Approximability of Economic Equilibrium in Housing Markets with Duplicate Houses

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and

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Basic notions

Definition

A housing market is a quadruple $\mathcal{M} = (A, H, \omega, \mathcal{P})$ where

- *A is a set of n agents, H is a set of m house types*
- *$\omega : A \to H$ is the endowment function*
- *preference profile $\mathcal{P}$ is an n-tuple of agents' preferences, i.e. linearly ordered lists $P(a)$ of acceptable house types*

Example.

$A = \{a_1, a_2, \ldots, a_7\}$;
$H = \{h_1, h_2, h_3, h_4\}$

$$
\begin{array}{ll}
\omega(a_1) = h_1; & P(a_1) : h_4, h_3, h_2, h_1 \\
\omega(a_2) = h_4; & P(a_2) : (h_1, h_3), h_4 \\
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$\omega(a_1) = h_1;$	$P(a_1) : h_4, h_3, h_2, h_1$	*acceptable houses*
$\omega(a_2) = h_4;$	$P(a_2) : (h_1, h_3), h_4$	*ties*
$\omega(a_3) = h_1;$	$P(a_3) : h_2, h_4, h_1$	*strict preferences*
$\omega(a_4) = h_2;$	$P(a_4) : (h_1, h_3), h_4, h_2$	*trichotomous preferences*
$\omega(a_5) = h_2;$	$P(a_5) : h_4, h_1, h_2$	
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$\omega(a_2) = h_4;$	$P(a_2) : (h_1, h_3), h_4$	*each agent has:*
$\omega(a_3) = h_1;$	$P(a_3) : (h_2, h_4), h_1$	*1. better house types*
$\omega(a_4) = h_2;$	$P(a_4) : (h_1, h_3, h_4), h_2$	*2. type of his own house*
$\omega(a_5) = h_2;$	$P(a_5) : (h_4, h_1), h_2$	*3. unacceptable houses*
$\omega(a_6) = h_3;$	$P(a_6) : h_4, h_3$	
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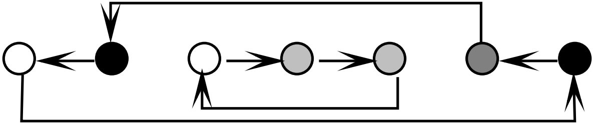
Further notation

Definition

A function $x : A \to H$ is an allocation *if there exists a bijection π on A such that $x(a) = \omega(\pi(a))$ for each $a \in A$.*

- Each allocation consists of trading cycles

$\omega(a_1) = h_1;$	$P(a_1) : h_4, h_3, h_2, h_1$	*take trading cycles*
$\omega(a_2) = h_4;$	$P(a_2) : (h_1, h_3), h_4$	$(a_1, a_7, a_6, a_2)(a_3, a_4, a_5)$
$\omega(a_3) = h_1;$	$P(a_3) : h_2, h_4, h_1$	*this means*
$\omega(a_4) = h_2;$	$P(a_4) : (h_1, h_3), h_4, h_2$	$x(a_1) = h_4;$
$\omega(a_5) = h_2;$	$P(a_5) : h_4, h_1, h_2$	$x(a_7) = h_3;$
$\omega(a_6) = h_3;$	$P(a_6) : h_4, h_3$	$x(a_6) = h_4;$
$\omega(a_7) = h_4;$	$P(a_7) : h_3, h_1, h_4$	$x(a_2) = h_1$ *etc.*



Economic equilibrium

Definition

A pair (p, x), where $p : H \to \mathbb{R}$ is a price function and x is an allocation on A is an economic equilibrium for market $\mathcal{M}$ if for each $a \in A$, house $x(a)$ is of type that is among the most preferred house types in his budget set, i.e.

$$S = B_a(p) = \{h \in H; p(h) \leq p(\omega(a))\}.$$

Lema

If (p, x) is an economic equilibrium for market $\mathcal{M}$ then $p(x(a)) = p(\omega(a))$ for each $a \in A$.

Example: equilibrium

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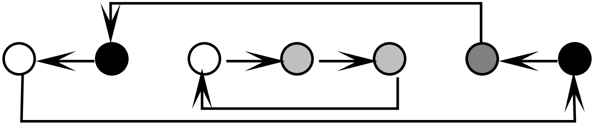
Take $p(h_j) = p$ for all j and $(a_1, a_7, a_6, a_2)(a_3, a_4, a_5)$

Not equilibrium, since $x(a_5) = h_1$ and this is the second-choice house



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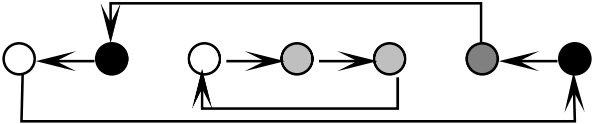

Observation: In this example there is no equilibrium with equal prices, as demand for houses of type h_4 is 3, while the supply is only 2.

