Approximability of the path-distance-width for AT-free graphs

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Outline

1. Introduction
   - Definition
   - Background
   - Our results
2. Approximation for AT-free graphs
   - AT-free graphs
   - Sketch of proof
3. Concluding remarks
   - Conclusions & Open problems

Outline

Definition: Path-distance-width (1 of 2)

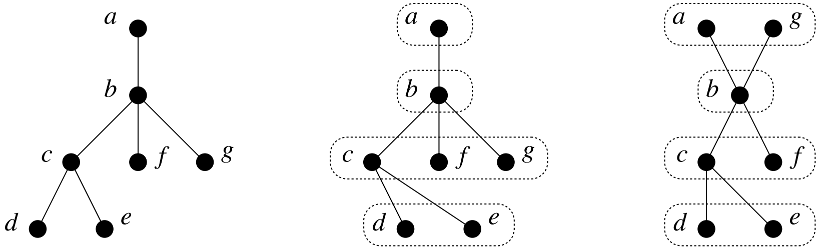
Definition (Distance)

The distance between two vertices u and v in G is denoted by $d_G(u, v)$. The *distance* between $S \subseteq V(G)$ and $v \in V(G)$ in G is defined as $d_G(S, v) = \min_{u \in S} d_G(u, v)$.

Definition (Distance structure)

The *distance structure* of G rooted at S, denoted by $D_G(S)$, is $(L_1, L_2, \ldots, L_t)$ s.t.

- $\bigcup_{1 \le i \le t} L_i = V(G)$ and
- $L_i = \{v \in V(G) \mid d_G(S, v) = i - 1\}$.



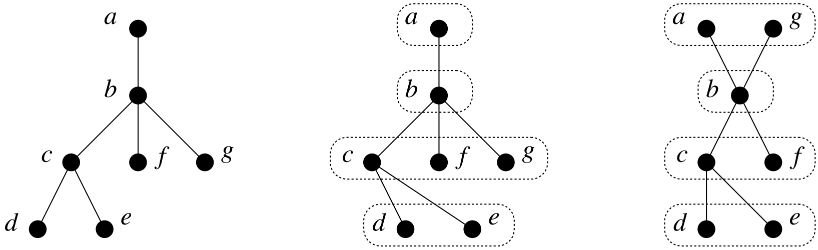
Definition: Path-distance-width (2 of 2)

Definition (Path-distance-width)

Let G be a graph, $S \subseteq V(G)$, and $D_G(S) = (L_1, L_2, \ldots, L_t)$. The *path-distance-width* of S in G, denoted by $pdw_G(S)$, is $\max_i |L_i|$. The *path-distance-width* of G is defined as

$$pdw(G) = \min_{S \subseteq V(G)} pdw_G(S).$$

Path-distance-width is defined for connected graphs only.



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Relations to other important graph parameters

Theorem (Corollary to some known results)

For any connected graph G,

$$treewidth(G) \leq pathwidth(G) \leq bandwidth(G) < 2 \cdot pdw(G).$$

- The “bounded pdw” constraint is very strong.
- One can use very simple structures of bounded pdw graphs.
- Even if a problem is NP-hard for graphs of bounded (treewidth | pathwidth | bandwidth), it may be in P for graphs of bounded path-distance-width.
- One can use very simple structures of bounded pdw graphs.

Known results (1 of 2)

The complexity for GRAPH ISOMORPHISM is not known: in P or NP-hard?

Theorem (Bodlaender 1990)

If G and H have treewidth at most k, then GRAPH ISOMORPHISM *of G and H can be solved in $O(n^{k+4.5})$ time.*

Theorem (Yamazaki *et al.* 1999)

If G and H have path-distance-width at most k, then GRAPH ISOMORPHISM *of G and H can be solved in $O(n^{k+1})$ time.*

Known results (2 of 2)

Theorem (Yamazaki *et al.* 1999)

Given a tree T and an integer k, deciding whether $pdw(T) \leq k$ is NP-complete.

Theorem (Yamazaki 2001)

It is NP-hard to approximate pdw of a tree within a factor $4/3 - \epsilon$ for any $\epsilon > 0$.

- Tree-like structures do not help.
- How about graphs with *chain-like* (or *path-like*) structures?

Related results (Bandwidth of chain-like graphs)

Theorem (Sprague 1994)

The bandwidth of an interval graph can be determined in $O(n \log n)$ time.

