The Cinderella game on holes and anti-holes.

Marijke H.L. Bodlaender[1] Cor A.J. Hurkens[2]
Gerhard J. Woeginger[2]

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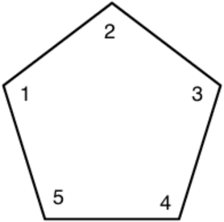
[1]Department of Information and Computing Sciences, Universiteit Utrecht, The Netherlands
[2]Department of Mathematics and Computer Science, TU Eindhoven, The Netherlands

Introduction to the game



- Proposed problem for the International Mathematical Olympiad
- We study variant where water arrives in rounds and the game board is an undirected graph

The game

- Game played on undirected simple graph $G = (V, E)$
- Every vertex v contains a bucket
- Every edge $[u, v] \in E$ indicates an incompatibility
- In every round the Stepmother distributes a liter of water in the buckets
- Cinderella empties the buckets in an independent set
- Stepmother tries to reach an overflow
- Cinderella wants to avoid an overflow

Some definitions and notation

- $\texttt{bucket}'(G)$: infimum of all bucket sizes Cinderella needs to win
- $\texttt{bucket}(G) : \texttt{bucket}'(G) - 1$ is the *bucket number* of G
- GREEDY: empty maximum weight independent set every turn
- $\texttt{g-bucket}(G)$: bucket number of G when Cinderella uses a GREEDY strategy

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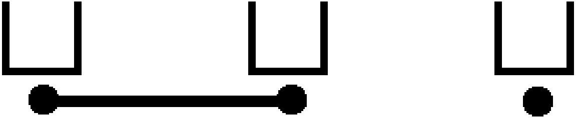
Definitions

- $x = (x_v)\ v \in V$ where x_v is the contents of bucket v at the start of a round
- $x(S) = \sum_{v \in S} x_v$
- y_v the contents of v after the Stepmother moved
- $\chi(G)$ the chromatic number of G
- $\omega(G)$ the clique number of G
- For $S \subseteq V$ we write $\chi(S)$
- $\mathcal{H}\langle k \rangle = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ \ldots\ldots \ + \frac{1}{k}$

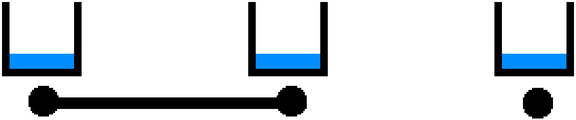
Example with bucket size 1.6

- $x_1 = x_2 = x_3 = 0$



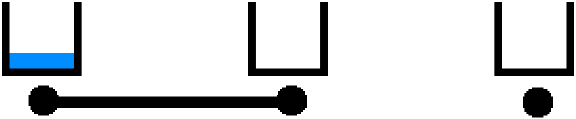
Example with bucket size 1.6

- $y_1 = y_2 = y_3 = 1/3$



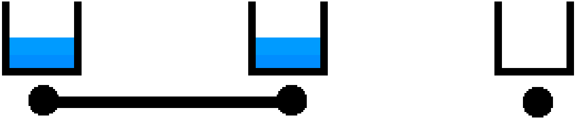
Example with bucket size 1.6

- $x_1 = 1/3$
- $x_2 = x_3 = 0$



Example with bucket size 1.6

- $y_1 = y_2 = 2/3$
- $y_3 = 0$



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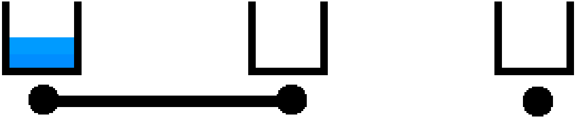
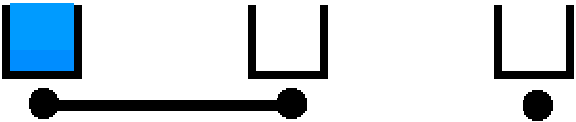
Example with bucket size 1.6

