On computing an optimal semi-matching

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joint work with

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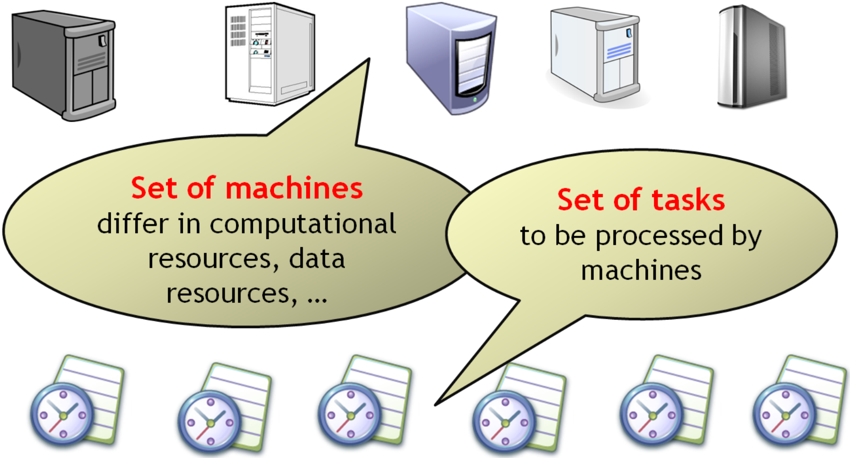
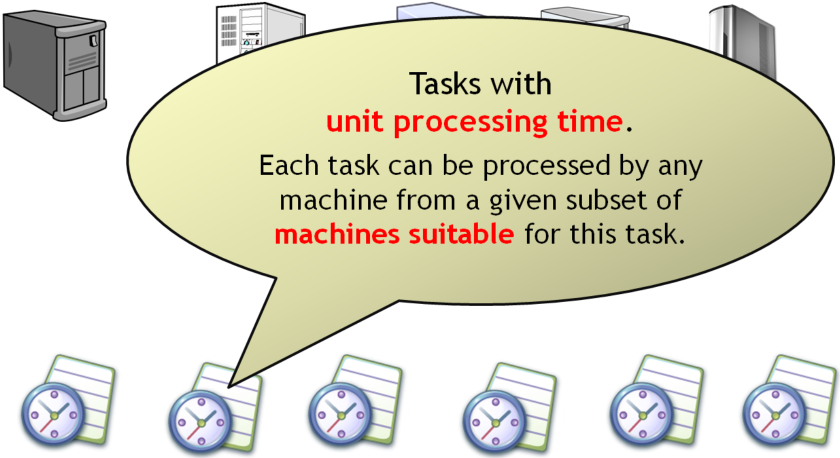
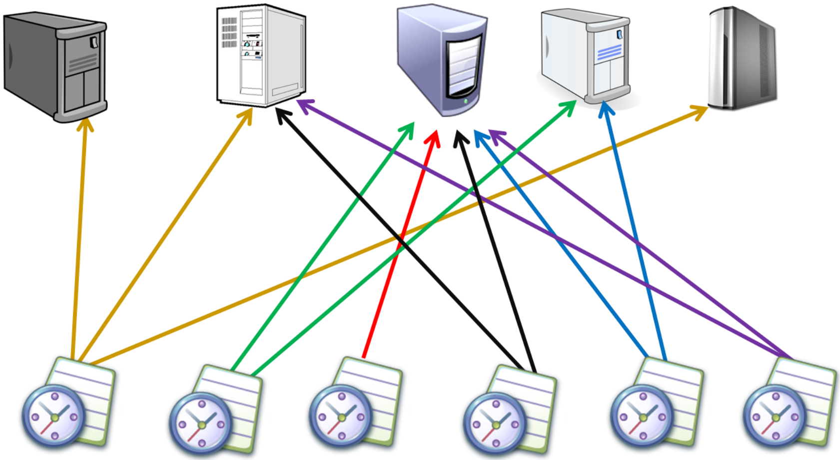
Motivation


Set of machines
differ in computational resources, data resources, ...
Set of tasks
to be processed by machines

