BIFURCATIONS OF HALO ORBITS - RIGOROUS NUMERICAL APPROACH

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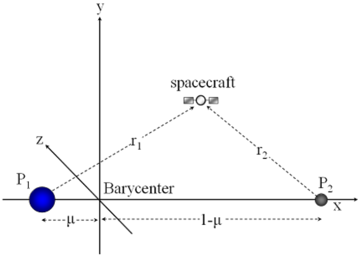
Circular Restricted Three Body Problem



1. the sum of the masses of the two primaries is normalized to one;
2. the distance between the two primaries is normalized to one;
3. the angular velocity of the primaries around their center of mass is normalized to one (the period is equal to 2π);

Some familiar systems are: Earth-Moon(0.012), Sun-Jupiter(0.0009537)

EQUATIONS OF MOTION

$$\begin{cases} \ddot{x} - 2\dot{y} = \frac{\partial\Omega(x,y,z)}{\partial x} \\ \ddot{y} + 2\dot{x} = \frac{\partial\Omega(x,y,z)}{\partial y} \\ \ddot{z} = \frac{\partial\Omega(x,y,z)}{\partial z} \end{cases} \tag{1}$$

where

$$\Omega(x, y, z) = \frac{1}{2}(x^2 + y^2) + \frac{1-\mu}{r_1} + \frac{\mu}{r_2} \tag{2}$$

and

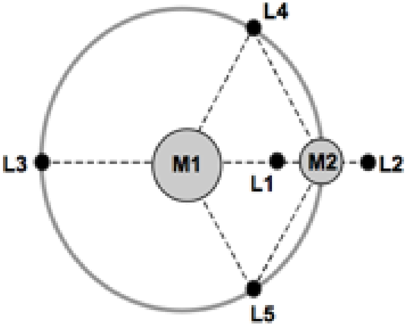
$$\begin{aligned} r_1 &= \sqrt{(x+\mu)^2 + y^2 + z^2}, \\ r_2 &= \sqrt{(x-1+\mu)^2 + y^2 + z^2}. \end{aligned}$$

Rewriting system into six first order diferential equations $\dot{u} = F(u)$, where $u = (x, y, z, \dot{x}, \dot{y}, \dot{z}) \in \mathbb{R}^6$

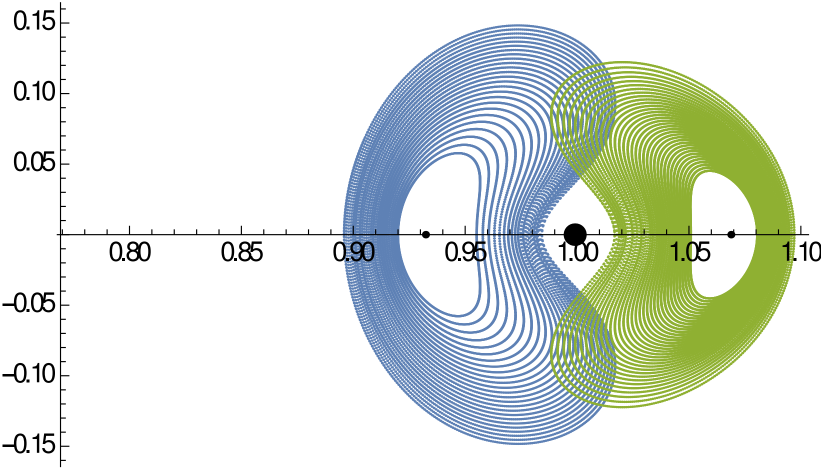
The system is hamiltonian and it admits a first integral called *Jacobi constant*

$$C(x, y, z, \dot{x}, \dot{y}, \dot{z}) = 2\Omega(x, y, z) - (\dot{x}^2 + \dot{y}^2 + \dot{z}^2).$$

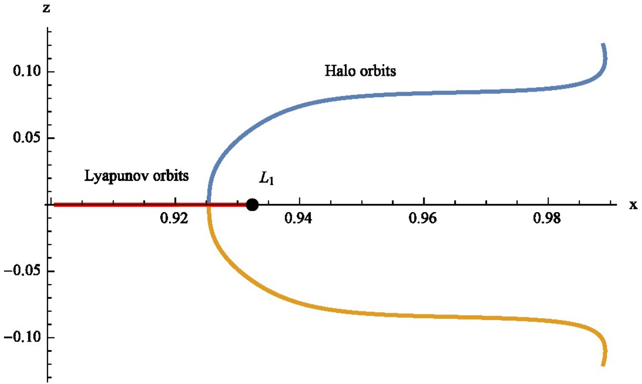
LIBRATION POINTS



Typical shape of families of *Lyapunov* orbits around L_1 and L_2 in Sun-Jupiter system.



