Counting independent sets in claw-free graphs

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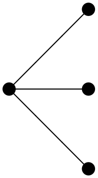

Problem definition

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Question: What is the number of independent sets in G?

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Björklund and Husfeldt 2006

If independent sets in a graph with n vertices can be counted in time $O^*(c^n)$ then this graph can be colored in time $O^*((1+c)^n)$ and polynomial space.

What is known

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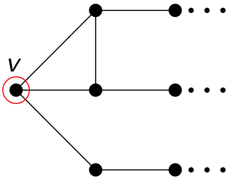
Wahlström 2008	$O^*(1.15..^n)$	$O^*(1.23..^n)$

Branching



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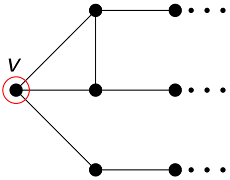

v not in S

v in S

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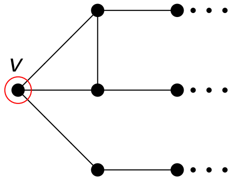

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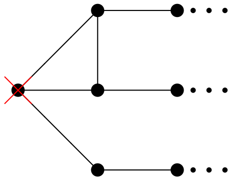

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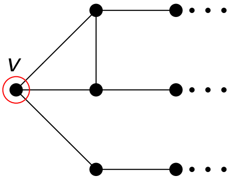

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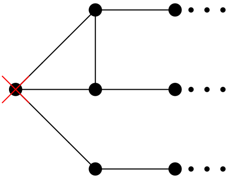

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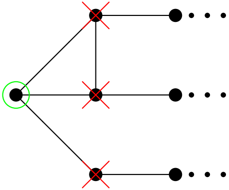


Branching

Recursion

$|IS(G)| = |IS(G-v)| + |IS(G-N[v])|$

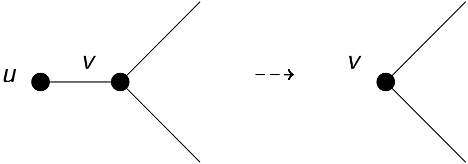
$$T(n) = T(n-1) + T(n-4)$$

$$1 = \frac{1}{\tau^1} + \frac{1}{\tau^4}$$

$$\tau_0 = 1.38..$$

$$T(n) = O^*(\tau_0^n) = O^*(1.38..^n)$$

Reduction



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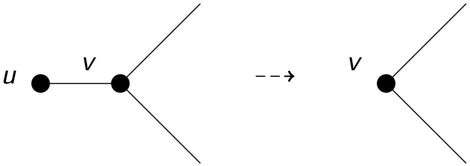

$$c_1(v) := c_1(v) \cdot c_0(u)$$
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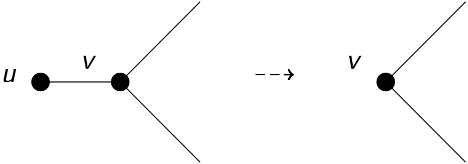

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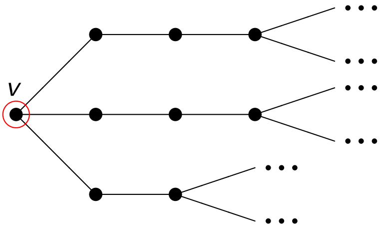
$$\sum_S \prod_{w: w \in S} \mathbf{c_1}(w) \prod_{w: w \notin S} \mathbf{c_0}(w)$$

For $c_1 = 1, \ c_0 = 1$ the sum is equal to the number of ISs.

Branching with reduction



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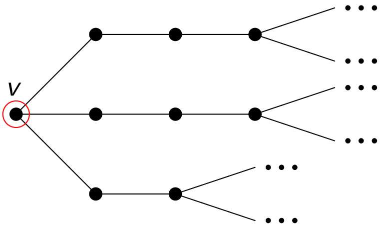

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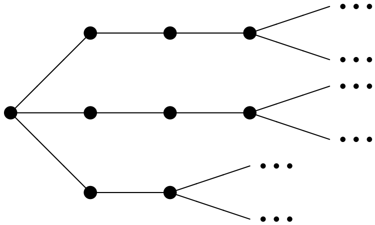

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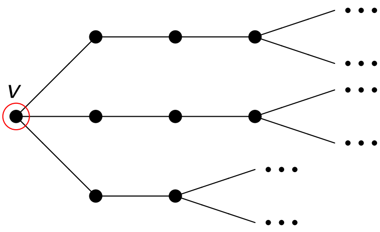