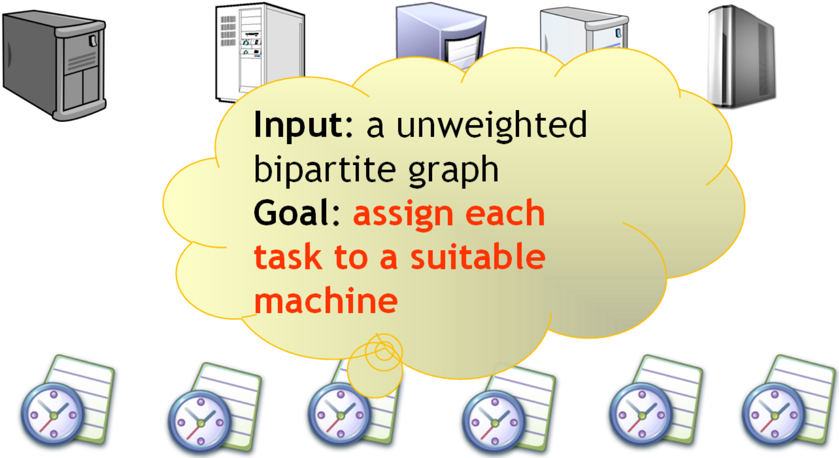
Motivation


Tasks with
unit processing time.
Each task can be processed by any machine from a given subset of **machines suitable** for this task.

Motivation

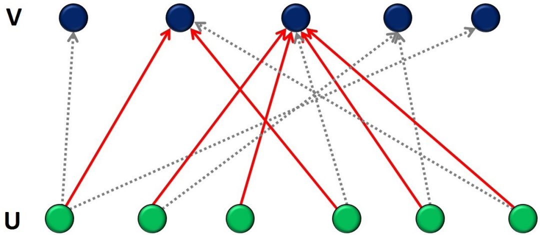


Motivation


Input: a unweighted bipartite graph
Goal: assign each task to a suitable machine

Semi-matching in a bipartite graph $G = (U, V, E)$:

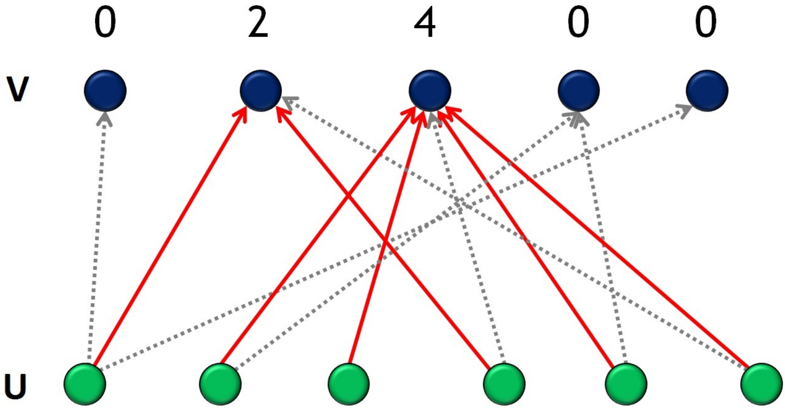
- any subset $M \subseteq E$ such that $deg_M(u) \leq 1$ for all $u \in U$
- *each task is assigned to at most one machine*



Maximum semi-matching - maximizes the number of assigned tasks; if there is *no other restriction* then

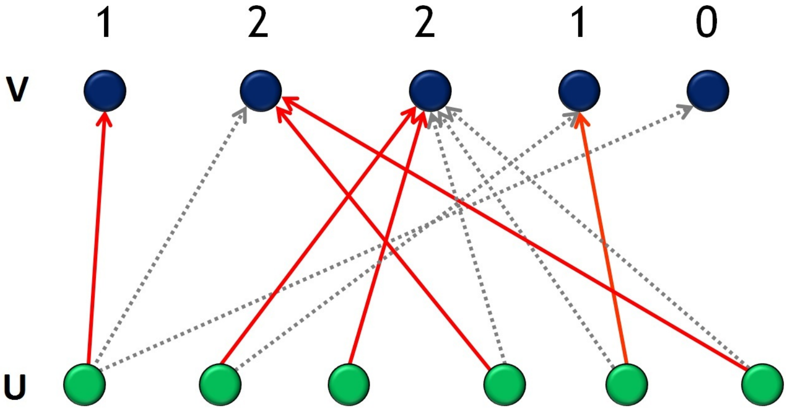
- any subset $M \subseteq E$ such that $deg_M(u) = 1$ for all $u \in U$
- always exists, many maximum semi-matchings

Which semi-matching is better?



Workload distribution (sorted loads): **4, 2, 0, 0, 0**

Which semi-matching is better?



