Characterizations of Deque and Queue Graphs

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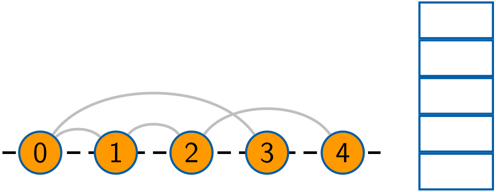
Introduction and Motivation

Deque Graphs

Proper Leveled-Planar Graphs

Conclusion and Future Work

- Graph layouts
  - Undirected graph: $G = (V, E)$
  - Linear layout $\pi : V \to \{0, \ldots, n-1\}$: positioning of the vertices
- Example: Stack layout



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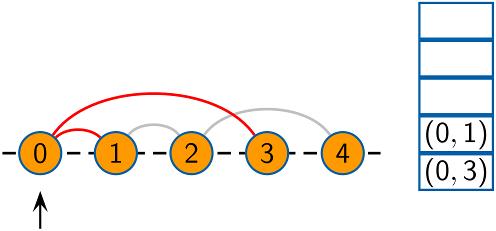

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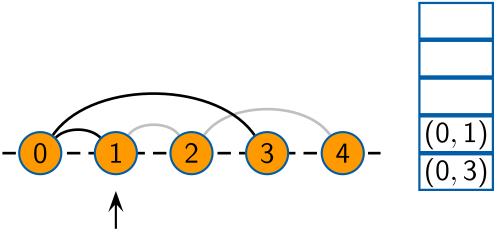

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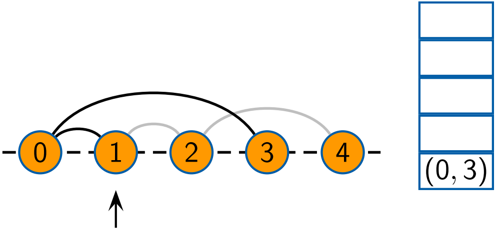

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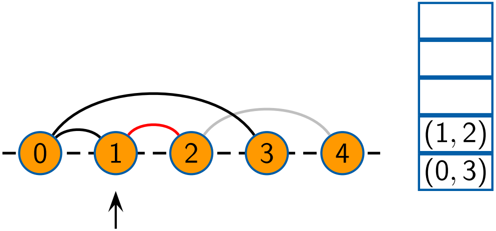

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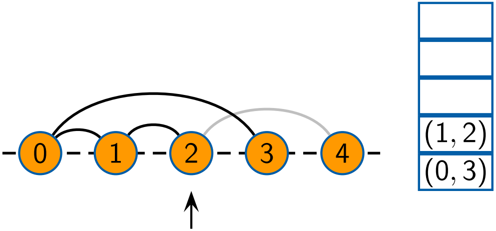

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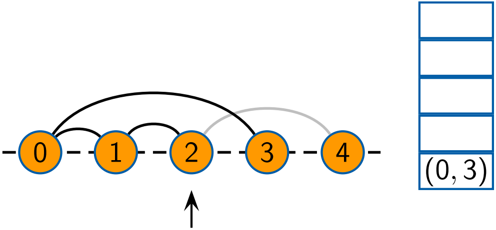

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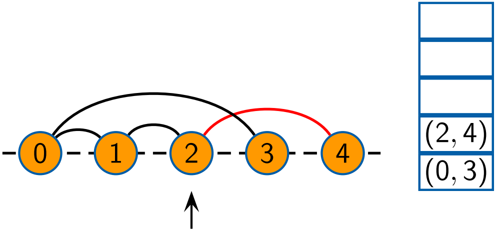

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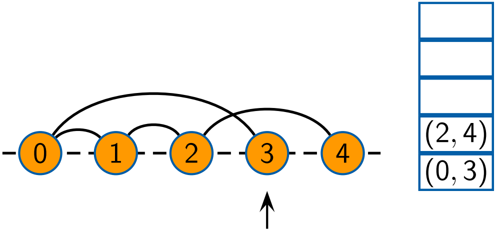

