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Parameterized Two-Player Nash Equilibrium

Danny Hermelin, Chien-Chung Huang,
Stefan Kratsch, and Magnus Wahlström

Bimatrix Game

- Played by two players: Row and Column
  - Two payoff matrices. A,B $\in$ $Q^{n\times n}$.

0	1	-2
0	2	2
1	2	-1

Row chooses i

0	2	0
0	-2	2
1	1	1

Column chooses j

Row payoff A[i,j] = -2

Column payoff B[i,j] = 0

Bimatrix Game

- Example:
  - Rock, paper, scissors:

	0	-1	1
	1	0	-1
	-1	1	0

	0	1	-1
	-1	0	1
	1	-1	0

- This example is a zero-sum game:
  - Row and column payoffs sum up to zero.
- General bimatrix games are not necessarily such.
  - In fact, the interesting cases (to us) are not zero-sum.

Bimatrix Game

- Players can play mixed strategies.
  - Distribution over rows and columns.

x

x			
1/2	0	1	-2
1/2	0	2	2
0	1	2	-1

y

0	0	1
0	2	0
0	-2	2
1	1	1

Row chooses distribution x

Column chooses distribution y

Row expected payoff

$x^T A y = 0$

Column expected payoff

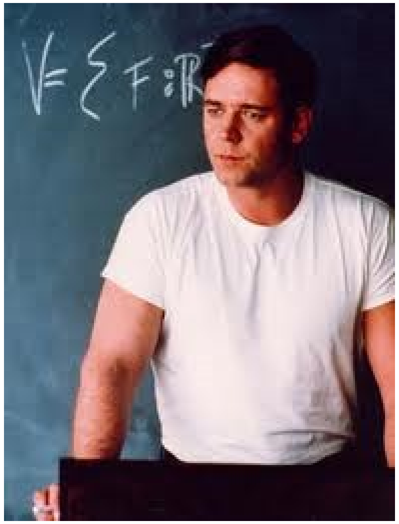
$x^T B y = 1$

Nash Equilibrium

- Neither player can improve their payoff, assuming the other player plays the same.

0	1	-2
0	2	2
1	2	-1

0	2	0
0	-2	2
1	1	1



Not Nash !

Row can improve by switching to row 2.

Nash Equilibrium

- Neither player can improve their payoff, assuming the other player plays the same.

0	1	-2
0	2	2
1	2	-1

0	2	0
0	-2	2
1	1	1

Nash !

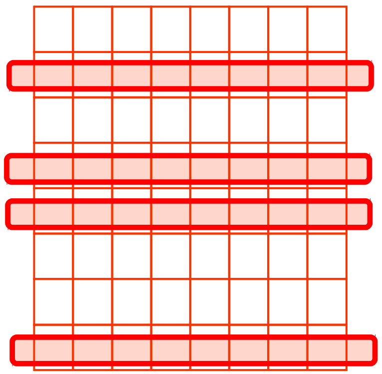
<u>Theorem (Nash)</u>: Any bimatrix <u>rational</u> game has a <u>mixed</u> equilibrium.

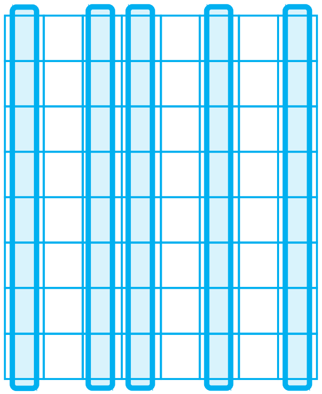
Computing Nash Equilibrium

- **<u>The Nash Equilibrium (NE) problem:</u>** Given a bimatrix rational game, find an equilibrium.
- NP-completeness theory does not apply because solution always exists.
- PPAD-complete by a series of papers:
  - Daskalakis, Goldberg, and Papadimitriou [STOC'06,STOC'06].
  - Daskalakis and Papadimitriou [ECCC'05]
  - Chen and Deng [ECCC'05]
  - Chen and Deng [FOCS'06]
- The 3-SAT of algorithmic game theory !

Computing Nash Equilibrium

- **<u>Support:</u>** Set of strategies played with non-zero probability.
- When support of both players is known, NE is easy.





- Solve LP with the following constraints:
  - $x_s > 0 \Rightarrow (Ay)_s \geq (Ay)_j$ for all $j \neq s$.
  - $y_s > 0 \Rightarrow (x^T B)_s \geq (x^T B)_j$ for all $j \neq s$

Computing Nash Equilibrium

<u>Theorem</u>: NE can be solved in $n^{O(k)}$ time, when the supports of each player are bounded by k.

