In the name of

Absolute Power & Absolute Knowledge

Goal Programming Approach for Solving Interval MOLP Problems

S. Rivaz, M.A. Yaghoobi and M. Hladík

8th Small Workshop on Interval Methods
Charles University
Prague, Czech Republic, June 9-11, 2015

Motivation

In many real-world situations, the parameters of decision-making problems are not deterministic. In this context, it is convenient to extend the traditional mathematical programming models incorporating their uncertainty. One of the tools for tackling vagueness is interval programming in which every uncertain parameter is determined with a closed interval. Moreover, many real-world problems inherently impose the need to investigate multiple and conflicting objective functions. This paper treats multiobjective linear programming problems with interval objective functions coefficients and target intervals.

MOLP Problems with Interval Objective Functions Coefficients and Target Intervals

$$\sum_{j=1}^{n} c_{kj} x_j = t_k, \quad k = 1, \cdots, p,$$

s.t.

$$\begin{aligned} Ax &\leq b, \\ c_{kj} &\in C_{kj}, \quad k = 1, \cdots, p, \quad j = 1, \cdots, n, \\ t_k &\in T_k, \quad k = 1, \cdots, p, \\ x &\geq 0, \end{aligned}$$

where C_{kj} is the closed interval $[c_{kj}^l, c_{kj}^u]$ representing a region the coefficients c_{kj} possibly take. T_k is the closed interval $[t_k^l, t_k^u]$ representing a region the target value t_k possibly take. The constraints of the problem are as in conventional MOLP problem.

The Work of Inuiguchi & Kume (1991)

Possible Deviation $D_k = [d_k^l, d_k^u]$ of $[\sum_{j=1}^n c_{kj}^l x_j, \sum_{j=1}^n c_{kj}^u x_j]$ from T_k

$$D_k = \begin{cases} [t_k^l - \sum_{j=1}^n c_{kj}^u x_j, t_k^u - \sum_{j=1}^n c_{kj}^l x_j] \\ \text{if } \ t_k^l - \sum_{j=1}^n c_{kj}^u x_j \geq 0, \\ \\ [0, (\sum_{j=1}^n c_{kj}^u x_j - t_k^l) \bigvee (t_k^u - \sum_{j=1}^n c_{kj}^l x_j)] \\ \text{if } \ t_k^l - \sum_{j=1}^n c_{kj}^u x_j < 0 < t_k^u - \sum_{j=1}^n c_{kj}^l x_j, \\ \\ [\sum_{j=1}^n c_{kj}^l x_j - t_k^u, \sum_{j=1}^n c_{kj}^u x_j - t_k^l] \\ \text{if } \ t_k^u - \sum_{j=1}^n c_{kj}^l x_j \leq 0, \end{cases}$$

where $\bigvee$ denotes the maximum.

$$0 \leq \lambda \leq 1, \quad w_k \geq 0 \ \text{ and } \ \sum_{k=1}^{p} w_k = 1.$$

The Work of Inuiguchi & Kume (1991)

$$min \;\; \lambda \sum_{k=1}^{p} w_k (d_k^{l^-} + d_k^{u^+}) + (1-\lambda) v^l$$

s.t.

$$\begin{aligned}
\sum_{j=1}^{n} c_{kj}^{u} x_j + d_k^{l^-} - d_k^{l^+} &= t_k^l, \quad k = 1, \cdots, p, \\
\sum_{j=1}^{n} c_{kj}^{l} x_j + d_k^{u^-} - d_k^{u^+} &= t_k^u, \quad k = 1, \cdots, p, \\
Ax &\leq b, \\
d_k^{l^-} + d_k^{u^+} &\leq v^l, \quad k = 1, \cdots, p, \\
x_j &\geq 0, \quad j = 1, \cdots, n, \\
d_k^{l^-}, d_k^{l^+}, d_k^{u^-}, d_k^{u^+}, v^l &\geq 0, \quad k = 1, \cdots, p.
\end{aligned}$$