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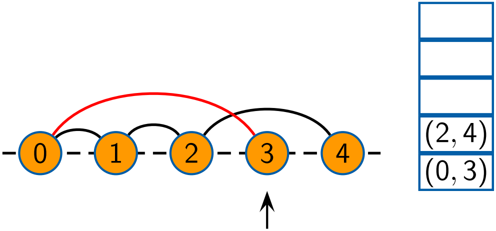

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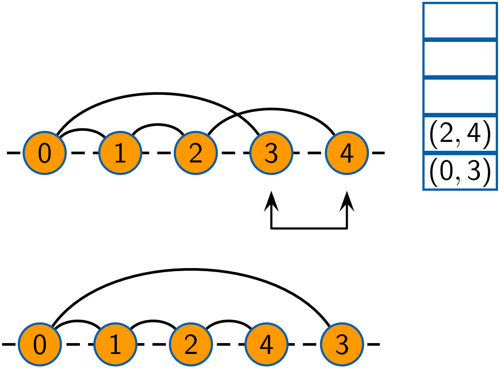

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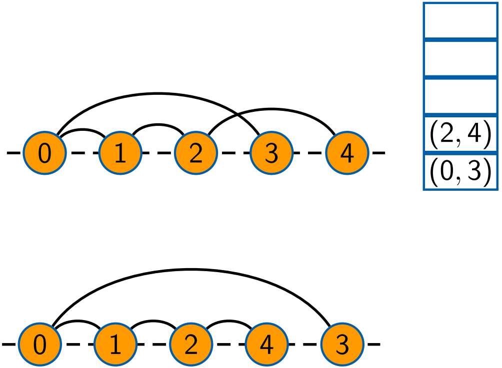
- Example: Stack layout
- Strong relationship between graph layouts and planarity



Bernhart, Kainen, '79: A graph is a. . .

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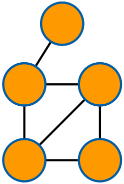

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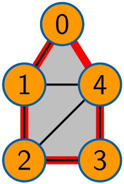

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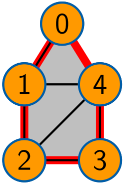

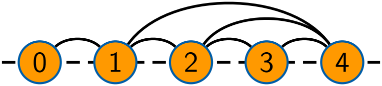

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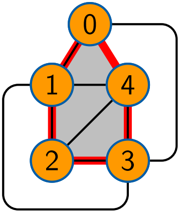

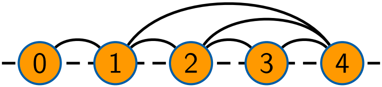

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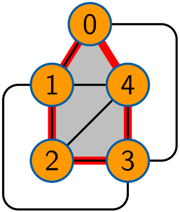
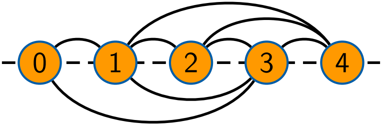
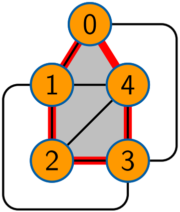
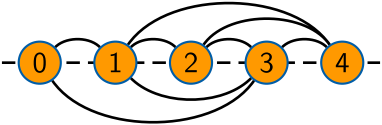




Bernhart, Kainen, '79: A graph is a...

- ...stack graph $\Longleftrightarrow$ it is outer-planar
- ...2-stack graph $\Longleftrightarrow$ it is subgraph of planar graph with a Hamiltonian cycle





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- Two sides: Head h and Tail t

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- Linear I/O layout:
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  - Input assignment $\alpha : E \to \{\mathtt{h}, \mathtt{t}\}$
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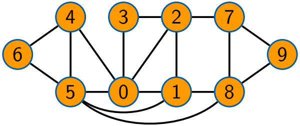
  - ...allows queue items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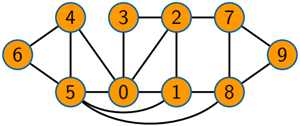

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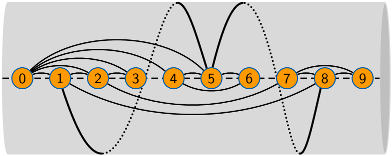