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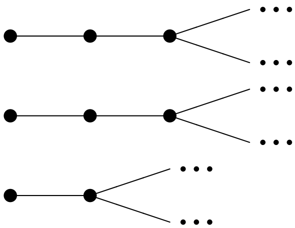

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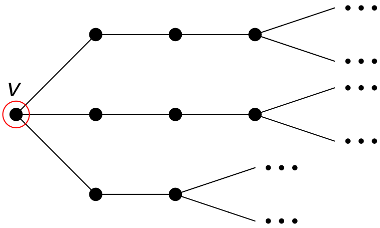

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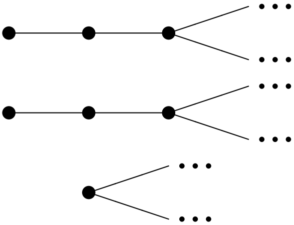

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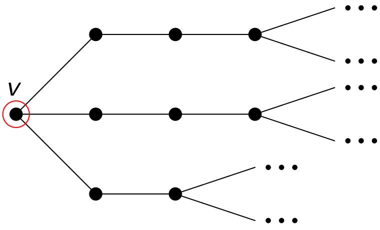

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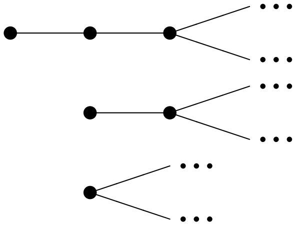

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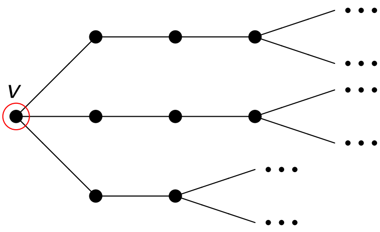

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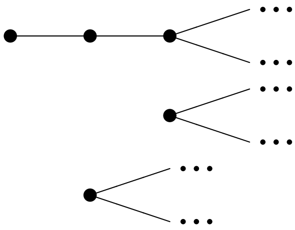

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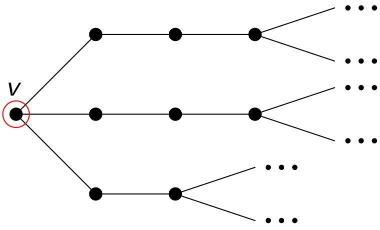

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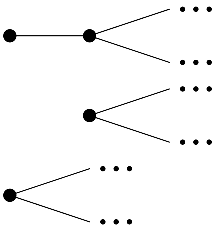

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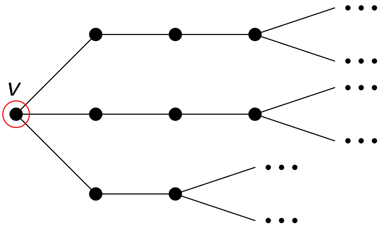

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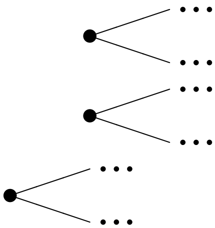

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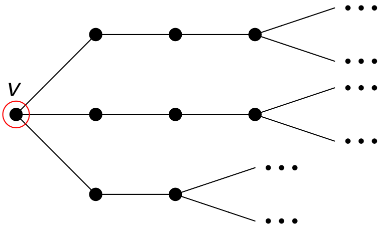

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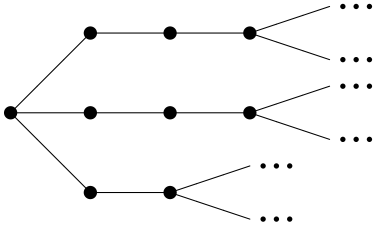

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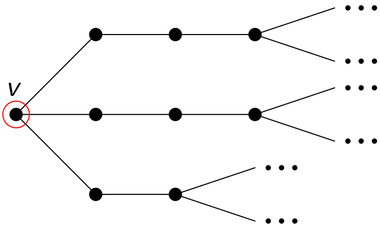

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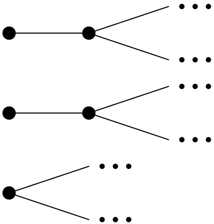