- $x_1 = 2/3$
- $x_2 = x_3 = 0$



Example with bucket size 1.6

- $y_1 = 5/3 > 1.6$
- $y_2 = y_3 = 0$



Results in this paper

- $\texttt{g-bucket}(G) \leq \mathcal{H}\left\langle \chi(G) - 1 \right\rangle$
- $\texttt{bucket}(G) \geq \mathcal{H}\left\langle \omega(G) - 1 \right\rangle$
- $\texttt{bucket}(G) = \mathcal{H}\left\langle \omega(G) - 1 \right\rangle \quad \forall$ graphs on $n \leq 6$ vertices
- $\texttt{bucket}(C_{2m+1}) = 1$
- $\texttt{g-bucket}(C_{2m+1}) = 1 + \frac{1}{m} \cdot 2^{-m}$
- $\texttt{g-bucket}\left(\overline{C_{2m+1}}\right) \leq \mathcal{H}\left\langle m \right\rangle - 1/(2m)$
- $\texttt{g-bucket}\left(\overline{C_{2m+1}}\right) \geq \mathcal{H}\left\langle m - 1 \right\rangle + \frac{m^2 - 3m + 1}{2m^2(m-1)}$

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Upper bound on general graphs

Theorem

Every graph $G = (V, E)$ *satisfies* $\texttt{g-bucket}(G) \leq \mathcal{H}\langle\chi(G) - 1\rangle$

Proof.

GREEDY maintains the following system of invariants

$$x(S) < \chi(S)\cdot(1 + \mathcal{H}\langle\chi(G) - 1\rangle - \mathcal{H}\langle\chi(S)\rangle) \text{ for all sets } S \subseteq V \quad (1)$$

Apply (1) to $S = \{v\}$ to show $x_v < \mathcal{H}\langle\chi(G) - 1\rangle$

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Upper bound for GREEDY on general graphs continued

If $\chi(S) = \chi(G)$, then

$$\begin{aligned} y(S) &\leq y(V - I) \leq \frac{\chi(G) - 1}{\chi(G)} y(V) \\ &\leq \frac{\chi(G) - 1}{\chi(G)} (x(V) + 1) < \chi(G) - 1 \end{aligned}$$

Assume that $\chi(S) < \chi(G)$ observe that

$$y(S) \leq \chi(S) \cdot y(I) \tag{2}$$

Upper bound for GREEDY on general graphs continued

Furthermore

$$x(S \cup I) \ < \ (\chi(S) + 1) \cdot (1 + \mathcal{H}\langle \chi(G) - 1 \rangle - \mathcal{H}\langle \chi(S) + 1 \rangle) \tag{3}$$

Applying (2) and (3) we derive

$$\begin{aligned} y(S) \quad &\leq \quad \frac{\chi(S)}{\chi(S)+1}\,(y(S) + y(I)) \ \leq \ \frac{\chi(S)}{\chi(S)+1}\,(x(S \cup I) + 1) \\ &< \quad \chi(S) \cdot \left(1 + \mathcal{H}\langle \chi(G) - 1 \rangle - \mathcal{H}\langle \chi(S) + 1 \rangle + \frac{1}{\chi(S)+1}\right) \\ &= \quad \chi(S) \cdot (1 + \mathcal{H}\langle \chi(G) - 1 \rangle - \mathcal{H}\langle \chi(S) \rangle) \end{aligned}$$

Lower bound on general graphs

Theorem

Every graph $G = (V, E)$ *satisfies* $\texttt{bucket}(G) \geq \mathcal{H}\langle\omega(G) - 1\rangle$

Let $\omega(G) = n$

Define a strategy for the Stepmother:

- Play game on the the largest clique, K
- At the first phase:
  - Fill repeatedly all buckets in K to the same level
  - This converges to $1 - \epsilon$
- In second phase
  - In r-th round fill $n - r$ fullest buckets to the same level
  - At the end of round $n - 2$ at least one bucket contains $\mathcal{H}\langle n - 1\rangle - \epsilon$