- Can this be improved substantially?
- Can we remove k out of the exponent?

<u>Theorem (Estivill-Castro, Parsa)</u>: NE cannot be solved in $n^{o(k)}$ time unless FPT=W[1].

<u>GOAL:</u> find interesting special cases that circumvent this

Graph Representation of Bimatrix Games

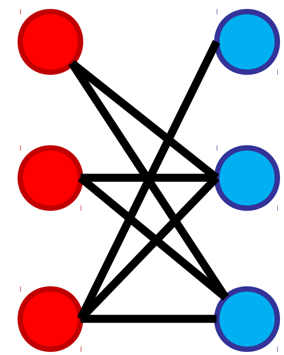
- Bipartite graph on rows and columns

0	1	-2
0	2	2
1	2	-1

+

0	2	0
0	-2	2
1	1	1

$\Rightarrow$

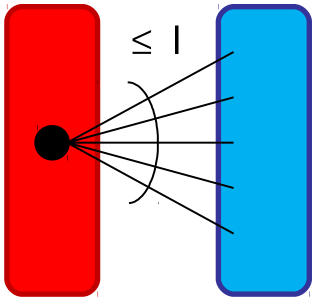


(i,j) is an edge $\Leftrightarrow$ $A[i,j] \neq 0$ or $B[i,j] \neq 0$

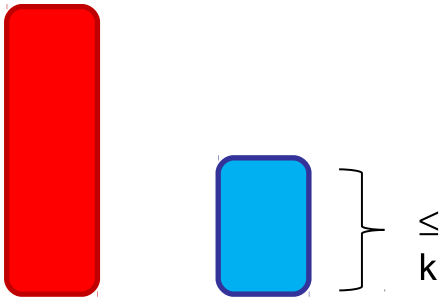
Interesting Special Cases

1. l-sparse games:
   – Degrees ≤ l.
1. k-unbalanced games:
   – One side has ≤ k vertices.
1. Locally bounded treewidth:
   – Every d-neighborhood has treewidth ≤ f(d).
   – Generalizes both previous cases.

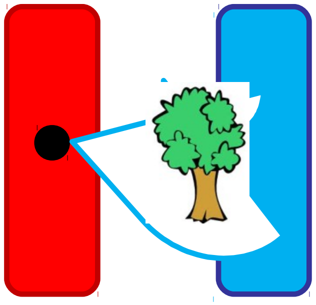
previously studied games



(1)



(2)



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Our Results

<u>Theorem</u>: NE in l-sparse games, where the support is bounded by k, can be solved in $l^{O(kl)}\, n^{O(1)}$ time.

- Without the restriction on the support size the problem is PPAD-complete [Chen, Deng, and Teng ‘06].

<u>Theorem</u>: NE in k-unbalanced games, where the row player’s payoff matrix has l different values, can be solved in $l^{O(k^2)}\, n^{O(1)}$ time.

- General k-sparse games is not known to be FPT.
- But how do we show its not ?

<u>Theorem</u>: NE in locally bounded treewidth games, where the support is bounded by k, and <u>both</u> payoff matrices have l different values, can be solved in $f(l, k)\, n^{O(1)}$ time for some computable f().

l-Sparse Games

- Recall l := max-degree and k:= support size.
- Two easy observations:
    1. Enough to search for minimal equilibriums.