$$\min \ \lambda \sum_{k=1}^{p} w_k v_k + (1-\lambda) v^u$$

$$\text{s.t.} \quad \begin{aligned}
\sum_{j=1}^{n} c_{kj}^{u} x_j + d_k^{l^-} - d_k^{l^+} &= t_k^l, \quad k = 1, \cdots, p,\\
\sum_{j=1}^{n} c_{kj}^{l} x_j + d_k^{u^-} - d_k^{u^+} &= t_k^u, \quad k = 1, \cdots, p,\\
Ax &\leq b,\\
d_k^{l^+} &\leq v_k, \quad k = 1, \cdots, p,\\
d_k^{u^-} &\leq v_k, \quad k = 1, \cdots, p,\\
v_k &\leq v^u, \quad k = 1, \cdots, p,\\
x_j &\geq 0, \quad j = 1, \cdots, n,\\
v^u &\geq 0,\\
d_k^{l^-}, d_k^{l^+}, d_k^{u^-}, d_k^{u^+}, v_k &\geq 0, k = 1, \cdots, p.
\end{aligned}$$

The Work of Inuiguchi & Kume (1991)

Necessary Deviation $E_k = [e_k^l, e_k^u]$ of $[\sum_{j=1}^n c_{kj}^l x_j, \sum_{j=1}^n c_{kj}^u x_j]$ from T_k

$$E_k = \begin{cases} [t_k^l - \sum_{j=1}^n c_{kj}^l x_j, t_k^u - \sum_{j=1}^n c_{kj}^u x_j] \\ \text{if } \; t_k^u - \sum_{j=1}^n c_{kj}^u x_j \geq t_k^l - \sum_{j=1}^n c_{kj}^l x_j \geq 0, \\ \\ [0, (t_k^u - \sum_{j=1}^n c_{kj}^u x_j) \bigvee (\sum_{j=1}^n c_{kj}^l x_j - t_k^l)] \\ \text{if } \; t_k^l - \sum_{j=1}^n c_{kj}^l x_j < 0 < t_k^u - \sum_{j=1}^n c_{kj}^u x_j, \\ \\ [\sum_{j=1}^n c_{kj}^u x_j - t_k^u, \sum_{j=1}^n c_{kj}^l x_j - t_k^l] \\ \text{if } \; t_k^l - \sum_{j=1}^n c_{kj}^l x_j \leq t_k^u - \sum_{j=1}^n c_{kj}^u x_j \leq 0, \\ \\ [\sum_{j=1}^n c_{kj}^l x_j - t_k^l, \sum_{j=1}^n c_{kj}^u x_j - t_k^u] \\ \text{if } \; \sum_{j=1}^n c_{kj}^u x_j - t_k^u \geq \sum_{j=1}^n c_{kj}^u x_j - t_k^l \geq 0, \\ \\ [0, (\sum_{j=1}^n c_{kj}^u x_j - t_k^u) \bigvee (t_k^l - \sum_{j=1}^n c_{kj}^l x_j)] \\ \text{if } \; \sum_{j=1}^n c_{kj}^l x_j - t_k^l < 0 < \sum_{j=1}^n c_{kj}^u x_j t_k^u, \\ \\ [t_k^u - \sum_{j=1}^n c_{kj}^u x_j, t_k^l - \sum_{j=1}^n c_{kj}^l x_j] \\ \text{if } \; \sum_{j=1}^n c_{kj}^l x_j - t_k^l \leq \sum_{j=1}^n c_{kj}^u x_j - t_k^u \leq 0. \end{cases}$$

The Work of Inuiguchi & Kume (1991)

$$min \ \ \lambda \sum_{k=1}^{p} w_k((e_k^{l^-} + e_k^{u^+}) \bigwedge (e_k^{l^+} + e_k^{u^-})) + (1-\lambda)u^l$$

s.t.

