Yet another method for solving interval linear equations

Milan Hladík

Department of Applied Mathematics
Faculty of Mathematics and Physics,
Charles University in Prague, Czech Republic
http://kam.mff.cuni.cz/~hladik/

SWIM 2015
Prague
June 9–11, 2015

Introduction

Notation

- An interval matrix $\mathbf{A}$ is defined as

$$\mathbf{A} := [\underline{A}, \overline{A}] = \{A \in \mathbb{R}^{m\times n} : \underline{A} \le A \le \overline{A}\},$$

- The center and radius of $\mathbf{A}$ are respectively defined as

$$\mathbf{A}_c := \frac{1}{2}(\underline{A} + \overline{A}), \quad \mathbf{A}_\Delta := \frac{1}{2}(\overline{A} - \underline{A}).$$

- The set of all m-by-n interval matrices is denoted by $\mathbb{IR}^{m\times n}$.

Introduction

Notation

- An interval matrix $\mathbf{A}$ is defined as

$$\mathbf{A} := [\underline{A}, \overline{A}] = \{A \in \mathbb{R}^{m\times n} : \underline{A} \le A \le \overline{A}\},$$

- The center and radius of $\mathbf{A}$ are respectively defined as

$$\mathbf{A}_c := \frac{1}{2}(\underline{A} + \overline{A}), \quad \mathbf{A}_\Delta := \frac{1}{2}(\overline{A} - \underline{A}).$$

- The set of all m-by-n interval matrices is denoted by $\mathbb{IR}^{m\times n}$.
- The magnitude of an $\mathbf{A} \in \mathbb{IR}^{m\times n}$ is defined as

$$\mathrm{mag}(\mathbf{A}) := \max(|\underline{A}|, |\overline{A}|).$$

Introduction

Notation

- An interval matrix $\mathbf{A}$ is defined as

$$\mathbf{A} := [\underline{A}, \overline{A}] = \{A \in \mathbb{R}^{m\times n} : \underline{A} \le A \le \overline{A}\},$$

- The center and radius of $\mathbf{A}$ are respectively defined as

$$\mathbf{A}_c := \frac{1}{2}(\underline{A} + \overline{A}), \quad \mathbf{A}_\Delta := \frac{1}{2}(\overline{A} - \underline{A}).$$

- The set of all m-by-n interval matrices is denoted by $\mathbb{IR}^{m\times n}$.
- The magnitude of an $\mathbf{A} \in \mathbb{IR}^{m\times n}$ is defined as

$$\operatorname{mag}(\mathbf{A}) := \max(|\underline{A}|, |\overline{A}|).$$

- The comparison matrix of $\mathbf{A} \in \mathbb{IR}^{n\times n}$ is the matrix $\langle \mathbf{A} \rangle \in \mathbb{R}^{n\times n}$ with entries

$$\langle \mathbf{A} \rangle_{ii} := \min\{|a| : a \in \mathbf{a}_{ii}\}, \quad i = 1, \dots, n,$$
$$\langle \mathbf{A} \rangle_{ij} := -\operatorname{mag}(\mathbf{a}_{ij}), \quad i \ne j.$$

Interval linear equations

Definition

Let $\mathbf{A} \in \mathbb{IR}^{n\times n}$, $\mathbf{b} \in \mathbb{IR}^{n}$, and consider a set of systems of linear equations

$$Ax = b, \quad A \in \mathbf{A},\ b \in \mathbf{b},$$

The corresponding solution set is defined as

$$\Sigma := \{x \in \mathbb{R}^n : \exists A \in \mathbf{A} \exists b \in \mathbf{b} : Ax = b\}.$$

By $\boldsymbol{\Sigma}$ we denote the interval hull of Σ, i.e., the smallest interval enclosure of Σ with respect to inclusion.

Interval linear equations

Definition

Let $\mathbf{A} \in \mathbb{IR}^{n \times n}$, $\mathbf{b} \in \mathbb{IR}^n$, and consider a set of systems of linear equations

$$Ax = b, \quad A \in \mathbf{A},\ b \in \mathbf{b},$$

The corresponding solution set is defined as

$$\Sigma := \{x \in \mathbb{R}^n : \exists A \in \mathbf{A} \exists b \in \mathbf{b} : Ax = b\}.$$

By $\boldsymbol{\Sigma}$ we denote the interval hull of Σ, i.e., the smallest interval enclosure of Σ with respect to inclusion.