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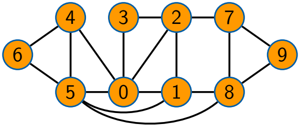
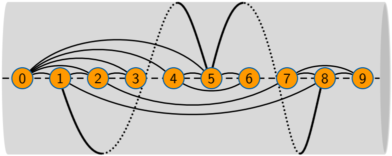
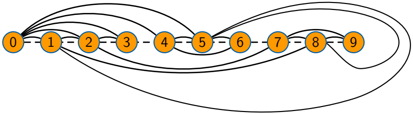
Deque Graphs







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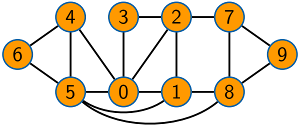
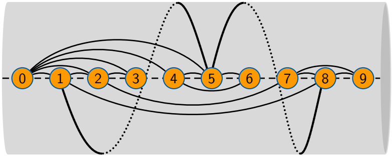
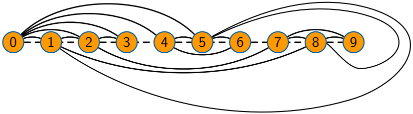
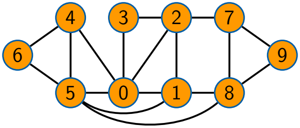


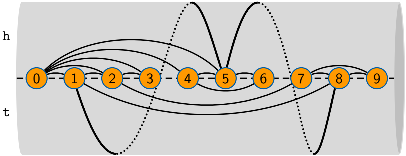


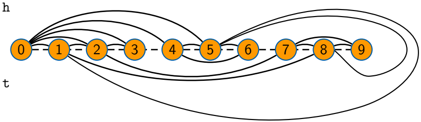


- A graph is a deque graph $\Longleftrightarrow$ it is linear cylindric planar

Deque Graphs







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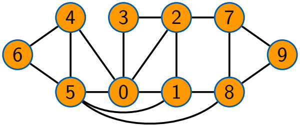

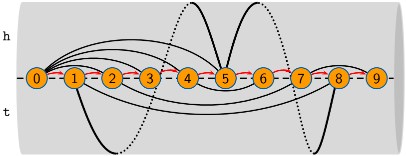

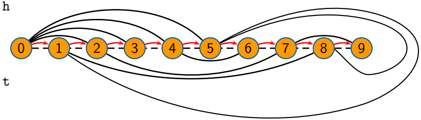

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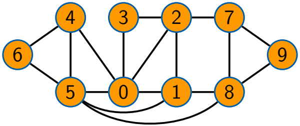
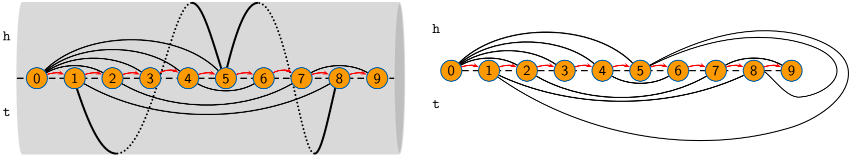
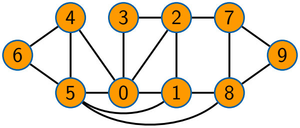


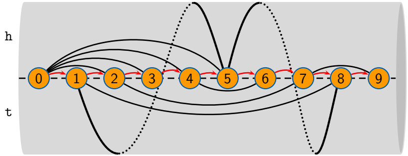


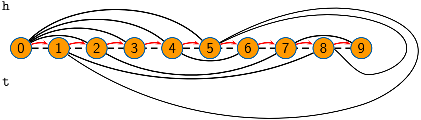
- A graph is a deque graph $\iff$ it is linear cylindric planar
  - G is a deque graph