$$\begin{aligned}
\sum_{j=1}^{n} c_{kj}^l x_j + e_k^{l^-} - e_k^{l^+} &= t_k^l, k = 1, \cdots, p, \\
\sum_{j=1}^{n} c_{kj}^u x_j + e_k^{u^-} - e_k^{u^+} &= t_k^u, k = 1, \cdots, p, \\
Ax &\leq b, \\
((e_k^{l^-} + e_k^{u^+}) \bigwedge (e_k^{l^+} + e_k^{u^-})) &\leq u^l, k = 1, \cdots, p, \\
x_j &\geq 0, j = 1, \cdots, n, \\
e_k^{l^-}, e_k^{l^+}, e_k^{u^-}, e_k^{u^+}, u^l &\geq 0, k = 1, \cdots, p,
\end{aligned}$$

where $\bigwedge$ denotes the minimum.

The Work of Inuiguchi & Kume (1991)

$$
\begin{aligned}
& \min \quad \lambda \sum_{k=1}^{p} w_k u_k + (1-\lambda) u^u \\
\text{s.t.} \quad & \sum_{j=1}^{n} c_{kj}^{l} x_j + e_k^{l^-} - e_k^{l^+} = t_k^l, \quad k = 1, \cdots, p, \\
& \sum_{j=1}^{n} c_{kj}^{u} x_j + e_k^{u^-} - e_k^{u^+} = t_k^u, \quad k = 1, \cdots, p, \\
& Ax \leq b, \\
& e_k^{l^-} + e_k^{l^+} \leq u_k, \quad k = 1, \cdots, p, \\
& e_k^{u^-} + e_k^{u^+} \leq u_k, \quad k = 1, \cdots, p, \\
& u_k \leq u^u, \quad k = 1, \cdots, p, \\
& x_j \geq 0, \quad j = 1, \cdots, n, \\
& u^u \geq 0, \\
& e_k^{l^-}, e_k^{l^+}, e_k^{u^-}, e_k^{u^+}, u_k \geq 0, k = 1, \cdots, p.
\end{aligned}
$$

Our Aim ...

Minimizing the total distance of planned intervals, $[\sum_{j=1}^{n} c_{kj}^{l} x_j, \sum_{j=1}^{n} c_{kj}^{u} x_j]$, from target intervals, $[t_k^l, t_k^u]$, $k = 1, \cdots, p$. In order to do that, we use the following distance concept between two intervals.

The Distance Between Two Intervals

Definition

Let $[a^l, a^u]$ and $[b^l, b^u]$ be two intervals. Then, the distance between them is defined as (Moore, 2009):

$$d([a^l, a^u], [b^l, b^u]) = max\{|a^l - b^l|, |a^u - b^u|\}.$$

It is easy to check that the distance d, defined in the above, introduces a metric on the space of closed intervals in the set of real numbers.

The Distance Between the kth Planned Interval and Its Target Interval $(k = 1, \cdots, p)$

$$D_k = d([\sum_{j=1}^{n} c_{kj}^{l} x_j, \sum_{j=1}^{n} c_{kj}^{u} x_j], [t_k^l, t_k^u]) =$$

$$max(|\sum_{j=1}^{n} c_{kj}^{l} x_j - t_k^l|, |\sum_{j=1}^{n} c_{kj}^{u} x_j - t_k^u|),$$

$$k = 1, \cdots, p,$$

Using the deviational variables $d_k^{l^-}$, $d_k^{l^+}$, $d_k^{u^-}$ and $d_k^{u^+}$ as:

$$\sum_{j=1}^{n} c_{kj}^{l} x_j + d_k^{l^-} - d_k^{l^+} = t_k^l;$$

$$\sum_{j=1}^{n} c_{kj}^{u} x_j + d_k^{u^-} - d_k^{u^+} = t_k^u;$$

D_k can be written as follows:

$$D_k = max(|d_k^{l^-} - d_k^{l^+}|, |d_k^{u^-} - d_k^{u^+}|),$$

$$k = 1, \cdots, p.$$

Our Model for Solving an Interval MOLP Problem with Target Intervals

$$min \;\; \lambda \sum_{k=1}^{p} w_k v_k + (1-\lambda)u \tag{1}$$

s.t.