Workload distribution (sorted loads): **2, 2, 1, 1, 0**

Optimal semi-matchings

Cost of a semi-matching *M* (the total completition time):

$$cost(M) = \sum_{v \in V} \frac{deg_M(v).(deg_M(v) + 1)}{2}$$

Optimal semi-matching

- a maximum semi-matching *M* such that $cost(M)$ is minimal
- a maximum semi-matching *M* such that its degree (workload) distribution is lexicographically minimal
  - shown by Bokal et al. to be equivalent with *cost*-minimal semi-matching (and also other cost measures)
  - in the previous example: $(4, 2, 0, 0, 0)$ vs. $(2, 2, 1, 1, 0)$

Our optimality criterion: **lexicographical minimality**

Previous work

Algorithms for computing **an optimal semi-matchings**:

- $O(n^3)$ by Horn (1973) and Bruno et al. (1974)
- $O(n \cdot m)$ by Lovász et al. (2006, JAlgor)
- $O(\min\{n^{3/2}, m \cdot n\} \cdot m)$ by Lovász et al. (2006, JAlgor)
- $O(n \cdot m)$ by Bokal et al. (2009) for generalized setting
- $O(\sqrt{n} \cdot m \cdot \log n)$ by Fakcharoenphol et al. (2010, ICALP)

Algorithms are based on finding (cost-reducing) alternating paths with some properties.

Maximum matchings in bipartite graphs:

- $O(\sqrt{n} \cdot m)$ by Micali and Vazirani (1980)
- $O(n^\omega)$ by Mucha and Sankowski (2004)
  - ω is the exponent of the best known matrix multiplication algorithm
  - randomized algorithm, better for dense graphs

Our work

Can we construct an algorithm for computing an optimal semi-matching that breaks through $O(n^{2.5})$ barrier for dense graphs?

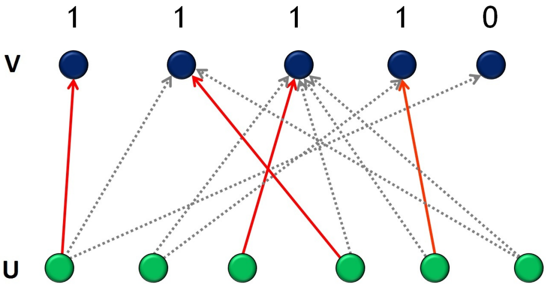
Answer: YES, we can

And moreover (side results):

- **new approach** for computing an optimal semi-matching: divide and conquer strategy instead of cost-reducing alternating paths
  - divide and conquer = more suitable for parallel computation
- **reduction** to a variant of *maximum bounded-degree semi-matching*
  - can be solved by different algorithms and approaches (e.g. maximum matchings, reduction to matrix multiplication)

Limited workload for V-vertices

Restriction: a machine can process only limited number of tasks, e.g. 1 task:

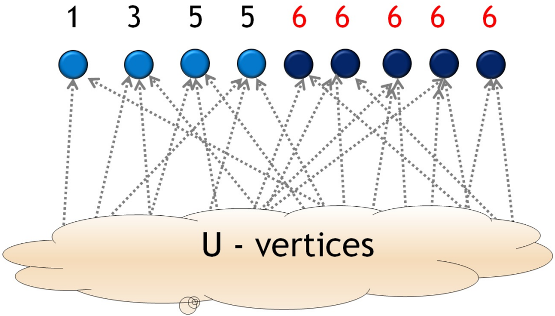


Intuition:

- there can be unassigned tasks
  - U-vertices not incident to a matching edge
- larger workload limit for machines = more assigned tasks

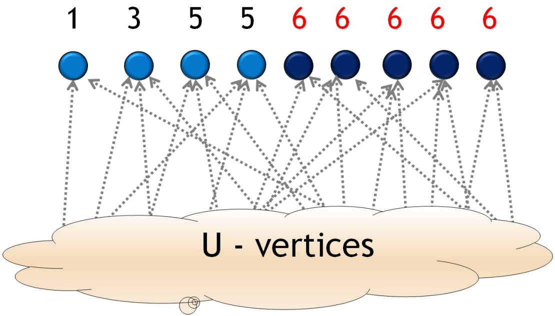
Limited workload for V-vertices

Maximum semi-matching with workload limit 6
(max. 6 tasks per machine):