Brief history

- Walras 1874: notion of equilibrium
- Arrow, Debreu 1954: notion of exchange economy
- equilibrium exist if commodites are infinitely divisible
- Deng, Papadimitriou, Safra 2002: if commodities are indivisible, decision about the equilibrium existence is NPC
- Shapley, Scarf 1974: housing market
- $m = n$; each house different
- Gale 1974: proof of equilibrium existence by TTC algorithm

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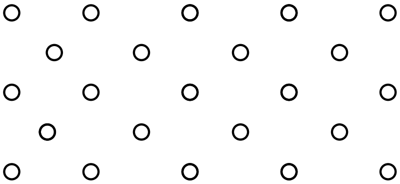

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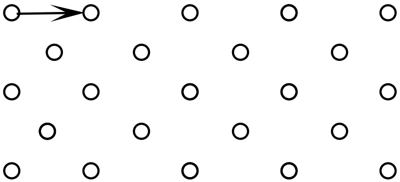

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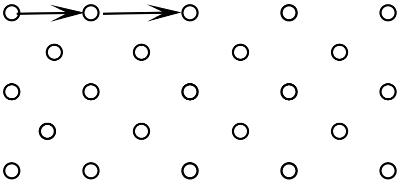

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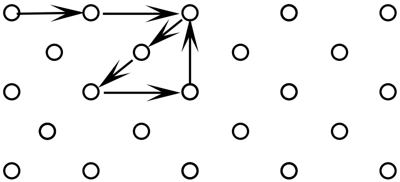

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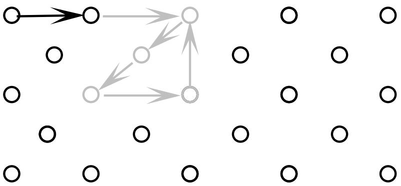

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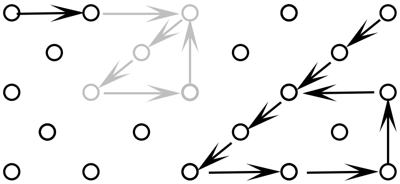

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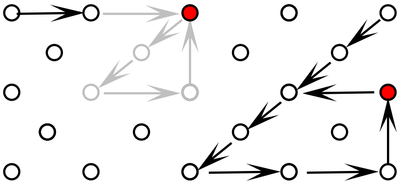

Duplicate houses make the difference

Theorem (Fekete, Skutella and Woeginger 2003)

If the housing market contains duplicate hoses, it is NP-complete to decide whether an economic equilibrium exists.

Theorem (KC & Fleiner 2008)

If preferences over house types are strict, the existence of equilibrium can be decided in polynomial time.

- $O(L)$ implementation – DFS algorithm (KC & Jelínková)

Theorem (Cechlárová & Fleiner 2008)

If preferences are trichotomous, the existence problem remains NP-complete.

Approximate equilibrium

Definition

An agent a is unsatisfied with respect to (p, x) if $x(a)$ is not among the most preferred house types in his budget set according to p; otherwise he is said to be satisfied.

$\mathcal{D}_{\mathcal{M}}(p, x)$ ... the set of unsatisfied agents in $\mathcal{M}$ w.r.t. (p, x)
$\mathcal{S}_{\mathcal{M}}(p, x)$ ... the set of satisfied agents in $\mathcal{M}$ w.r.t. (p, x)

Definition

(p, x) is an α-deficient equilibrium, if $|\mathcal{D}_{\mathcal{M}}(p, x)| = \alpha$. Deficiency $\mathcal{D}(\mathcal{M})$ of a housing market $\mathcal{M}$, is the minimum α such that $\mathcal{M}$ admits an α-deficient equilibrium.

$$opt(\mathcal{M}) = n - \mathcal{D}(\mathcal{M})$$

- housing market $\mathcal{M}$ admits an equilibrium iff $opt(\mathcal{M}) = n$

Easy cases

- *an acyclic market always has an equilibrium*
- *If $m = 2$, then $opt(\mathcal{M}) = \max\{2\min\{n_1, n_2\}, n_1, n_2\}$, where $|A(h_1)| = n_1$, $|A(h_2)| = n_2$.*

- $p_1 = p_2$: trading cycles alternate h_1 and h_2; so $\mathcal{S} = 2min\{n_1, n_2\}$
- $p_1 \neq p_2$: no trading, but all agents with cheaper house are satisfied

For $n_2 = 2n_1$ we have $opt(\mathcal{M}) = 2/3n$.

Theorem (KC & Schlotter 2010)

If preferences are arbitrary and the number m of house types fixed then $\mathcal{D}(\mathcal{M})$ can be computed in $O(m^m\sqrt{n}L)$ time, where L is the total length of preference lists of all agents .

Approximating the number of satisfied agents

Theorem (KC & Jelínková 2011)

If preferences are trichotomous then there is a 2-approximation algorithm for $opt(\mathcal{M})$. Moreover, this guarantee is tight.