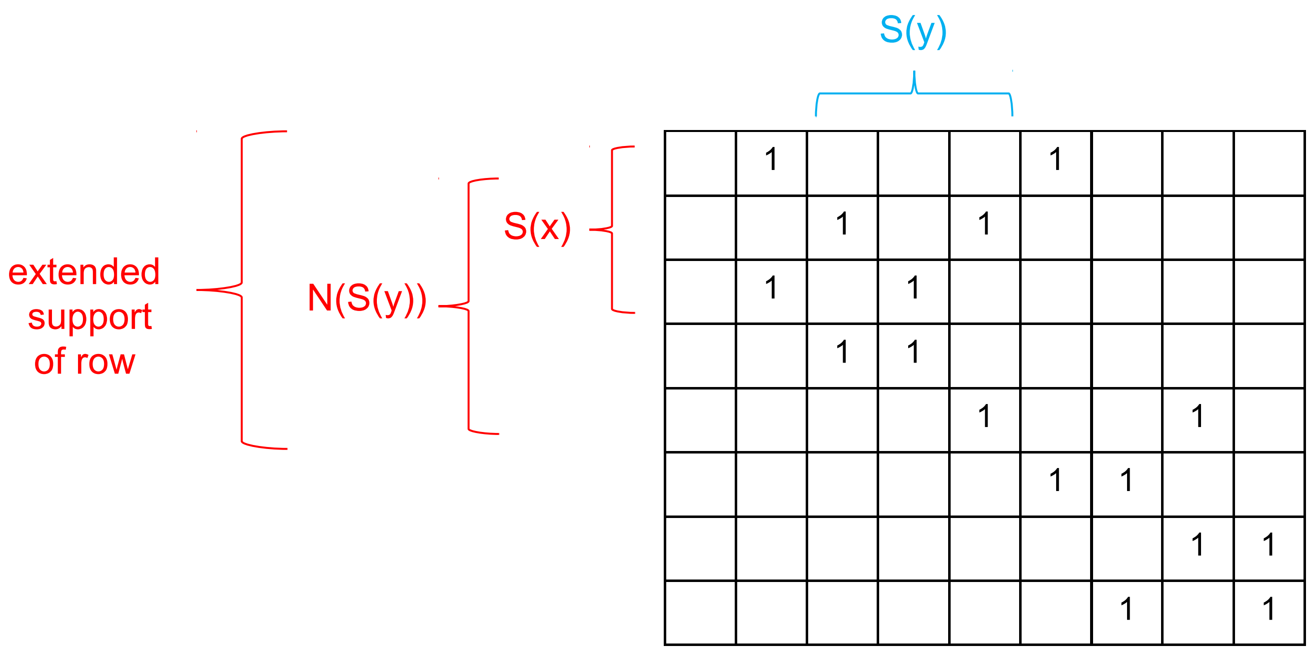
Definition: An equilibrium (x,y) is ***minimal*** if for any equilibrium (x',y') with $S(x') \subseteq S(x)$ and $S(y') \subseteq S(y)$, we have S(x') = S(x) and S(y') = S(y).

2. If n > kl , then both players receive non-negative payoffs on any $k \times k$ equilibrium.

If a player get negative payoff and n > kl , there will always be a zero-payoff strategy to switch to.

l-Sparse Games

Definition: The *extended support* of (x,y) is S(x) ∪ N(S(y)) for the row player, and S(y) ∪ N(S(x)) for the column player.



The size of the extended support of each player ≤ k + kl.

l-Sparse Games

- Main technical lemma:

Lemma: If (x,y) is minimal equilibrium, then the subgraph $H \subseteq G$ induced by the extended supports has at most 2 connected components.

Proof sketch:

1. Prove separately for the case where $A_{s(x),s(y)}$ = **0** and $B_{s(x),s(y)}$ = **0**, and for the case when one of these matrices is not all-zero.

2. In the latter case, normalize probabilities on some connected component of H.

3. In the former case, argue the same on G[N(S(x))] and G[N(S(y))].

I-Sparse Games

- Folklore FPT lemma:

Lemma: Let G be a graph on n vertices of maximum degree Δ. Then one can enumerate all induced subgraphs H on h vertices and c connected components in $H \subseteq G$ in $\Delta^{O(h)} n^{O(c)}$ time.

Proof sketch:

1. Guess c vertices S in G to be the targets of vertices in different connected components of H.

2. Branch on the h-neighborhood of S to enumerate all $H \subseteq G$.

3. The size of each branch-tree is $\Delta^{O(h)}$.

I-Sparse Games

- The algorithm:

1.Guess the number h of strategies in both extended support.

2.Guess the number of connected components c $\in \{1,2\}$ in the corresponding induced subgraph.

3.Enumerate all induced subgraphs on h vertices and c connected components.

4.For each such subgraph, the supports of both players are known. Thus, one can use LP to determine if it corresponds to an equilbrium.

l-Sparse Games

- Extensions:

1. We can improve running-time to $l^{O(kl)} n^{O(1)}$ in case both payoff matrices are non-negative.

2. Another route to a well-known PTAS.

3. Connectivity lemma can be used to show that the problem has no “polynomial kernel”.

Open questions

1. k-unbalanced games with an arbitrary number of payoffs.
2. Bounded treewidth games with an arbitrary number of payoffs.
3. Parameterized analog of the PPAD class.

<u>Conjecture:</u> NE parameterized by k in k-unbalanced games is Para-PPAD-Complete.