$$\Longrightarrow$$

G has a planar supergraph with a Hamiltonian path







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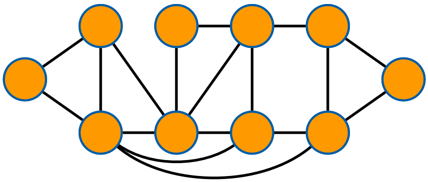
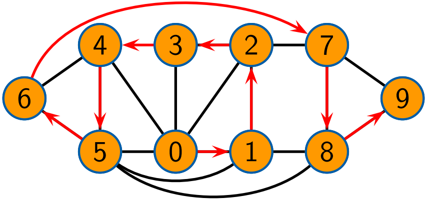
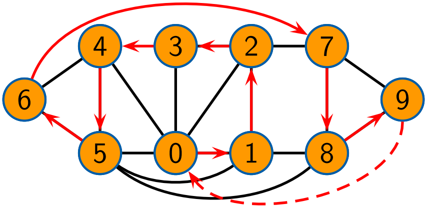
- Proof idea



- Proof idea
  - "Cut" along Hamiltonian path



- Proof idea
  - "Cut" along Hamiltonian path and back to start vertex



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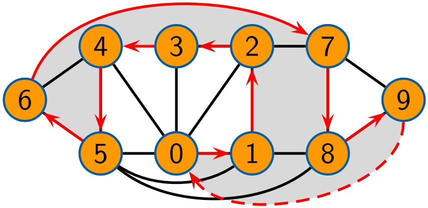
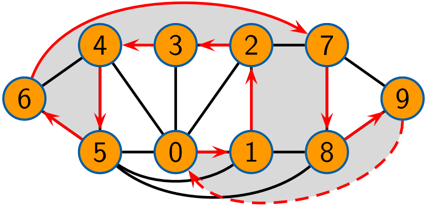
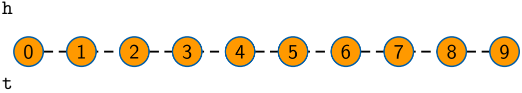


- Proof idea
  - "Cut" along Hamiltonian path and back to start vertex
  - Linear layout: Hamiltonian path





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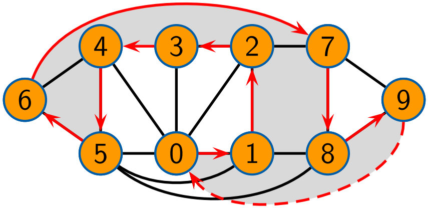
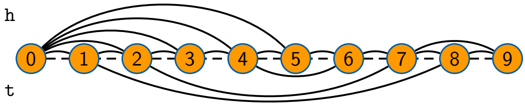
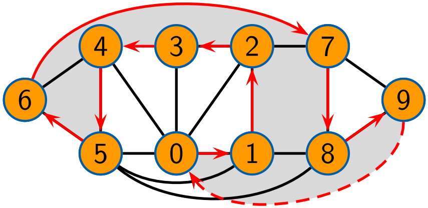
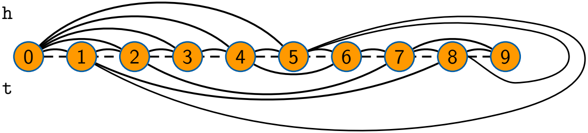
  - Stack edges: within one region

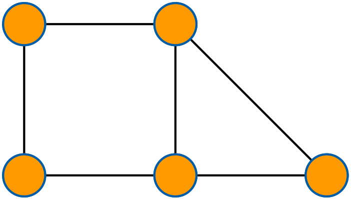


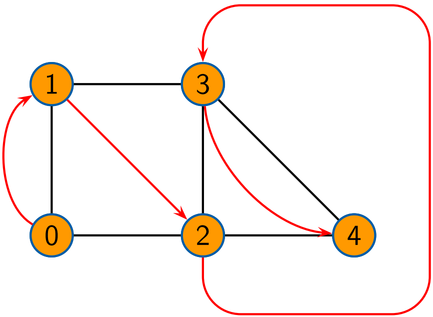


- Proof idea
  - "Cut" along Hamiltonian path and back to start vertex
  - Linear layout: Hamiltonian path
  - Stack edges: within one region
  - Queue edges: different regions