Theorem (Kloks *et al.* 1999)

Given an AT-free graph G and an integer k, deciding whether bandwidth$(G) \leq k$ is NP-complete. The bandwidth of an AT-free graph can be approximated within a factor 2 in $O(mn)$ time.

Theorem (Golovach *et al.* 2009)

k-BANDWIDTH for AT-free graphs is in FPT.

Outline

Our results

Theorem

The path-distance-width of a k-cocomparability graph can be approximated within a factor $2k + 1$ in $O(mn)$ time.

Theorem

The path-distance-width of an AT-free graph can be approximated within a factor 3 in $O(m + n)$ time.

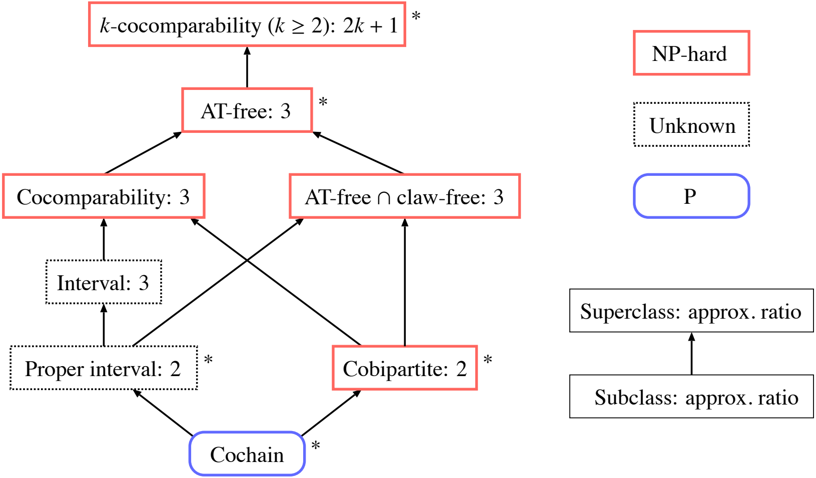
Theorem

The problem to determine the path-distance-width of a cobipartite graph is NP-hard.

Cobipartite $\subset$ AT-free $\subset$ k-cocomparability ($k \geq 2$).

n: the number of vertices. m: the number of edges.

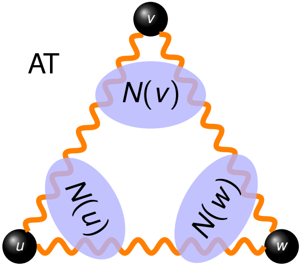
Summary of results



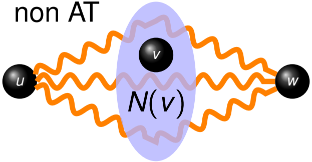
Outline

Definition: AT-free graphs

AT



non AT



Definition (Asteroidal Triple (AT))

An *asteroidal triple* (*AT*) is a vertex triple such that there is a path between any two of them avoiding the neighbors of the third.

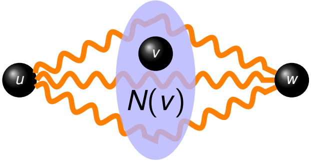
Definition (AT-free graphs)

A graph is *AT-free* if it contains no AT.

Roughly speaking, AT-free graphs have *chain-like* structures. Interval graphs and permutation graphs are AT-free.

Property of AT-free graphs

(u, w) is a dominating pair if the following holds $\forall v$



Definition (Dominating pair)

A vertex pair (u, v) in G is a *dominating pair* of G if the vertex set of every u–v path in G is a dominating set of G.

Theorem (Corneil *et al.* 1995 & 1999)

Every connected AT-free graph has a dominating pair. A dominating pair of a connected AT-free graph can be found in linear time.

$(*)$ "AT-free" $\neq$ "having a dominating pair". Consider the wheel graph W_n for $n \geq 7$.

Outline

Key lemma

For $S \subseteq V(G)$, $diam_G(S) = \max_{u,v \in S} d_G(u, v)$.

Lemma

Let $S \subseteq V(G)$. Then, $pdw(G) \geq |S|/(diam_G(S) + 1)$.

Proof.

For any distance structure of G, S can intersect at most $diam_G(S) + 1$ levels. □

Corollary

Let $D_S(G) = (S = X_1, \ldots, X_t)$ be a distance structure of G. If $diam_G(X_i) \leq k$ for $1 \leq i \leq t$, then $pdw_G(S) \leq (k + 1)pdw(G)$.