- trichotomous market represented by graph $G = (V, E)$ where vertices correspond to agents and $(i, j) \in E$ if agent i accepts house $\omega(j)$
- let C be a maximum cycle packing of G, covering agents A_C
- If $|A_C| \geq n/2$: all houses the same price, cycles of C trading
- If $|A_C| < n/2$: then A_C is a feedback vertes set and submarket generated by $A \setminus A_C$ acyclic.
- Satisfy all agents in $A \setminus A_C$, by setting prices according to a topological ordering in acyclic graph

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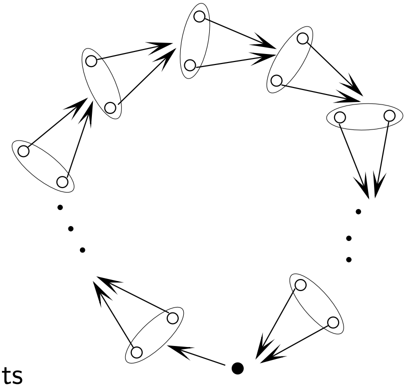

$2q + 1$ agents

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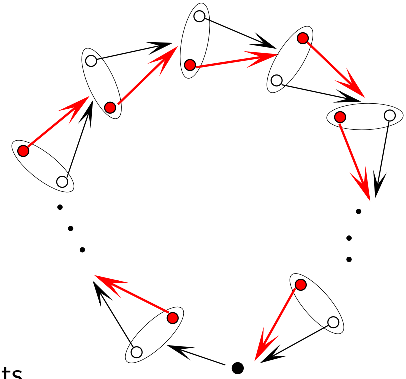

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each cycle packing satisfies $q + 1$ agents

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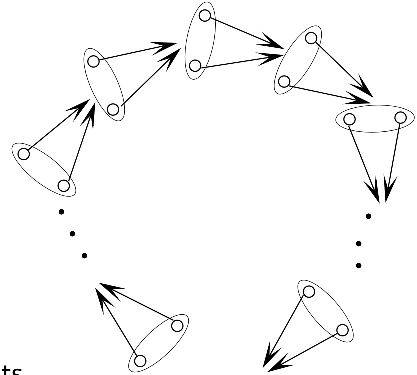
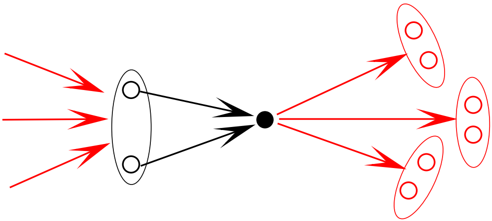


$2q + 1$ agents
each cycle packing satisfies $q + 1$ agents,
but $2q$ agents can be satisfies

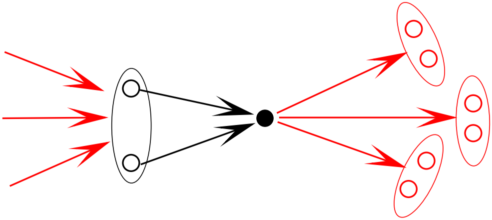
Inapproximability – transformation

- for graph G construct a market $\mathcal{M}$.
- for $v \in G$: 2 in-agents $I_v = \{i_{v,1}, i_{v,2}\}$ and one out-agent o_v
- $\omega(i_{v,1}) = \omega(i_{v,2}) = h_v$; $\omega(o_v) = h_v^*$
- in-agents I_v desire house of out-agent o_v
- agent o_v desires houses h_w such that $\{v, w\} \in E(G)$



Inapproximability – transformation – properties

- constructed market $\mathcal{M}$ is trichotomous, $n = 3|V(G)|$
- F vertex cover in G iff $\{o_v; v \in F\}$ feedback vertex set in $\mathcal{M}$.
- There exists an optimal (p, x) with no trading
- (p, x) optimal with no trading then all in-agents are satisfied



Inapproximability – result

Theorem

The construction yields for each graph G a trichotomous housing market $\mathcal{M}$ with $n = 3|V(G)|$ agents such that $opt(\mathcal{M}) = 3|V(G)| - min\{|W|, W$ vertex cover in $G\}$.

Halldórsson, Iwama, Miyazaki, Yanagisawa, *Improved approximation results for the stable marriage problem*, ACM Trans. Alg., 2007

- construction: to each graph $G = (V, E)$ a stable marriage instance I such that the # men= # women= $3|V(G)|$ and $|opt(I)| = 3|V(G)| - min\{|W|, W$ vertex cover in $G\}$.
- we get by the same computations the following result

Theorem

It is NP-hard to approximate $opt(\mathcal{M})$ for trichotomic markets with an approximation factor smaller than $21/19$

Inapproximability – extensions

Theorem

It is NP-hard to approximate $opt(\mathcal{M})$ for general markets

1. *within a factor smaller than 1.2, and*
2. *within a factor smaller than 1.5, if UGC is true.*

Open problems:

- Better approximation algorithms?
- For general preferences?