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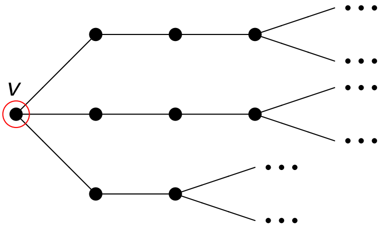

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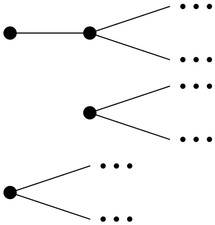

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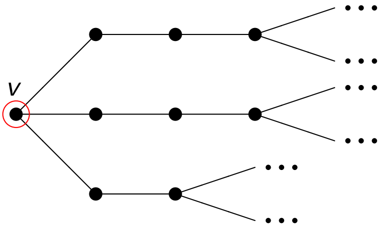

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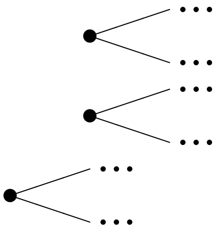


Measure and conquer

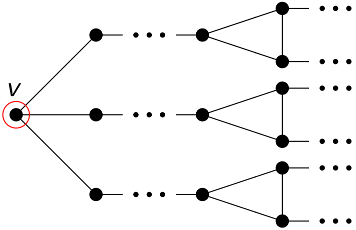
$\mu(H)$ - the measure of a subgraph H of G
H_0, H_1 - subgraphs for which the algorithm applied to H recursively call its-self.
$\Delta_0 = \mu(H) - \mu(H_0)$, $\Delta_1 = \mu(H) - \mu(H_1)$,
$\tau_H = \tau(\Delta_0, \Delta_1)$ be the solution of the equation

$$1 = \frac{1}{\tau^{\Delta_0}} + \frac{1}{\tau^{\Delta_1}}$$

τ_0 - maximum τ_H over all subgraphs H appearing in the algorithm applied to G
The complexity of the algorithm

$$T(G) = O^*(\tau_0^{\mu(G)})$$

Branching with reduction



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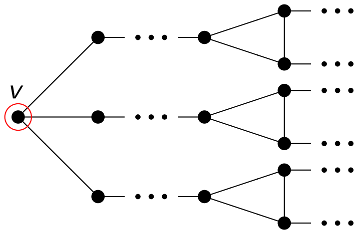
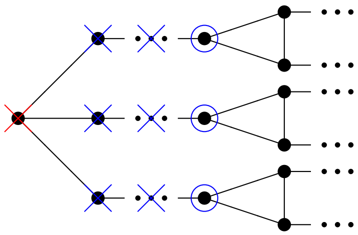


v not in *S*



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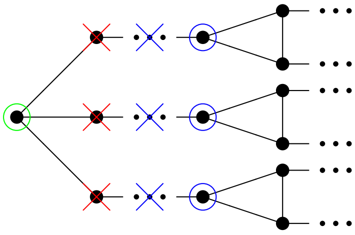
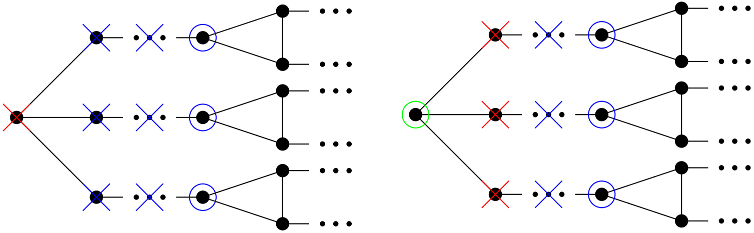


Graphs with $\Delta(G) = 3$



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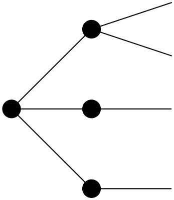
$$1 = \frac{1}{\tau^4} + \frac{1}{\tau^4}$$

$$\tau_0 = 1.18..$$

$$T(G) = t(n_3(G)) = O^*(\tau_0^{n_3(G)}) = O^*(1.18..^{n_3(G)}) = O^*(1.18..^{n(G)})$$

Density

$\frac{2m}{n} > 2.4$



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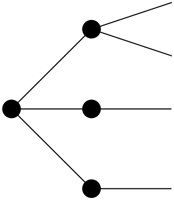
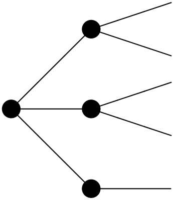