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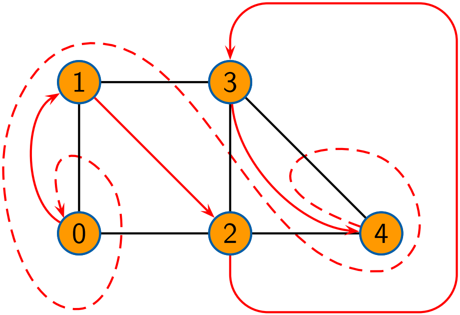
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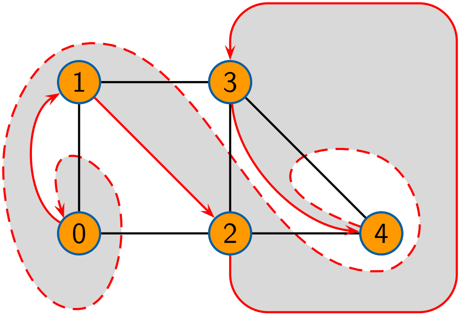
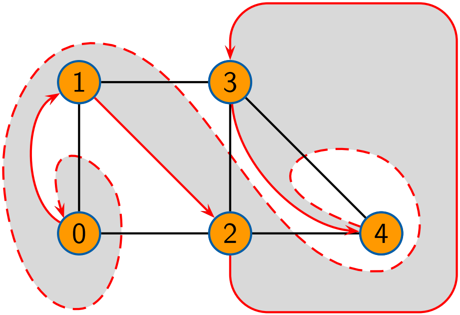
Deque Graphs

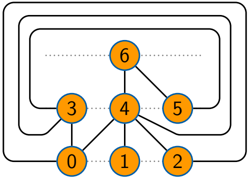


- All edges are queue edges $\Longrightarrow$ queue graph

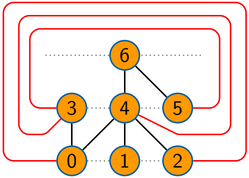


- All edges are queue edges $\Longrightarrow$ queue graph
- Consequence: The dual of an embedded queue graph has a Eularian path

- G is a queue graph $\iff$ G is arched leveled-planar (Heath and Rosenberg)



- G is a queue graph $\iff$ G is arched leveled-planar (Heath and Rosenberg)
- "almost" proper leveled-planar



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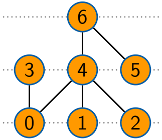

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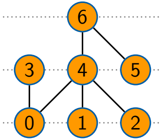

Proposition:

$$G \text{ is proper leveled-planar} \Longrightarrow G \text{ is a bipartite queue graph}$$

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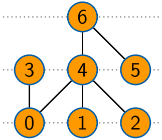
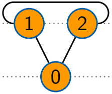
- arched leveled-planar $\not\Rightarrow$ bipartite





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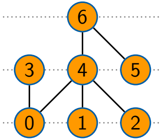
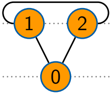


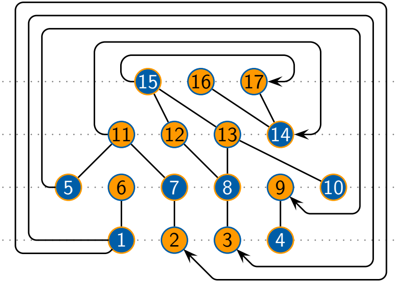


Theorem:

G is proper leveled-planar

$\Longleftrightarrow$

G is a bipartite queue graph



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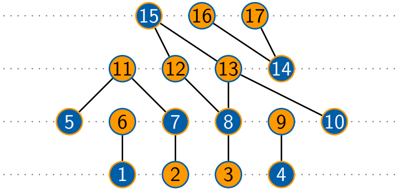
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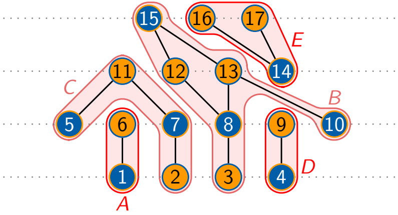
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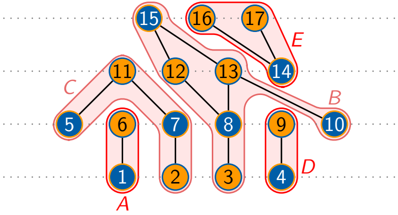
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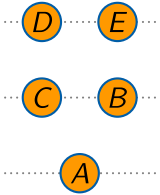
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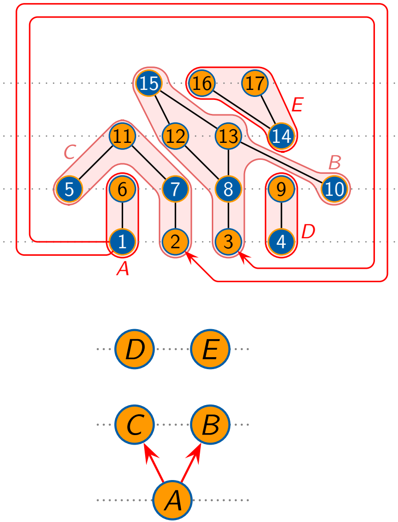

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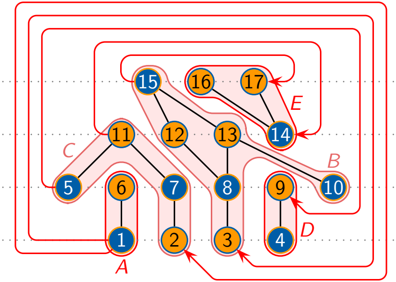
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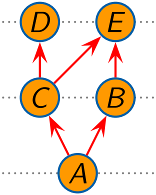
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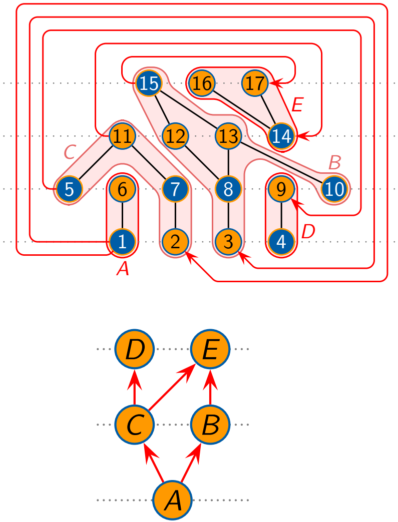

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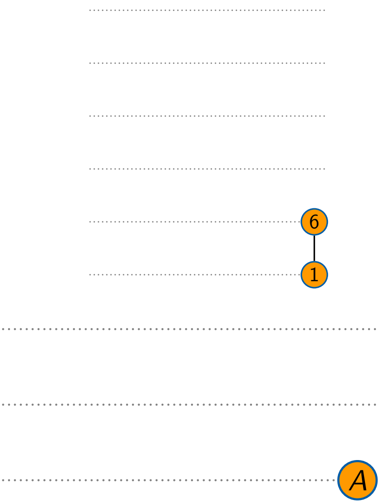

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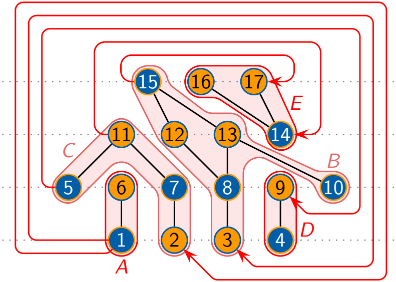

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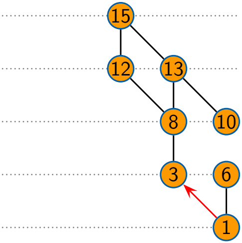

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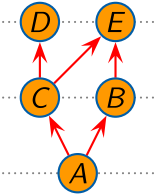