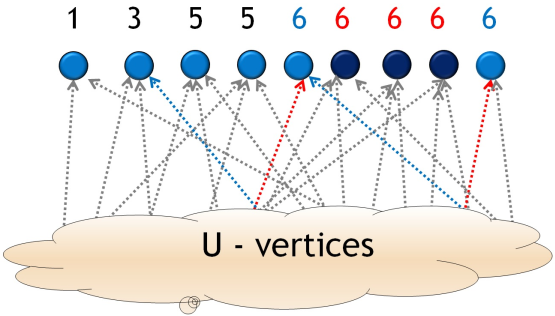
Is it necessary to increase workload limit for all V-vertices (machines) in order to match all U-vertices?

Intuition related to limited workload



- **no sense** to increase the workload limit for vertices (machines) that are not fully loaded in a given maximum semi-matching

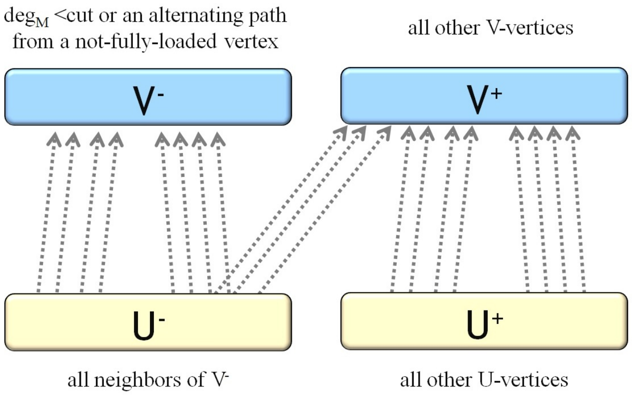
Are all fully-loaded vertices good candidates?



- **no sense** to increase the workload limit for fully loaded vertices (machines) that are endpoints of an alternating path starting in a non-fully loaded vertex

Intuition: How to divide the problem

Maximum semi-matching M respecting a workload limit *cut*:



Find an optimal semi-matching

- in $G^- = (U^-, V^-, E^-)$ by "decreasing" workload limits
- in $G^+ = (U^+, V^+, E^+)$ by "increasing" workload limits

(Sub)problem instances

$LSM(G)$ - a set of all optimal semi-matchings for G

Input/problem instances: $(G, down, up, M_f)$

- an input bipartite graph $G = (U, V, E)$ such that
  - $\forall M \in LSM(G), \forall v \in V : down \leq deg_M(v) \leq up$
- a semi-matching M_f in G such that
  - $\forall v \in V : deg_{M_f}(v) \geq down$

Goal: if $(G, down, up, M_f)$ is an input, compute an optimal semi-matching for G

Starting point: $(G, 0, \infty, \emptyset)$

- G is a graph, in which we want to find an optimal semi-matching
- all preconditions are satisfied

(Sub)problem instances

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Divide phase for cut ($down \leq cut \leq up$):

$$\begin{array}{ccc} & (G, down, up, M_f) & \\ \swarrow & & \searrow \\ (G^-, down, cut, M_f^-) & & (G^+, cut, up, M_f^+) \end{array}$$

Key property:

- $\forall M^- \in LSM(G^-), \forall M^+ \in LSM(G^+)$: $M^- \cup M^+ \in LSM(G)$

Trivial case (or why is M_f required)

Input: $(G, down, up, M_f)$, where $up - down \leq 1$
Problem: How to compute $M \in LSM(G)$?

First idea:

- compute a maximum semi-matching M for load limit up
- it can happen that $M \notin LSM(G)$:
  - $(3, 2, 2, 2, 2, 2) \in LSM(G)$ vs. $(3, 3, 3, 3, 1, 0) \notin LSM(G)$

Solution:

- utilizing M_f with $deg_{M_f}(v) \geq down$ for all $v \in V$, **transform** semi-matching M to a semi-matching M_B such that
  - $|M| = |M_B|$
  - $down \leq deg_{M_B}(v) \leq up$ for all $v \in V$
- it can be shown that $M_B \in LSM(G)$
- transformation can be realized in the linear time