Bounding the diameter of each level

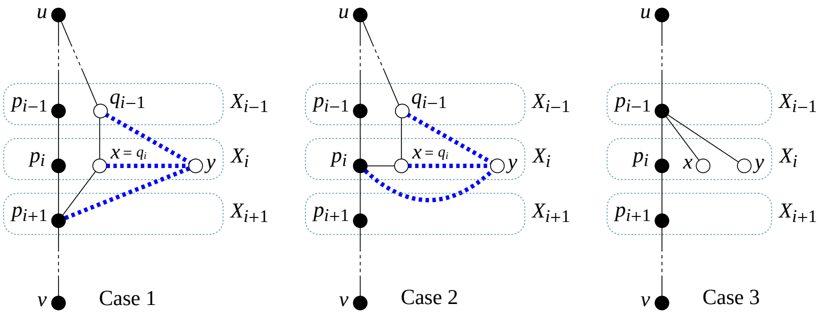
- (u, v): a dominating pair.
- $D(\{u\}) = (X_1, \dots, X_t)$.
- x, y: vertices in X_i.
- $(p_1, \dots, p_\ell)$: a shortest u–v path.

Lemma

$diam_G(X_i) \leq 2$ *for* $1 \leq i \leq t$

Proof.

Figure shows $d_G(x, y) \leq 2$. □



Proof

Theorem

The path-distance-width of an AT-free graph can be approximated within a factor 3 *in* $O(m+n)$ *time.*

Proof.

1. Find a dominating pair (u, v) in $O(m+n)$ time.
2. Construct the distance structure $D(\{u\})$ in $O(m+n)$ time.
3. Output the maximum size of the levels of $D(\{u\})$ in $O(n)$ time.

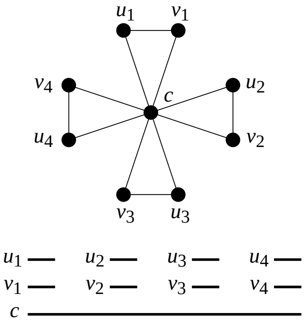
Since each level of $D(\{u\})$ has diameter at most two,

$$pdw_G(\{u\}) \leq 3 \cdot pdw(G).$$

This completes the proof. □

Note on the approximation factor 3

The factor 3 is best possible even for interval graphs (and thus for AT-free graphs) in the following sense.



The friendship graph F_4 and its interval representation.

Theorem

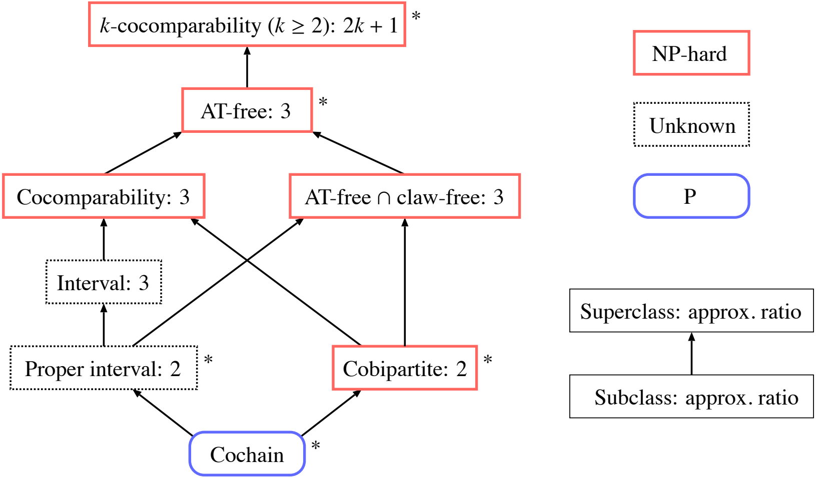
The approximation ratio 3 cannot be improved if we select only one vertex as the initial set.

Proof.

Let A be the algorithm examining all distance structures rooted at a single vertex. For friendship graphs, the approximation ratio of A is $3 - o(1)$. Since friendship graphs are interval graphs, the theorem holds. □

Outline

Conclusions



Open problems

- The complexity for (proper) interval graphs.
- Approximation for trees or chordal graphs.
  (Recall: better than 4/3-approximation is NP-hard.)
- When parametrized by path-distance-width, is Graph Isomorphism FPT?