$\frac{2m}{n} > 2.5$



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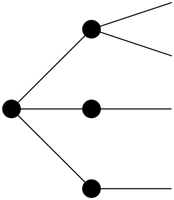
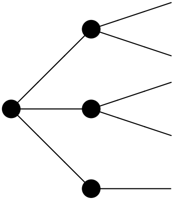
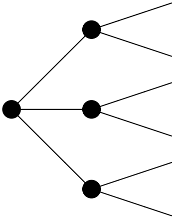


$\frac{2m}{n} > 2.5$



$\frac{2m}{n} > 2\frac{2}{3}$



Density

$$\mu(G) = \begin{cases} -2n(G) + 2m(G) & \text{for } \frac{2m}{n} \in (2; 2.4] \\ -1.40..n(G) + 1.50..m(G) & \text{for } \frac{2m}{n} \in (2.4; 2.5] \\ \dots & \\ a_i n(G) + b_i m(G) & \text{for } \frac{2m}{n} \in (k_i; k_{i+1}] \end{cases}$$

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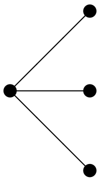
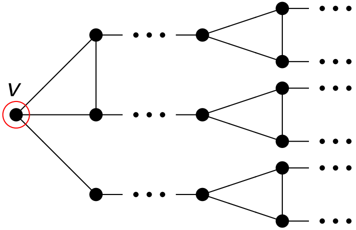
$$T(G) = O^*(\tau_0^{\mu(G)}) = O^*(1.18..^{\mu(G)}) = O^*(1.15..^{n(G)})$$

Claw-free graphs

A graph is claw-free if it does not contain $K_{1,3}$ as induced subgraph.



Claw-free graphs with $\Delta(G) = 3$



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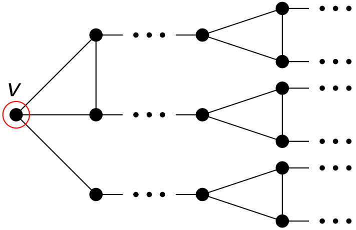

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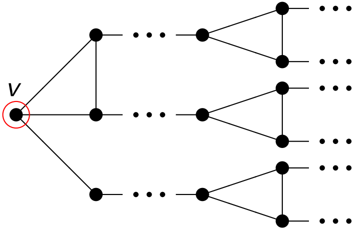

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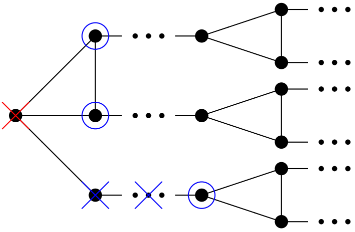

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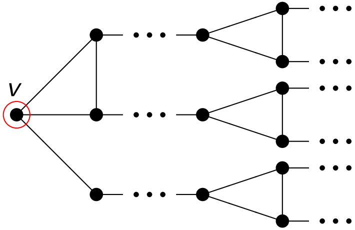

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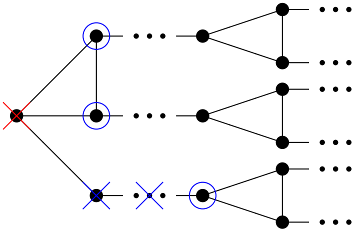

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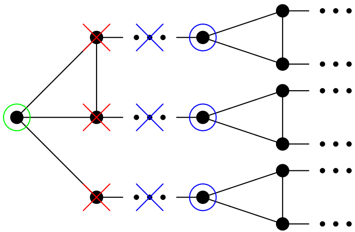

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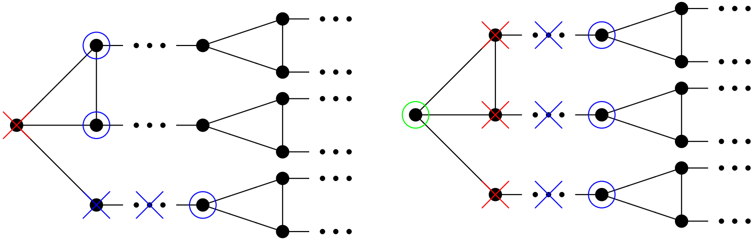