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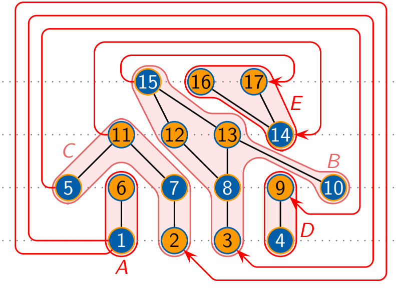
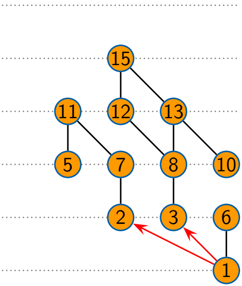
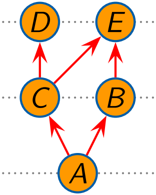
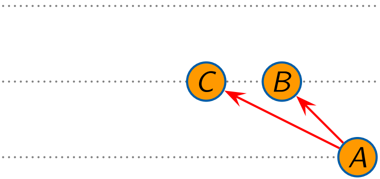
Proper Leveled-Planar Graphs









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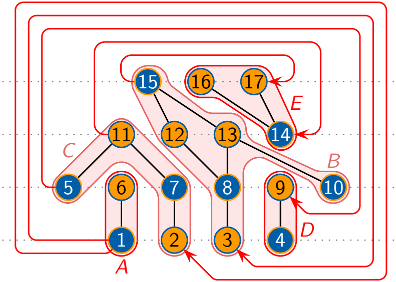

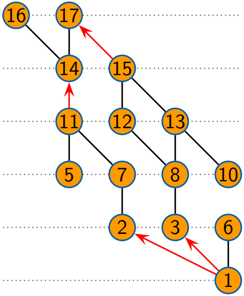

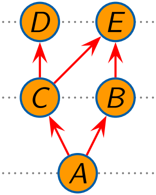

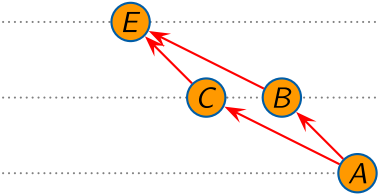

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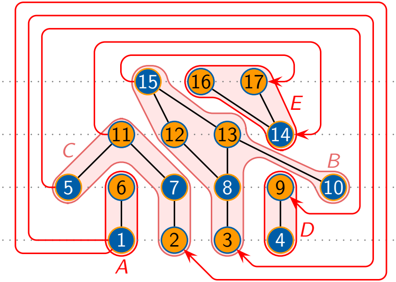
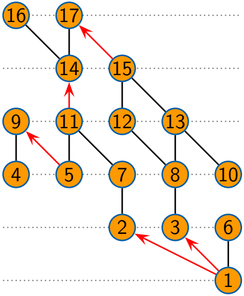
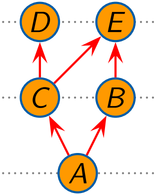
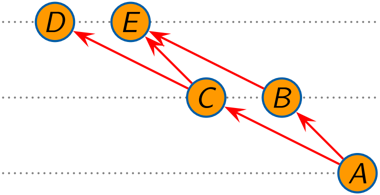
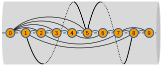


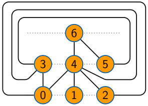






- Deque layouts
  - Planar and Hamiltonian cycle $\Longleftrightarrow$ 2-Stacks
  - Planar and Hamiltonian path $\Longleftrightarrow$ Deque





Conclusion and Future Work

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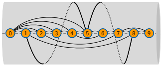
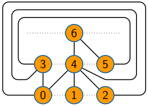
  - Future: Planar $\Longleftrightarrow$ Extended Deque





Conclusion and Future Work

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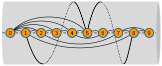

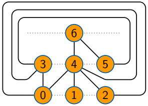

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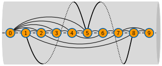

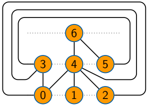

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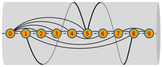

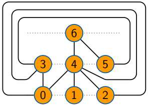

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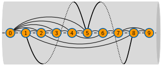

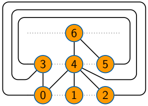

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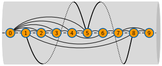

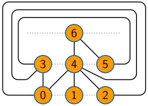

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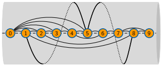

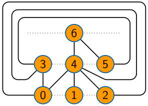

Thank You!