(a) planar Lyapunov orbits


z
0.10
0.05
-0.05
-0.10
Halo orbits
Lyapunov orbits
L_1
0.92
0.94
0.96
0.98
x

The system possesses several symmetries that map trajectories onto other trajectories

$$\begin{aligned} S &: \quad (x(t), y(t), z(t)) \longrightarrow (x(t), y(t), -z(t)) \quad (3) \\ R &: \quad (x(t), y(t), z(t)) \longrightarrow (x(-t), -y(-t), z(-t)). \quad (4) \end{aligned}$$

Given this symmetry, a trajectory that crosses perpendicularly the $y = 0$ plane twice is a periodic orbit.
Let us define the following Poincaré sections

$$\begin{aligned} \Pi_+ &= \{(x, y, z, \dot{x}, \dot{y}, \dot{z}) \in \mathbb{R}^6 : y = 0 \text{ and } \dot{y} > 0\}, \\ \Pi_- &= \{(x, y, z, \dot{x}, \dot{y}, \dot{z}) \in \mathbb{R}^6 : y = 0 \text{ and } \dot{y} < 0\}. \end{aligned}$$

We define the following Poincaré maps between sections

$$P_+ : \Pi_+ \to \Pi_+, \quad P_- : \Pi_- \to \Pi_-, \quad P_{\frac{1}{2},+} : \Pi_+ \to \Pi_-, \quad P_{\frac{1}{2},-} : \Pi_- \to \Pi_+.$$

The sections Π_+, Π_- are invariant under the time-reversing symmetry R.

ALGORITHM

If $(x, \dot{y})$ are such that

$$P_{\frac{1}{2},+}(x, z, \dot{x} = 0, \dot{y}, \dot{z} = 0) = (x_1, z_1, 0, \dot{y}_1, 0) \tag{5}$$

and this fixed point might correspond to a Halo orbit. Therefore, we are interested in computation of the solution set to the following nonlinear equation

$$f(z, x, \dot{y}) := \pi_{(\dot{x},\dot{z})}\left(P_{\frac{1}{2},+}(x, z, 0, \dot{y}, 0)\right) = 0, \tag{6}$$

where $\pi_{(\dot{x},\dot{z})}$ denotes canonical projection onto $(\dot{x}, \dot{z})$ variables. We will call the function f a *bifurcation function* and by $f_{\dot{x}}, f_{\dot{z}}$ we will denote the components of f.

Let us define a new *reduced bifurcation function*

$$\widetilde{f}(z,x,\dot{y}) \;=\; \left(f_{\dot{x}}(z,x,\dot{y}), \int_0^1 \frac{\partial}{\partial z} f_{\dot{z}}(tz,x,\dot{y}) \right).$$

Lemma

Let us fix a set

$$W = Z \times X \times \dot{Y} := Z \times [x_0 - \Delta x, x_0 + \Delta x] \times [\dot{y}_0 - \Delta \dot{y}, \dot{y}_0 + \Delta \dot{y}]$$

such that Z is an open interval containing zero. Put

$$N_1 \;=\; \begin{bmatrix} x_0 \\ \dot{y}_0 \end{bmatrix} - \begin{bmatrix} \frac{\partial f_{\dot{x}}(W)}{\partial x} & \frac{\partial f_{\dot{x}}(W)}{\partial \dot{y}} \\ \frac{\partial^2 f_{\dot{z}}(W)}{\partial x \partial z} & \frac{\partial^2 f_{\dot{z}}(W)}{\partial \dot{y} \partial z} \end{bmatrix}_I^{-1} \cdot \begin{bmatrix} f_{\dot{x}}(Z,x_0,y_0) \\ \frac{\partial f_{\dot{z}}(Z,x_0,y_0)}{\partial z} \end{bmatrix} \qquad (7)$$

If $N_1 \subset \mathrm{int}(X \times \dot{Y})$ then the solution set to new reduced bifurcation function restricted to W is the graph of smooth functions

$$Z \ni z \to (x(z), \dot{y}(z)) \in X \times \dot{Y} \qquad (8)$$

CONTINUATIONS OF HALO ORBITS

Lemma

Let us fix a set

$$W = Z \times X \times \dot{Y} := Z \times ([x_0 - \Delta x, x_0 + \Delta x] \times [\dot{y}_0 - \Delta \dot{y}, \dot{y}_0 + \Delta \dot{y}]) .$$

Put

$$N = \begin{bmatrix} x_0 \\ \dot{y}_0 \end{bmatrix} - \left[D_{(x,\dot{y})} f(W) \right]_I^{-1} \cdot f(Z, x_0, \dot{y}_0)^T .$$