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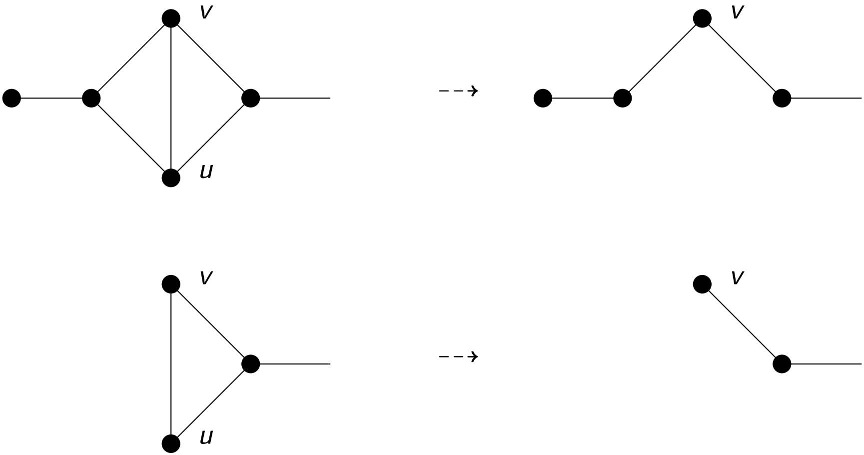
$$1 = \frac{1}{\tau^4} + \frac{1}{\tau^6}$$

$$\tau_0 = 1.15..$$

$$T(G) = t(n_3(G)) = O^*(\tau_0^{n_3(G)}) = O^*(1.15..^{n_3(G)}) = O^*(1.15..^{n(G)})$$

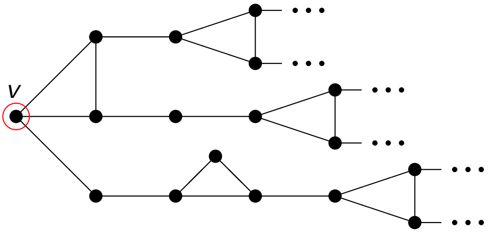
measure with density $T(G) = O^*(1.14..^{n(G)})$