Dividing subroutine - idea

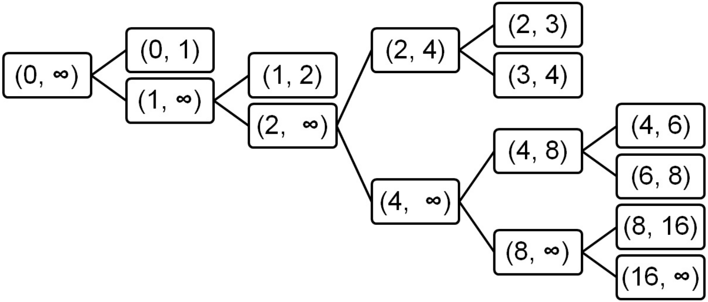
Input instance: $(G, down, up, M_f)$

Computation:

1. compute a maximum semi-matching M for workload limit cut
2. compute M_B by rebalancing M with respect to M_f
3. compute V^-, V^+, U^-, and U^+ considering workload of V-vertices
4. compute induced subgraphs $G^- = (U^-, V^-, E^-)$ and $G^+ = (U^-, V^+, E^+)$
5. compute $M_f^- = M_B \cap E^-$ and $M_f^+ = M_B \cap E^+$
6. return $(G^-, down, cut, M_f^-)$ and (G^+, cut, up, M_f^+)

Main algorithm - Divide and conquer

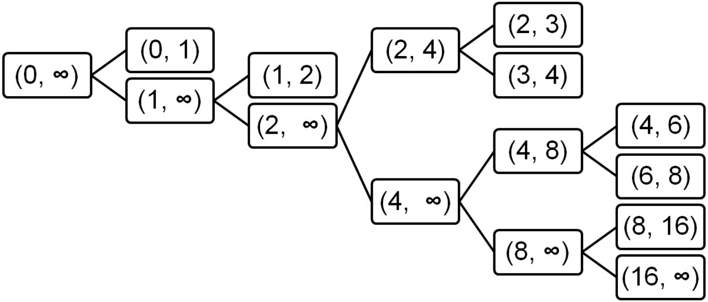
Computational tree starting with $(G, 0, \infty, \emptyset)$:



- Divide and conquer: $(down, up)$ is always divided into 2 subintervals (of almost equal size)
- Doubling: $(down, \infty)$ is divided to $(down, 2 \cdot down)$ and $(2 \cdot down, \infty)$

Main algorithm - Computation

Computational tree starting with $(G, 0, \infty, \emptyset)$:



- after $O(\log n)$ levels, graphs of subproblems are empty
  - there is no subgraph of G for which a semi-matching with load of a V-vertex at least $n + 1$ exists

Maximum semi-matching with workload limits?

- in each step of the algorithm, we need a maximum semi-matching that respects the workload limits

Problem (Bounded-degree semi-matching)

Instance: *A bipartite graph* $G = (U, V, E)$ *with* $n = |U| + |V|$ *vertices and* $m = |E|$ *edges; a capacity mapping* $c : V \to \mathbb{N}$ *satisfying* $\sum_{v \in V} c(v) \leq 2 \cdot n$.

Question: *Find a semi-matching* M *in* G *with maximum number of edges such that* $deg_M(v) \leq c(v)$ *for all* $v \in V$.

Time complexity notation: $T_{BDSM}(n, m)$ *for a graph n vertices and m edges.*

Total time for computing an optimal semi-matching:

$$O((n + m + T_{BDSM}(n, m)) \cdot \log n)$$

Bounded-degree semi-matching

Reduction to maximum matching:

- make $c(v)$ **copies** of each V-vertex v
- new graph has at most $3 \cdot n$ vertices
- apply algorithm for maximum matching in $O(n^{\omega})$ by Mucha and Sankowski

$$O(n^{\omega} \cdot \log n)$$

Reduction to $(1, c)$**-semi-matchings**:

- $(1, c)$-semi-matching is bounded-degree semi-matching **without** condition $\sum_{v \in V} c(v) \leq 2 \cdot n$
- due to algorithm by Katrenič and Semanišin, $(1, c)$-semi-matching can be computed in time $O(\sqrt{n} \cdot m)$

$$O(\sqrt{n} \cdot m \cdot \log n)$$

Conclusion

- algorithm for computing an optimal semi-matching in time $O(n^\omega)$ with high probability
  - since $\omega \leq 2.38$, this algorithms breaks through $O(n^{2.5})$ barrier for **dense graphs**
- **new algorithm** for computing an optimal semi-matching based on **divide and conquer strategy** and working in time $O(\sqrt{n} \cdot m \cdot \log n)$
  - divide and conquer strategy promises **efficient parallelization**

Thank you for your attention