If $N \subset \mathrm{int}(X \times \dot{Y})$ then the solution set to (6) restricted to W is a graph of a smooth function

$$Z \ni z \to (x(z), \dot{y}(z)) \in X \times \dot{Y} .$$

NUMERICAL RESULTS - bifurcations

Computer assisted prove of the single bifurcation for $\mu = 0.0009537$

$$B1(0.92538773918106597, 0, 0, 0, 0.057714472776115309, 0)$$
$$B1[0]+ = 1e - 10 * (-1, 1);$$
$$B1[2]+ = 6e - 10 * (-1, 1);$$
$$B1[4]+ = 1e - 10 * (-1, 1);$$

Computer assisted prove of the bifurcation for the range $\mu \in [0.0166015625, 1 - 0.0166015625]$

NUMERICAL RESULTS - halo orbits

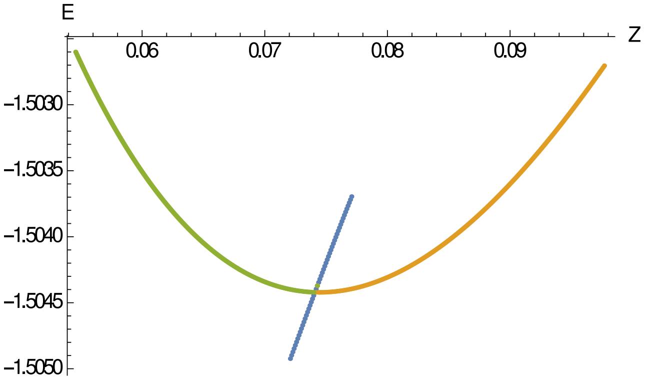
Computer assisted prove of the single halo orbit $\mu = 0.0009537$

$$H1 = (0.9253885387616267, 0, 0.001, 0, 0.057743435584918982, 0)$$

$$H1[0]+ = 4e - 9 * (-1, 1);$$
$$H1[2]+ = 4e - 10 * (0, 1);$$
$$H1[4]+ = 4e - 9 * (-1, 1);$$

Computer assisted prove of the bifurcation for the range $z \in [-0.083664781253492707, 0.083664781253492707]$

PERIOD DOUBLING BIFURCATION OF THE HALO ORBITS



PERIOD DOUBLING BIFURCATION FUNCTION - DEFINITION

Consider a fixed point curve $h_{fp} : Z \ni z \to (x_{fp}(z), \dot{y}_{fp}(z)) \in X \times \dot{Y}$ of Halo orbits.

$$\pi_{(\dot{x},\dot{z})}(P_+(x, z, \dot{x} = 0, \dot{y}, \dot{z} = 0)) = 0 \tag{9}$$

$$\pi_{\dot{x}}(P_+(x, z, \dot{x} = 0, \dot{y}, \dot{z} = 0)) = 0 \text{ for } (z, x, \dot{y}) \in V \text{ iff } x = x(z, \dot{y}) \tag{10}$$

Bifurcation function:

$$G_{\dot{z}}(z, x(z, \dot{y}), \dot{y}) := \pi_{\dot{z}}(P_+(x(z, \dot{y}), z, \dot{x} = 0, \dot{y}, \dot{z} = 0)) = 0 \tag{11}$$

$$g(z, \dot{y}) = \frac{G_{\dot{z}}(z, x(z, \dot{y}), \dot{y})}{\dot{y} - \dot{y}_{fp}(z)} \tag{12}$$

Lemma

Assume $H : \mathbb{R}^n \to \mathbb{R}$ is C^1. Let $x, y \in \mathbb{R}^n$ Then

$$H(x) - H(y) = \int_0^1 DH(t(x-y)+y)dt(x-y) \tag{13}$$

$G_{\dot{z}}(z, x, \dot{y}(z,x)) = (\dot{y} - \dot{y}_{fp}(z))g(z, \dot{y})$, where

$$g(z,\dot{y}) = \int_0^1 \frac{\partial}{\partial \dot{y}}(G_{\dot{z}}(z, x(z, t(\dot{y}-\dot{y}_{fp}(z))+\dot{y}_{fp}(z)), t(\dot{y}-\dot{y}_{fp}(z))+\dot{y}_{fp}(z))dt \tag{14}$$

Therefore, we want to determine the solution set of the following equation:

$$g(z,\dot{y}) = 0 \text{ for } (z,\dot{y}) \in U \tag{15}$$

$$N = [z_0] - \left[\frac{\partial g(Z,\dot{Y})}{\partial z}\right]_I^{-1} \cdot \left[g(z_0,\dot{Y})\right] \tag{16}$$