Problem formulation

The aim is to compute $\boldsymbol{\Sigma}$ or an as tight as possible enclosure of Σ by an interval vector $\mathbf{x} \in \mathbb{IR}^n$, meaning that $\Sigma \subseteq \mathbf{x}$.

Assumption

Assumption

Assume that $\mathbf{A}_c = I_n$.

Assumption

Assumption

Assume that $\mathbf{A}_c = I_n$.

- Easily satisfied by preconditioning $\mathbf{A} = \mathbf{b}$ by $\mathbf{A}_c^{-1}$.
- Rigorously precondition as

$$A'x = b', \quad A' \in [I_n - \mathrm{mag}(I_n - R\mathbf{A}), I_n + \mathrm{mag}(I_n - R\mathbf{A})], \; b' \in R\mathbf{b}.$$

where $R \approx \mathbf{A}_c^{-1}$.

Assumption

Assumption

Assume that $\mathbf{A}_c = I_n$.

- Easily satisfied by preconditioning $\mathbf{A} = \mathbf{b}$ by $\mathbf{A}_c^{-1}$.
- Rigorously precondition as

$$A'x = b', \quad A' \in [I_n - \mathrm{mag}(I_n - R\mathbf{A}), I_n + \mathrm{mag}(I_n - R\mathbf{A})], \; b' \in R\mathbf{b}.$$

where $R \approx \mathbf{A}_c^{-1}$.

Consequences

- Σ is bounded (i.e., $\mathbf{A}$ contains no singular matrix) if and only if the spectral radius $\rho(\mathbf{A}_\Delta) < 1$,
- Σ can be determined in polynomial time.

Interval hull computation

Two (equivalent) formulas for computing the interval hull Σ:

- Hansen–Bliek–Rohn method (1993),
- Ning–Kearfott formula (1997).

Interval hull computation

Two (equivalent) formulas for computing the interval hull Σ:

- Hansen–Bliek–Rohn method (1993),
- Ning–Kearfott formula (1997).

Denote:

$$u := \langle \mathbf{A} \rangle^{-1} \operatorname{mag}(\mathbf{b}),$$
$$d_i := (\langle \mathbf{A} \rangle^{-1})_{ii}, \quad i = 1, \ldots, n,$$
$$\alpha_i := \langle \mathbf{a}_{ii} \rangle - 1/d_i, \quad i = 1, \ldots, n.$$

Interval hull computation

Two (equivalent) formulas for computing the interval hull Σ:

- Hansen–Bliek–Rohn method (1993),
- Ning–Kearfott formula (1997).

Denote:

$$u := \langle \mathbf{A} \rangle^{-1} \operatorname{mag}(\mathbf{b}),$$
$$d_i := (\langle \mathbf{A} \rangle^{-1})_{ii}, \quad i = 1, \ldots, n,$$
$$\alpha_i := \langle \mathbf{a}_{ii} \rangle - 1/d_i, \quad i = 1, \ldots, n.$$

Theorem (Ning–Kearfott, 1997)

$$\boldsymbol{\Sigma}_i = \frac{\mathbf{b}_i + (u_i/d_i - \operatorname{mag}(\mathbf{b}_i))[-1, 1]}{\mathbf{a}_{ii} + \alpha_i[-1, 1]}, \quad i = 1, \ldots, n.$$

Interval hull computation

Two (equivalent) formulas for computing the interval hull Σ:

- Hansen–Bliek–Rohn method (1993),
- Ning–Kearfott formula (1997).

Denote:

$$u := \langle \mathbf{A} \rangle^{-1} \operatorname{mag}(\mathbf{b}),$$

$$d_i := (\langle \mathbf{A} \rangle^{-1})_{ii}, \quad i = 1, \ldots, n,$$

$$\alpha_i := \langle \mathbf{a}_{ii} \rangle - 1/d_i, \quad i = 1, \ldots, n.$$

Theorem (Ning–Kearfott, 1997)

$$\boldsymbol{\Sigma}_i = \frac{\mathbf{b}_i + (u_i/d_i - \operatorname{mag}(\mathbf{b}_i))[-1, 1]}{\mathbf{a}_{ii} + \alpha_i[-1, 1]}, \quad i = 1, \ldots, n.$$

Disadvantage

- We have to safely compute the inverse of $\langle \mathbf{A} \rangle$.