$$\begin{aligned}
\sum_{j=1}^{n} c_{kj}^{l} x_j + d_k^{l^-} - d_k^{l^+} &= t_k^l, \quad k = 1, \cdots, p, \\
\sum_{j=1}^{n} c_{kj}^{u} x_j + d_k^{u^-} - d_k^{u^+} &= t_k^u, \quad k = 1, \cdots, p, \\
Ax &\leq b, \\
|d_k^{l^-} - d_k^{l^+}| &\leq v_k, \quad k = 1, \cdots, p, \\
|d_k^{u^-} - d_k^{u^+}| &\leq v_k, \quad k = 1, \cdots, p,
\end{aligned}$$

Continued ...

$$
\begin{aligned}
v_k &\leq u, \quad k = 1, \cdots, p, \\
d_k^{l^-} . d_k^{l^+} &= 0, \quad k = 1, \cdots, p, \\
d_k^{u^-} . d_k^{u^+} &= 0, \quad k = 1, \cdots, p, \\
x_j &\geq 0, \quad j = 1, \cdots, n, \\
v_k &\geq 0, \quad k = 1, \cdots, p, \\
d_k^{l^-},\ d_k^{l^+},\ d_k^{u^-},\ d_k^{u^+} &\geq 0, \quad k = 1, \cdots, p, \\
u &\geq 0.
\end{aligned}
$$

The proposed model can be transformed to a linear programming problem. It is obvious that $|(.)| \leq v_k$, $k = 1, \cdots, p$, can be substituted with two linear constraints $(.) \leq v_k$ and $-(.) \leq v_k$. The nonlinear constraints $d_k^{l^-} . d_k^{l^+} = 0$ and $d_k^{u^-} . d_k^{u^+} = 0$, $k = 1, \cdots, p$, can be eliminated due to the dependency of columns of d_k^{l-}, d_k^{l+} and d_k^{u-}, d_k^{u+}, when the Simplex method is used for solving the model.

New Model

$$
\begin{aligned}
& min \quad \lambda \sum_{k=1}^{p} w_k v_k + (1-\lambda) u && (2)\\
\text{s.t.} \quad & \sum_{j=1}^{n} c_{kj}^{l} x_j + d_k^{l^-} - d_k^{l^+} = t_k^l, \quad k = 1, \cdots, p,\\
& \sum_{j=1}^{n} c_{kj}^{u} x_j + d_k^{u^-} - d_k^{u^+} = t_k^u, \quad k = 1, \cdots, p,\\
& Ax \leq b,\\
& d_k^{l^-} + d_k^{l^+} \leq v_k, \quad k = 1, \cdots, p,\\
& d_k^{u^-} + d_k^{u^+} \leq v_k, \quad k = 1, \cdots, p,\\
& v_k \leq u, \quad k = 1, \cdots, p,\\
& x_j \geq 0, \quad j = 1, \cdots, n,\\
& v_k \geq 0, \quad k = 1, \cdots, p,\\
& d_k^{l^-},\ d_k^{l^+},\ d_k^{u^-},\ d_k^{u^+} \geq 0, \quad k = 1, \cdots, p,\\
& u \geq 0.
\end{aligned}
$$

The Property of Model (2)

Lemma

Let model (2) be feasible then it has an optimal solution in which the constraints $d_k^{l^-}.d_k^{l^+} = 0$ *and* $d_k^{u^-}.d_k^{u^+} = 0$, $k = 1, \cdots, p$, *are satisfied.*

Theorem

The optimal solution of model (1) can be obtained by solving model (2).

