Exact Bounds for the Solution Sets of Parametric Interval Linear Systems

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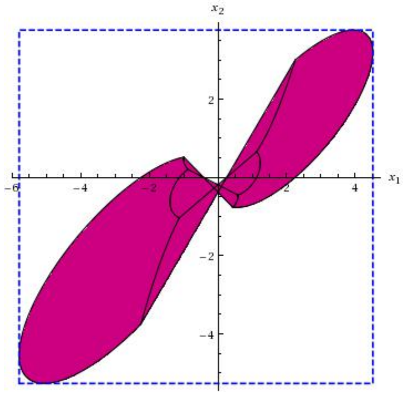

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Parametric Linear Systems

Consider the linear algebraic system

$$A(p) \cdot x \;=\; b(p),$$

where

$$A(p) := A_0 + \sum_{i=1}^{k} A_i p_i, \qquad b(p) := b_0 + \sum_{i=1}^{k} b_i p_i$$

$$A_i \in \mathbb{R}^{n\times n},\; b_i \in \mathbb{R}^n, \quad i = 0, \ldots, k$$

the uncertain parameters p_i vary within given intervals

$$p \;\in\; [p] = ([\underline{p}_1, \overline{p}_1], \ldots, [\underline{p}_k, \overline{p}_k])^\top.$$

GOAL:

$$\boldsymbol{\Sigma}^{p}_{uni} := \{x \in \mathbb{R}^n \mid (\exists p \in [p])(A(p)x = b(p))\}\,.$$

If $\boldsymbol{\Sigma}^{p}_{uni}$ is bounded,
in worst-case analysis of uncertain systems, outer interval estimation is sought.

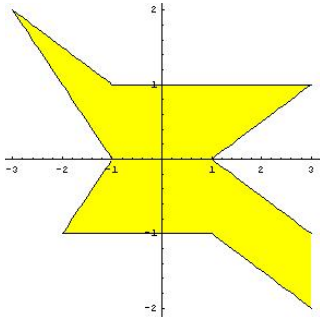
$$\text{Find} \quad [u] \in \mathbb{IR}^n \quad \text{such that} \quad \boldsymbol{\Sigma}^{p}_{uni} \subseteq [u].$$

Interval Hull Solution, $\square\boldsymbol{\Sigma}^{p}_{uni}$, is the minimal outer estimation:

$$\square\boldsymbol{\Sigma}^{p}_{uni} \;:=\; \bigcap\{u \in \mathbb{IR}^n \mid \boldsymbol{\Sigma}^{p}_{uni} \subseteq [u]\}.$$

Parametric Solution Sets — Classification

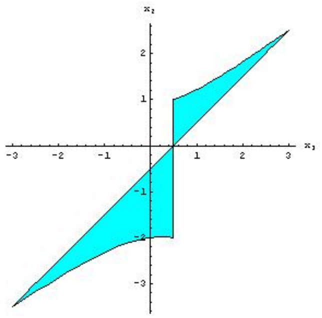
- Linear Boundary (the hull is attained at the interval end-points)



  - sufficient conditions in CAMWA 68, 2014
  - hull computation by :

    end-point search, local/global monotonicity proof, etc.

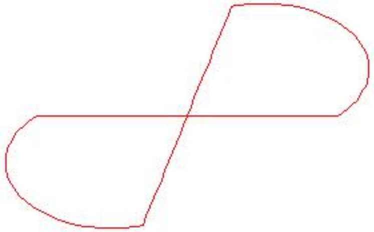
- NonLinear Boundary, the hull is attained at the interval end-points



hull computation by: local/global monotonicity proof,

etc.

- NonLinear Boundary the hull is attained at points in the interior of the intervals



We propose methodology for hull computation

applicable to this case, too.

Parametric Hypersurfaces

Definition. A hypersurface in n-dim projective space is an algebraic surface of dimension $n-1$. It is then defined by a single equation $f(x_1, \ldots, x_n) = 0$.

Definition. A hypersurface in n-dim space is called parametric hypersurface, $x(p)$, if it is defined by n coordinate functions

$$x_i(p) = x_i(p_1, \ldots, p_{n-1}), \quad i = 1, \ldots, n.$$

Definition. Restricted PHS, $x(p)\big|_{p\in[p]}$, is a piece (part) of PHS, obtained for a specified range of the parameters.