Perfect graphs

Theorem

Every perfect graph G has
$\mathtt{bucket}(G) = \mathtt{g\text{-}bucket}(G) = \mathcal{H}\left\langle \omega(G) - 1 \right\rangle$

GREEDY odd holes: Upper bound

Theorem

The odd cycle C_{2m+1} *has* $\texttt{g-bucket}(C_{2m+1}) \leq 1 + \frac{1}{m} \cdot 2^{-m}$

Proof(Upper bound).

GREEDY maintains the following invariants

$$\sum_{i=1}^{2m+1} x_i \ < \ \frac{m+1}{m}$$

$$\sum_{i=k}^{k+2t-1} x_i \ < \ 1 + \frac{1}{m} \cdot 2^{t-m} \qquad \text{for } 1 \leq k \leq 2m+1,\ 1 \leq t \leq m$$

$$x_k \ < \ 1 + \frac{1}{m} \cdot 2^{-m} \qquad \text{for } 1 \leq k \leq 2m+1$$

GREEDY odd holes: Lower bound

Theorem

The odd cycle C_{2m+1} has $\texttt{g-bucket}(C_{2m+1}) \geq 1 + \frac{1}{m} \cdot 2^{-m}$

Proof.

First phase: Fill repeatedly all buckets to the same level

Second phase:

	B_1	B_2	B_3	$\dots$	B_{2m-3}	B_{2m-2}	B_{2m-1}	B_{2m}
SL	$\frac{1}{m}$	$\frac{1}{m}$	$\frac{1}{m}$	$\dots$	$\frac{1}{m}$	$\frac{1}{m}$	$\frac{1}{m}$	$\frac{1}{m}$
CL	0	$\frac{1}{m}$	0	$\dots$	$\frac{1}{m}$	0	$\frac{1}{m}$	$\frac{1}{m}$
SL	0	$\frac{1}{m}$	$\frac{1}{m}$	$\dots$	α_1	α_1	α_1	α_1
CL	0	0	$\frac{1}{m}$	$\dots$	0	α_1	α_1	0

	$\dots$	B_{k+1}	B_{k+2}	$\dots$	B_{2m-k-1}	B_{2m-k}	B_{2m-k+1}	B_{2m-k+2}
CL	0	$\frac{1}{m}$	0	$\dots$	$\frac{1}{m}$	0	α_{k-1}	α_{k-1}
SL	0	$\frac{1}{m}$	$\frac{1}{m}$	$\dots$	α_k	α_k	α_k	α_k
CL	0	0	$\frac{1}{m}$	$\dots$	0	α_k	α_k	0

GREEDY odd holes: Lower bound (continued)

Third phase:

	...	B_{m+1}	B_{m+2}	...
CL	0	α_{m-1}	α_{m-1}	0
SL	0	$\alpha_{m-1} + \frac{1}{2}$	$\alpha_{m-1} + \frac{1}{2}$	0
CL	0	$\alpha_{m-1} + \frac{1}{2}$	0	0

The alpha values solve to

$$\begin{aligned} \alpha_k &= \frac{1}{2m}\left(k + 1 + 2^{-k}\right) \Rightarrow \\ \alpha_{m-1} + \frac{1}{2} &= 1 + \frac{1}{m} \cdot 2^{-m} \end{aligned}$$

Possible future research

Conjecture

Every graph G satisfies $\texttt{bucket}(G) = \mathcal{H}\left\langle \omega(G) - 1 \right\rangle$

Conjecture

A graph G is perfect, if and only if $\texttt{bucket}(G) = \texttt{g-bucket}(G)$

Conjecture

The difference between $\texttt{g-bucket}(G)$ *and* $\texttt{bucket}(G)$ *is bounded by an absolute constant (that does not depend on G)*