Reduction 2

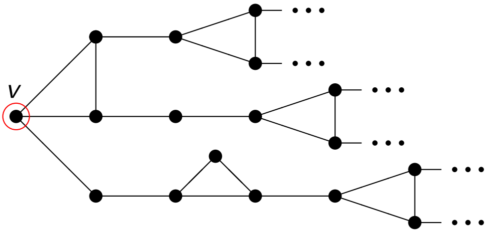


$$c_0(v) := c_0(v)c_0(u)$$
$$c_1(v) := c_0(v)c_1(u) + c_1(v)c_0(u)$$

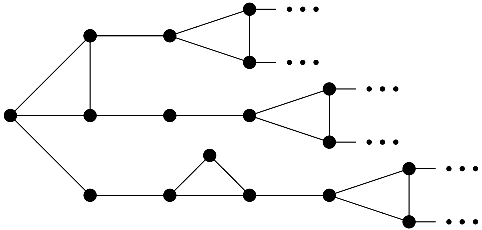
Branching with reduction 1 & 2



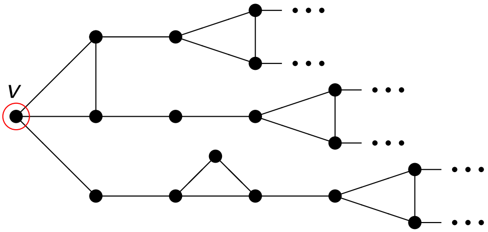
Branching with reduction 1 & 2



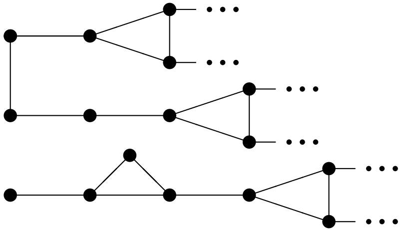
v not in S



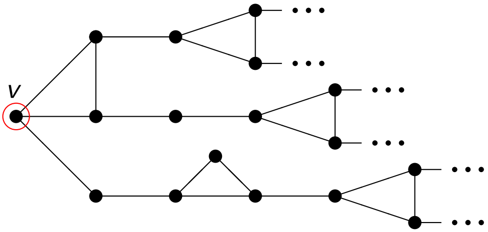
Branching with reduction 1 & 2



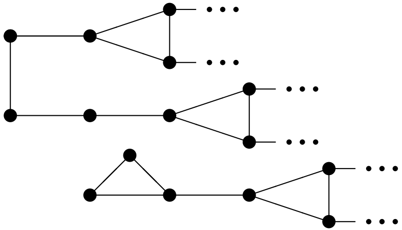
v not in S



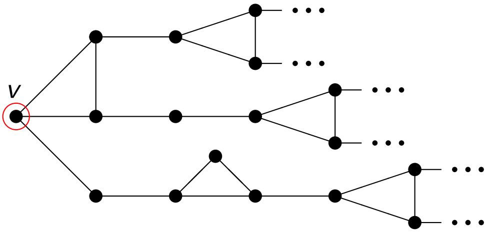
Branching with reduction 1 & 2



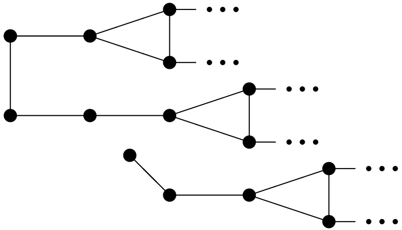
v not in S



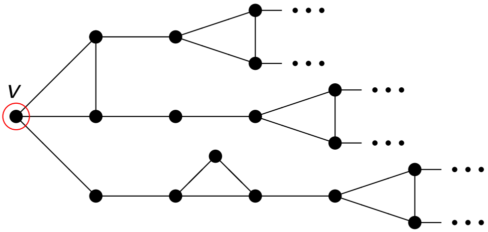
Branching with reduction 1 & 2



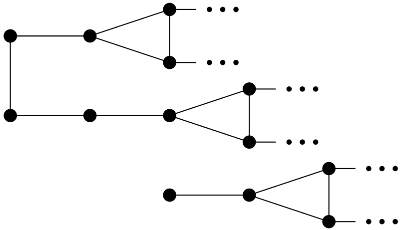
v not in S



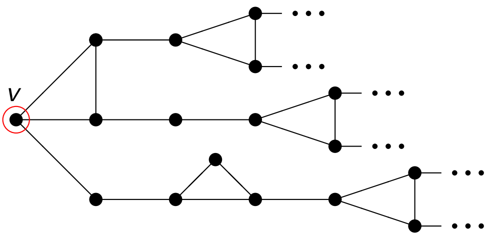
Branching with reduction 1 & 2



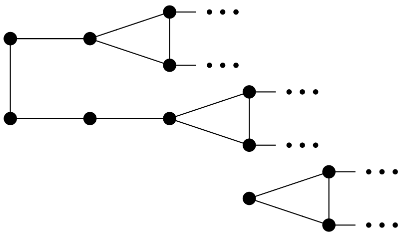
v not in S



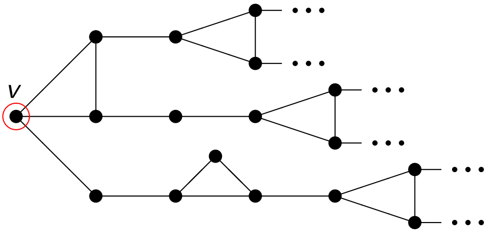
Branching with reduction 1 & 2



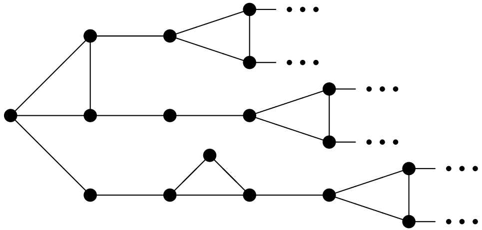
v not in S



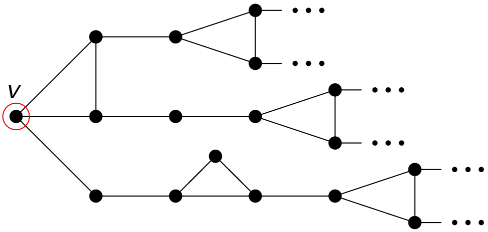
Branching with reduction 1 & 2