Consider PHSs $x(p)$, $x(p) \in \mathbb{R}^n$, $p \in \mathbb{R}^{n-1}$, where $x(p) := A^{-1}(p) \cdot b(p)$ is the analytic solution of $A(p)x = b(p)$, if it exists.

Notations

Let $p \in \mathbb{R}^k$, $A(p) \in \mathbb{R}^{n\times n}$, $b(p) \in \mathbb{R}^n$, $\mathcal{K} = \{1, \ldots, k\}$, $n \leq k$.

For an index set $\Pi = \{\pi_1, \ldots, \pi_k\}$, p_Π denotes $(p_{\pi_1}, \ldots, p_{\pi_k})$.

Define the set of all index sets of dimension $n-1$ by

$$Q(n-1,k) \ := \ \{q = (\pi_1, \ldots, \pi_{n-1}) \mid q \subset \mathcal{K}\}.$$

For a given $q \in Q(n-1,k)$, define $\tilde{q} := \mathcal{K} \setminus q$.

$$\{\pm 1\}^m \ := \ \{x \in \mathbb{R}^m \mid |x| = (1, \ldots, 1)\}$$

For $[a] = [\underline{a}, \overline{a}] \in \mathbb{IR}^m$ and $u \in \{\pm 1\}^m$, a^u is defined by

$$\{a^u\}_i := \begin{cases} \underline{a}_i & \text{if } u_i = -1 \\ \overline{a}_i & \text{if } u_i = 1 \end{cases}, \quad i = 1, \ldots, m.$$

Boundary of Σ^p: $\partial\Sigma^p$

E.Popova, BIT 48(2008):95–115.

Theorem 1. If $A^{-1}(p)$ exists,

$$\partial\Sigma^p \subseteq \bigcup_{q\in Q(n-1,k)} \; \bigcup_{u\in\{\pm 1\}^{k-n+1}} x(p_q, p_{\tilde{q}}^u)|_{p_q\in[p_q]}.$$

If $k \leq n-1$, Σ^p is degenerate and

$$\partial\Sigma^p = x(p)|_{p\in[p]} = \Sigma^p.$$

Interval Hull of Σ^p: $\square\Sigma^p$

Corollary 1. If $A^{-1}(p)$ exists, then

$$\square\Sigma^p = \bigcup_{q\in Q(n-1,k)} \; \bigcup_{u\in\{\pm1\}^{k-n+1}} \square\left\{x(p_q, p_{\tilde q}^u) \mid p_q \in [p_q]\right\}.$$

$x_i(p_q, p_{\tilde q}^u)$ is a rational function of $n-1$ variables p_q, of arbitrary degree.

$\mathrm{Card}(\bigcup_{q\in Q(n-1,k)} \bigcup_{u\in\{\pm1\}^{k-n+1}} x(p_q, p_{\tilde q}^u)) = \binom{k}{n-1}2^{k-n+1}$

Interval Hull of Σ^p by 2D projections

Corollary 2. If $A^{-1}(p)$ exists, then

$$\Box\Sigma^p = \bigcup_{q\in Q(1,k)} \bigcup_{u\in\{\pm 1\}^{k-1}} \Box\left\{x(p_q, p_{\tilde{q}}^u) \mid p_q \in [p_q]\right\}.$$

