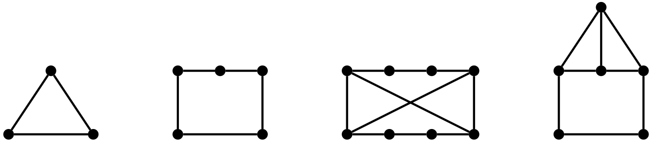
Op.: $n(H) = 20$,
$nt(H) = 14, |S_H| = 7$.

Phase-III

- Let G_2 be the resulting graph after the second phase
- It suffices to show that $\alpha(G_2) \geq n(G_2)/3 + nt(G_2)/42$
- We apply more operations to G_2 to remove *all* remaining triangles
- The removed subgraphs do not satisfy the local-ratio property
- We use a charging argument and amortized analysis to measure the impact of each of these operations on the resulting graph
- Our goal is to show that:
  $\alpha(G_2) \geq (23n(G_2) - 6e(G_2) + nt(G_2))/42$

Phase-III

- A block of a graph is called *difficult* [Harant et al.] if it is isomorphic to one of the following four graphs



- A connected graph is called *bad* [Harant et al.] if every block of the graph is either a difficult block or an edge between two difficult blocks

Phase-III

Lemma

A triangle-free graph G' that does not contain bad components satisfies $\alpha(G') \geq (4n(G') - e(G'))/7 = (23n(G') - 6e(G') + nt(G'))/42$

- After phase-II G_2 does not contain bad components
- For a subgraph H let $e^+(H)$ be the number of edges with at least one endpoint in H
- Call a vertex in H *internal* if all its neighbors are in H

Phase-III

- It suffices to show that:
  1. Each operation removes a subgraph H such that there exists an independent set consisting of internal vertices in H of size at least $(23n(H) - 6e^{+}(H) + nt(H))/42$; and
  2. the subgraph resulting from G_2 at the end of these operations is triangle-free and contains no bad components
- It will follow by additivity (using the above lemma) that:

$$\begin{aligned}\alpha(G_2) &\geq (23n(G_2) - 6e(G_2) + nt(G_2))/42 \\ &\geq n(G_2)/3 + nt(G_2)/42\end{aligned}$$

Phase-III

- For a subgraph H let $\phi(H) = |S_H| - (23n(H) - 6e^+(H) + nt(H))/42$, where S_H is a maximum independent set consisting of internal vertices in H
- We would like to show that each introduced operation that removes a subgraph H satisfies $\phi(H) \geq 0$
- This will be the case for most of the operations that we apply except few

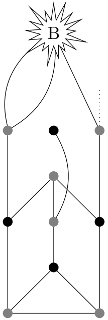
Phase-III

- We use amortized analysis: we show that each time one of these few operations applies, the “deficit" in the function ϕ caused by this operation can be “compensated for" by operations that *must* have occurred earlier in this phase
- For each operation that removes a subgraph H, we introduce a parameter $c(H)$, where $c(H)$ is the *debit* of operation H meant to possibly pay off the deficit of some later operations
- Let $s = 1/14$

Phase-III

- For each fringe edge $e = (u, v)$ to H between a boundary vertex u of H and a vertex $v \in G_2 - V(H)$, we define a debit $c(e) = s/2$ if v is a neighbor of some top vertex in a type-II steeple, and $c(e) = s/4$ otherwise
- We define $c(H)$ to be the sum of $c(e)$ over all fringe edges to H
- Finally, we extend the function $\phi(H)$ and define the function $\Phi(H) = \phi(H) - c(H)$
- It suffices to show that the sum of $\Phi(H)$ over all removed subgraphs H is positive

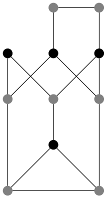
Example of a type-I steeple operation: no deficit



$e^+(H) \geq 15$
$nt(H) \leq 7$
$n(H) \leq 10$
$c(H) \leq \frac{s}{4}$
$\phi^-(B) = -\frac{1}{7}$

Phase-III

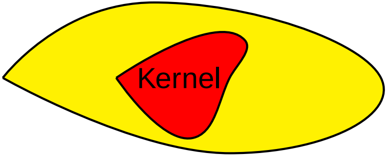
Example of a type-II steeple operation: deficit is *s* which is compensated for by the lack of the two edges incident to the neighbors of the top vertices in the steeple