A Numerical Example

In order to illustrate the proposed model, consider the following interval MOLP problem with interval targets which was proposed by Inuiguchi and Kume (1991):
Optimize:

$$c_1x_1 + c_4y_1 = t_1,$$
$$c_2x_2 + c_5y_2 = t_2,$$
$$c_3x_3 + c_6y_3 = t_3,$$

s.t.

$$\begin{aligned} x_1 + x_2 + x_3 &\le 9, \\ y_1 + y_2 + y_3 &\le 9, \\ c_1 \in [2,3],\ c_4 \in [5,7],\ t_1 &\in [28,32], \\ c_2 \in [2,3],\ c_5 \in [1,3],\ t_2 &\in [25,30], \\ c_3 \in [4,8],\ c_6 \in [2,3],\ t_3 &\in [31,37], \\ x_j \ge 0, \quad y_j \ge 0, \quad j &= 1,2,3. \end{aligned}$$

Continued ...

Applying the proposed model when $\lambda = \frac{1}{2}$ and $w_1 = w_2 = w_3 = \frac{1}{3}$, we have

$$min \quad \frac{1}{6}(v_1 + v_2 + v_3) + \frac{1}{2}u$$

s.t.

$$\begin{aligned}
2x_1 + 5y_1 + d_1^{l^-} - d_1^{l^+} &= 28,\\
2x_2 + y_2 + d_2^{l^-} - d_2^{l^+} &= 25,\\
4x_3 + 2y_3 + d_3^{l^-} - d_3^{l^+} &= 31,\\
3x_2 + 7y_2 + d_1^{u^-} - d_1^{u^+} &= 32,\\
3x_2 + 3y_2 + d_2^{u^-} - d_2^{u^+} &= 30,\\
8x_2 + 3y_2 + d_3^{u^-} - d_3^{u^+} &= 37,\\
x_1 + x_2 + x_3 &\leq 9,\\
y_1 + y_2 + y_3 &\leq 9,\\
|d_1^{l^-} - d_1^{l^+}| &\leq v_1,
\end{aligned}$$

$$
\begin{aligned}
|d_2^{l^-} - d_2^{l^+}| &\leq v_2, \\
|d_2^{u^-} - d_2^{u^+}| &\leq v_2, \\
|d_3^{l^-} - d_3^{l^+}| &\leq v_3, \\
|d_3^{u^-} - d_3^{u^+}| &\leq v_3, \\
v_k &\leq u, \quad k = 1, 2, 3, \\
x_j &\geq 0, \ y_j \geq 0, j = 1, 2, 3, \\
v_k &\geq 0, \quad k = 1, 2, 3, \\
d_k^{l^-}, \ d_k^{l^+}, \ d_k^{u^-}, \ d_k^{u^+} &\geq 0, \quad k = 1, 2, 3, \\
u &\geq 0.
\end{aligned}
$$

The problem solution is $x_1 = 0$, $x_2 = 4.5$, $x_3 = 4.5$, $y_1 = 5$, $y_2 = 3.6667$ and $y_3 = 0.3333$ where $D_1 = 3$, $D_2 = 12.3333$, $D_3 = 12.3334$ and D(x)=10.87.

References

- Bitran, G.R., Linear multiobjective problems with interval coefficients, Management Science, 26, 694-706 (1980)
- Chinneck, J.W., Ramadan, K., Linear programming with interval coefficients, Journal of Operational Research Society, 51, 209-220 (2000)
- Fiedler, M., Nedoma, J., Rohn, J., Zimmermann, K., Linear optimization problems with inexact data, Springer-Verlag, New York (2006)
- Inuiguchi, M., Kume, Y., Goal programming problems with interval coefficients and target intervals, European Journal of Operational Research, 52, 345-360 (1991)

References

- Jones, D.F., Tamiz, M., Practical goal programming, Springer (2010)
- Moore, R.E., Kearfott, R.B., Cloud, M.J., Introduction to interval analysis, Society for Industrial and Applied Mathematics, Philadelphia (2009)
- Oliveira, C. and Antunes, C. H., Multiple objective linear programming models with interval coefficients - an illustrated overview, European Journal of Operational Research, 181, 1434-1463 (2007)
- Oliveira, C. and Antunes, C. H., An interactive method of tackling uncertainty in interval multiple objective linear programming, Journal of Mathematical Sciences, 161, 854-866 (2009)

Thank You!