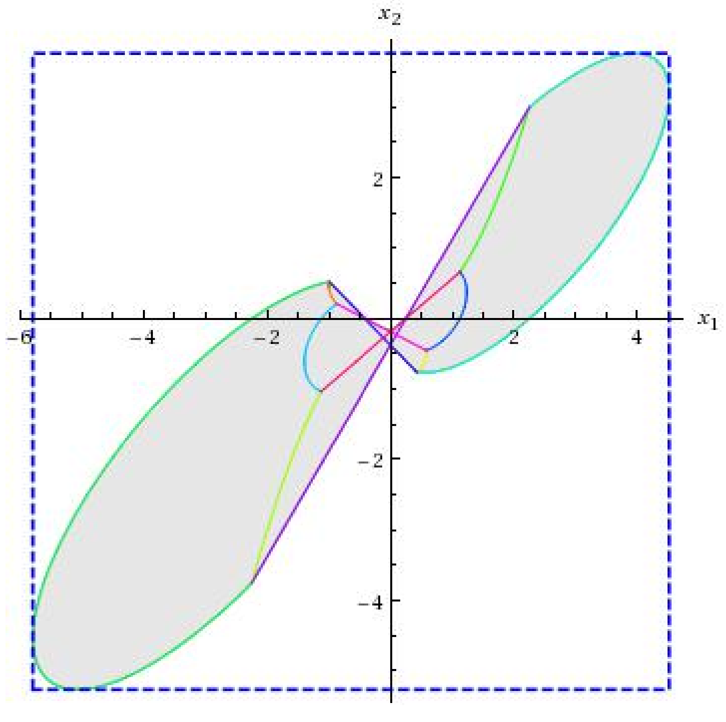
$x_i(p_q, p_{\tilde{q}}^u)$ is a rational function of 1 variable p_q, of arbitrary degree.

$$\Box\left\{x_i(p_q, p_{\tilde{q}}^u) \mid p_q \in [p_q]\right\} = \left[\min_{p_q\in[p_q]} x_i(p_q, p_{\tilde{q}}^u),\ \max_{p_q\in[p_q]} x_i(p_q, p_{\tilde{q}}^u)\right]$$

Some software systems, like *Mathematica*®, can find the exact global extremum for such problems with rational data.

Example 1.

$$\begin{pmatrix} 2p_1 & p_2 - 2 \\ -p_2 & 2p_1 \end{pmatrix} x = \begin{pmatrix} p_3 \\ -1/2 \end{pmatrix}, \qquad p \in ([\frac{2}{3}, \frac{4}{3}], [-\frac{12}{10}, 2], [-3, 3]).$$



$\boldsymbol{A(p)}$ is NOT strongly regular,

all other methods fail.

$$\text{hull} = \begin{pmatrix} [-5.7996, 4.5178] \\ [-5.2639, 3.7678] \end{pmatrix}$$

Unboundedness of Σ_{uni}^{p}

Theorem 2. $\boldsymbol{\Sigma_{uni}^{p}}$ is unbounded, if NullSpace$(\boldsymbol{A(p)}) = \{\mathbf{0}\}$ and there exist

$$\mathbf{1} \le \boldsymbol{i} \le \boldsymbol{n}, \quad \boldsymbol{q} \in \boldsymbol{Q}(\boldsymbol{n}-\mathbf{1},\boldsymbol{k}) \quad \text{and} \quad \boldsymbol{u} \in \{\pm\mathbf{1}\}^{\boldsymbol{k}-\boldsymbol{n}+\mathbf{1}},$$

such that

$$\max_{\boldsymbol{p_q} \in [\boldsymbol{p_q}]} \boldsymbol{x_i}(\boldsymbol{p_q}, \boldsymbol{p_{\tilde{q}}^{u}}) \;=\; \infty$$

or

$$\min_{\boldsymbol{p_q} \in [\boldsymbol{p_q}]} \boldsymbol{x_i}(\boldsymbol{p_q}, \boldsymbol{p_{\tilde{q}}^{u}}) \;=\; -\infty.$$

Empty/Unbounded Σ^p_{uni}

Theorem 3.

- If NullSpace$(\boldsymbol{A}(\boldsymbol{p})) \neq \{\boldsymbol{0}\}$, then $\boldsymbol{\Sigma^p_{uni}}$ is either empty or unbounded.

- If NullSpace$(\boldsymbol{A}(\boldsymbol{p})) \neq \{\boldsymbol{0}\}$ and $\boldsymbol{b}(\boldsymbol{p}) = \boldsymbol{0}$ for some $\boldsymbol{p} \in [\boldsymbol{p}]$,
 then $\boldsymbol{\Sigma^p_{uni}}$ is unbounded.

Example 2: Constraint Satisfaction

Garloff, Granvilliers, Smith, LNCS 3478, 2005.

Constraint satisfaction techniques have to be improved in order to process exponential-based models (often ill conditioned).

$$f(x,t) = \sum_{j=1}^{3} x_{2j-1} \exp(-x_{2j} t)$$

Find $x \in \mathbb{R}^6$, such that $\quad f(x, t_0 + ih) = \tilde{y}_i, \qquad i = 1, \ldots, 6.$

* With a large initial box RealPaver computes no reduction.