$e^+(H) \geq 15$
$nt(H) \leq 8$
$n(H) \leq 11$
$c(H) = 0$

Kernelization

- A *parameterized problem* is a set of instances (x, k) where x is the input and k is the parameter
- A parameterized problem has a *kernel* (or is *kernelizable*) if there exists a polynomial-time algorithm that for every instance (x, k) outputs an instance (x', k') such that:
   i. (x, k) and (x', k') are equivalent
   ii. $|x'| \leq g(k)$ for some function g
   iii. $k' \leq g(k)$



Kernelization

- IS-3: Given a graph G of maximum degree at most 3 and k, does G have an independent set of size $\geq k$
- Brook's Theorem states that every K_4-free graph G of maximum degree ≤ 3 has an independent set of size $\geq n(G)/3$
- Brook's theorem implies that IS-3 has a kernel of size $3k$: remove the K_4's
- Can we improve the $3k$ upper bound for IS-3?

Kernelization

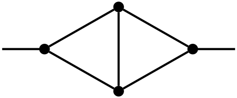
- One possible approach is to preprocess the graph so that to guarantee the presence of nontriangle vertices
- We can then use the presented combinatorial result

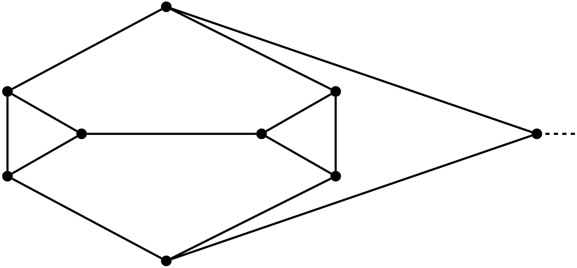
Kernelization algorithm for IS-3: Prelude

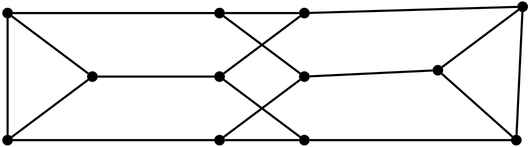
Reduction Rule

Remove the obstacle graphs

Kernelization algorithm for IS-3: Prelude



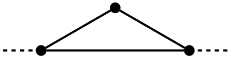




Kernelization algorithm for IS-3: Prelude

Reduction Rule

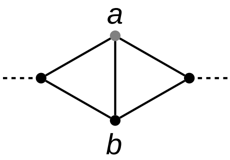
Remove every triangle with a degree-2 vertex



Kernelization algorithm for IS-3: Prelude

Reduction Rule

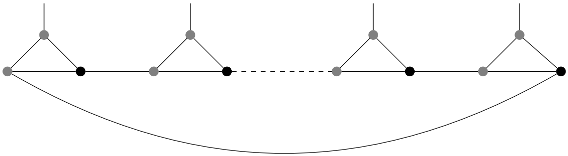
Break adjacent triangles



Kernelization algorithm for IS-3: Prelude

Reduction Rule

Remove cycles of triangles



Kernelization algorithm for IS-3: Prelude

- Let $n(G)$ be the number of vertices in G, and $nt(G)$ be the number of nontriangle vertices in G
- Suppose that none of the three previous reduction rules applies to G

Theorem

The number of nontriangle vertices $nt(G)$ satisfies $nt(G) \geq n(G)/10$

Kernelization algorithm for IS-3: Prelude

- Call a graph *reduced* if none of the above reductions applies to it
- We have the following theorem:

Theorem

Let G be a reduced graph then $\alpha(G) \geq 141n(G)/420$

Proof.

This follows from the combinatorial lower bound $\alpha(G) \geq n(G)/3 + nt(G)/42$ after noting that $nt(G) \geq n/10$ $\square$

Kernelization algorithm for IS-3: Prelude

Corollary

IS-3 *has a kernel of size* $420k/141$ *that is computable in* $O(k)$ *time*

Corollary

Unless P=NP, VC-3 *does not have a kernel of size smaller than* $420k/279 \approx 1.505$

Concluding Remarks

- Concrete open problems: Can we improve the upper bound on the kernel size for (planar) VC-3 or IS-3?
- Can we improve the lower bound on the kernel size for VC-3?
- Duality does not seem to be a very promising technique for tightening this gap