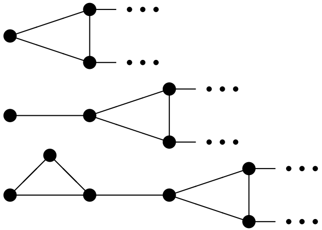
v in S



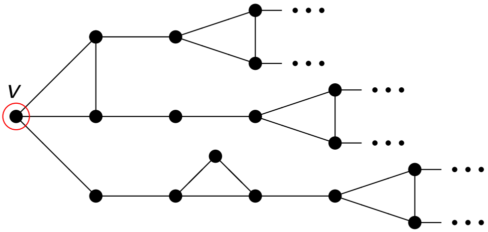
Branching with reduction 1 & 2



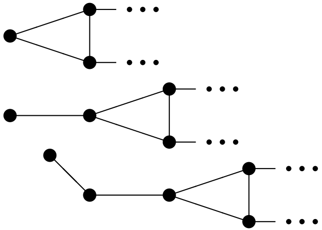
v in S



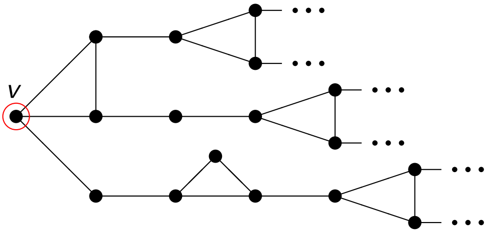
Branching with reduction 1 & 2



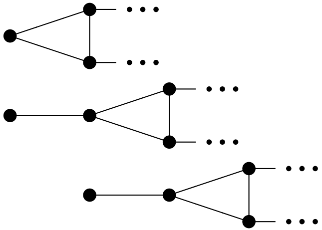
v in S



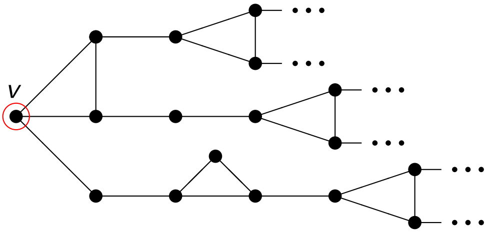
Branching with reduction 1 & 2



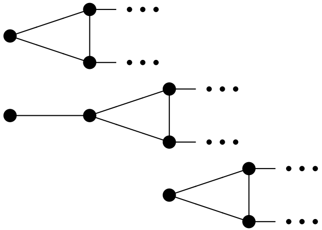
v in S



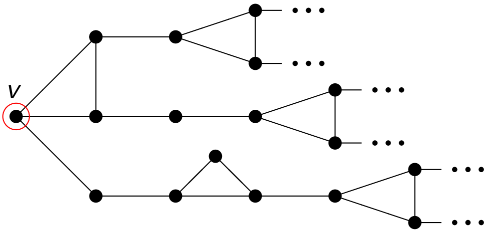
Branching with reduction 1 & 2



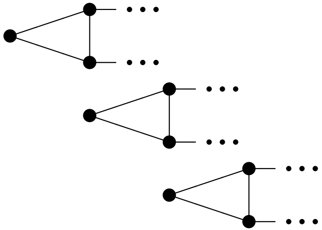
v in S



Branching with reduction 1 & 2



v in S



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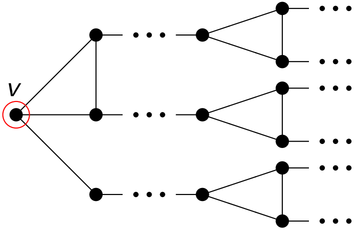
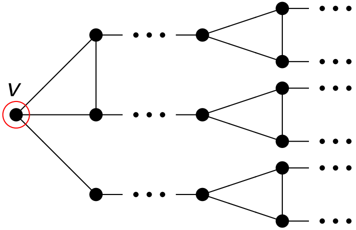


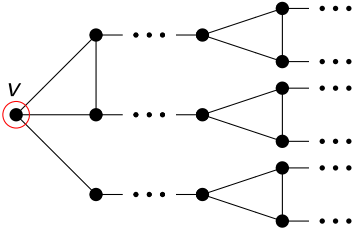
Branching with reduction 1 & 2



v not in S

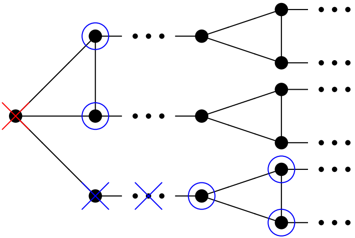
v in S

Branching with reduction 1 & 2



v not in S

v in S



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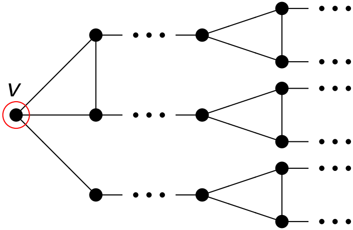
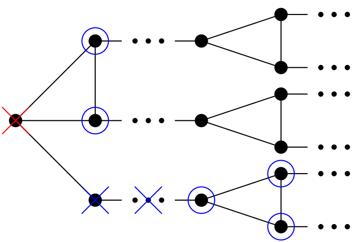