* With 6 redundant constraints the initial box is reduced slightly.

Interval Prony's method is used as a preprocessing step to deliver a suitable initial box.

Interval Prony's method (3-exp case)

$$\boldsymbol{f(x, t_0 + ih) \in [\underline{y}_i, \overline{y}_i]}, \qquad i = 1, \ldots, 6.$$

Since exponential sums may be very sensitive to changes in their parameters, special emphasis is put on finding sharp bounds for the solution set of

$$\begin{pmatrix} [y_1] & [y_2] & [y_3] \\ [y_2] & [y_3] & [y_4] \\ [y_3] & [y_4] & [y_5] \end{pmatrix} \begin{pmatrix} \zeta_1 \\ \zeta_2 \\ \zeta_3 \end{pmatrix} = - \begin{pmatrix} [y_4] \\ [y_5] \\ [y_6] \end{pmatrix}.$$

If the zeros of $\boldsymbol{u^3 + [\zeta_3]u^2 + [\zeta_2]u + [\zeta_1]}$ are positive and separate, find a tight enclosure of for the zero set by interval version of Cardano's formula.

Then, find sharp bounds for the solution set of

$$\begin{pmatrix} 1 & 1 & 1 \\ [u_1] & [u_2] & [u_3] \\ [u_1]^2 & [u_2]^2 & [u_3]^2 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = \begin{pmatrix} [y_1] \\ [y_2] \\ [y_3] \end{pmatrix}.$$

Interval Prony's method (3-exp case)

Garloff et al., 2006, solve the parametric interval system

$$\begin{pmatrix} y_1 & y_2 & y_3 \\ y_2 & y_3 & y_4 \\ y_3 & y_4 & y_5 \end{pmatrix} \begin{pmatrix} \zeta_1 \\ \zeta_2 \\ \zeta_3 \end{pmatrix} = - \begin{pmatrix} y_4 \\ y_5 \\ y_6 \end{pmatrix}$$

where $y_i \in [\underline{y}_i, \overline{y}_i]$, $i = 1, \ldots, 6,$ by:

i) the components of the symbolic solution are interval rational functions, e.g.,

$$\zeta_3 = \frac{2y_3y_4y_5 - y_4^3 - y_2y_5^2 + y_6(y_2y_4 - y_3^2)}{y_1(y_3y_5 - y_4^2) - y_2^2y_5 + 2y_2y_3y_4 - y_3^3}$$

ii) bound the range of ζ_i, $i = 1, 2, 3$, by Bernstein expansion of the interval function.

Guaranteed Interval Hull in fl. point

Combine: the method of PHSs in 2D space with

the Bernstein expansion method (or other guaranteed method)

to reduce the number of interval parameters

and obtain:

i) guaranteed tight enclosure of the hull in floating point

ii) expanded applicability to problems with

* bigger size

* larger parameter intervals

While Garloff et al. solve the Hankel-type system for 10^{-6} interval radius,

the method of PHSs can be applied with radius close to the radius of singularity.

Properties of Our Methodology

1. Do NOT require regularity or strong regularity of $A(p)$ on $[p]$.
   Our approach shows *a posteriori* the regularity or singularity of $A(p)$ on $[p]$.

2. It reduces to solving constraint global optimization problems, where
   the objective is rational function of one variable.
   In this case the exact extrema can be found for exact data
   by available software (e.g., *Mathematica*®).

3. Delivers exact bounds for each component of Σ^p
   if and only if Σ^p is bounded.

4. The methodology has exponential complexity.
   For $k \leq n-1$, and each component of Σ^p:
   2 global optimization problems with k variables,
   or $k2^k$ global optimization problems with 1 variable.

Applicability of Our Methodology:

Solving real-life problems, where:

- small number of parameters, or offline analysis;
- systems with very large parameter intervals;
- bound(s) for single solution components are required;
- no special interval software is available.

Allows, in general:

- constructing benchmark examples (together with visualization);
- estimating the quality of newly designed numerical methods;
- proving the existence of a singular matrix;
- combination with other self-verified methods;
- improving constraint satisfaction techniques by preprocessing, etc.