v not in S



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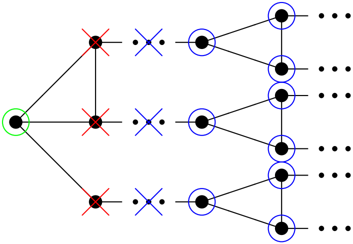
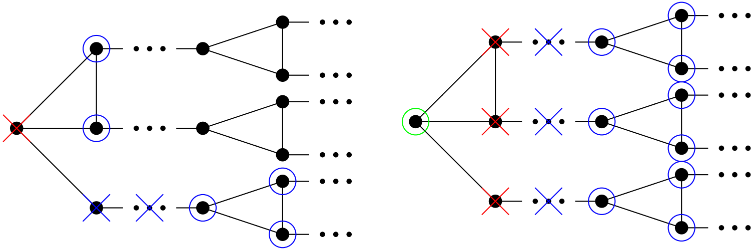
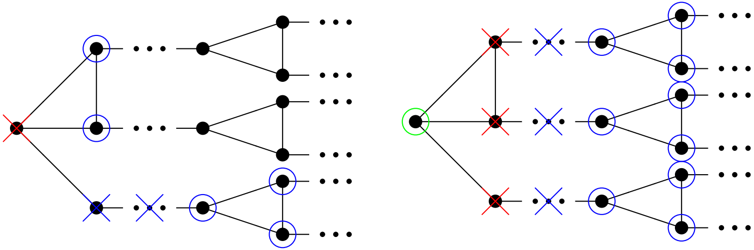


Claw-free graphs with $\Delta(G) = 3$ - New measure



$tr_3(G)$ - number of triangles in G with all vertices of degree 3

$\mu(G) = 3tr_3(G)$



$tr_3(G)$ - number of triangles in G with all vertices of degree 3

$\mu(G) = 3tr_3(G)$

$T(\mu(G)) = T(\mu(G) - 6) + T(\mu(G) - 12)$

Claw-free graphs with $\Delta(G) = 3$ - New measure

$tr_3(G)$ - number of triangles in G with all vertices of degree 3

$\mu(G) = 3tr_3(G)$

$T(\mu(G)) = T(\mu(G) - 6) + T(\mu(G) - 12)$

$1 = \frac{1}{\tau^6} + \frac{1}{\tau^{12}}$

Claw-free graphs with $\Delta(G) = 3$ - New measure

$tr_3(G)$ - number of triangles in G with all vertices of degree 3

$\mu(G) = 3tr_3(G)$

$T(\mu(G)) = T(\mu(G) - 6) + T(\mu(G) - 12)$

$1 = \frac{1}{\tau^6} + \frac{1}{\tau^{12}}$

$\tau_0 = 1.08..$

Claw-free graphs with $\Delta(G) = 3$ - New measure

$tr_3(G)$ - number of triangles in G with all vertices of degree 3

$\mu(G) = 3tr_3(G)$

$T(\mu(G)) = T(\mu(G) - 6) + T(\mu(G) - 12)$

$1 = \frac{1}{\tau^6} + \frac{1}{\tau^{12}}$

$\tau_0 = 1.08..$

$T(G) = O^*(1.08..^{\mu(G)}) = O^*(1.08..^{n_3(G)}) = O^*(1.08..^{n(G)})$

Graphs with arbitrary $\Delta(G)$

$$\mu(G) = \begin{cases} a_1 n(G) + b_1 m(G) & \text{for } \frac{2m}{n} \in (k_1, k_2] \\ a_2 n(G) + b_2 m(G) & \text{for } \frac{2m}{n} \in (k_2, k_3] \\ \dots \end{cases}$$

New algorithm - claw-free graphs - $O^*(1.2354^n)$

Graphs with arbitrary $\Delta(G)$

$$\mu(G) = \begin{cases} a_1 n(G) + b_1 m(G) & \text{for } \frac{2m}{n} \in (k_1, k_2] \\ a_2 n(G) + b_2 m(G) & \text{for } \frac{2m}{n} \in (k_2, k_3] \\ \dots \end{cases}$$

New algorithm - claw-free graphs - $O^*(1.2354^n)$

THANK YOU