Interval operators

Iteration methods can usually be expressed by an operator $\mathcal{P} : \mathbb{IR}^n \mapsto \mathbb{IR}^n$

$$(\mathbf{x} \cap \Sigma) \subseteq \mathcal{P}(\mathbf{x}).$$

Interval operators

Iteration methods can usually be expressed by an operator $\mathcal{P} : \mathbb{IR}^n \mapsto \mathbb{IR}^n$

$$(\mathbf{x} \cap \Sigma) \subseteq \mathcal{P}(\mathbf{x}).$$

Basically, iterations then can have the plain form $\mathbf{x} \mapsto \mathcal{P}(\mathbf{x})$, or the form with intersections $\mathbf{x} \mapsto \mathcal{P}(\mathbf{x}) \cap \mathbf{x}$.

Interval operators

Iteration methods can usually be expressed by an operator $\mathcal{P} : \mathbb{IR}^n \mapsto \mathbb{IR}^n$

$$(\mathbf{x} \cap \Sigma) \subseteq \mathcal{P}(\mathbf{x}).$$

Basically, iterations then can have the plain form $\mathbf{x} \mapsto \mathcal{P}(\mathbf{x})$, or the form with intersections $\mathbf{x} \mapsto \mathcal{P}(\mathbf{x}) \cap \mathbf{x}$.

Known operators

- The Krawczyk operator

$$\mathbf{x} \mapsto \mathbf{b} + (I_n - \mathbf{A})\mathbf{x}.$$

- Denote by $\mathbf{D}$ the interval diagonal matrix, whose diagonal is the same as that of $\mathbf{A}$, and $\mathbf{A}'$ is used for the interval matrix $\mathbf{A}$ with zero diagonal. The interval Jacobi operator reads

$$\mathbf{x} \mapsto \mathbf{D}^{-1}(\mathbf{b} - \mathbf{A}'\mathbf{x}).$$

- The interval Gauss–Seidel operator is similar to Jacobi, but evaluated raises by evaluating successively.

Limiting enclosures

By $\mathbf{x}^{\mathrm{GS}}$ and $\mathbf{x}^{\mathrm{K}}$ we denote the limit enclosures computed by the interval Gauss–Seidel and Krawczyk methods, respectively.

Limiting enclosures

By $\mathbf{x}^{\mathrm{GS}}$ and $\mathbf{x}^{\mathrm{K}}$ we denote the limit enclosures computed by the interval Gauss–Seidel and Krawczyk methods, respectively.

Theorem

Recall

$$u := \langle \mathbf{A} \rangle^{-1} \operatorname{mag}(\mathbf{b}).$$

We have

$$\mathbf{x}^{\mathrm{GS}} = \mathbf{D}^{-1}(\mathbf{b} + \operatorname{mag}(\mathbf{A}')u[-1, 1]),$$
$$\mathbf{x}^{\mathrm{K}} = \mathbf{b} + \mathbf{A}_{\Delta} u[-1, 1].$$

Moreover,

$$u = \operatorname{mag}(\mathbf{\Sigma}) = \operatorname{mag}(\mathbf{x}^{\mathrm{GS}}) = \operatorname{mag}(\mathbf{x}^{\mathrm{K}}).$$

Limiting enclosures

By $\mathbf{x}^{\mathrm{GS}}$ and $\mathbf{x}^{\mathrm{K}}$ we denote the limit enclosures computed by the interval Gauss–Seidel and Krawczyk methods, respectively.

Theorem

Recall

$$u := \langle \mathbf{A} \rangle^{-1} \operatorname{mag}(\mathbf{b}).$$

We have

$$\begin{aligned} \mathbf{x}^{\mathrm{GS}} &= \mathbf{D}^{-1}(\mathbf{b} + \operatorname{mag}(\mathbf{A}')u[-1,1]), \\ \mathbf{x}^{\mathrm{K}} &= \mathbf{b} + \mathbf{A}_{\Delta} u[-1,1]. \end{aligned}$$

Moreover,

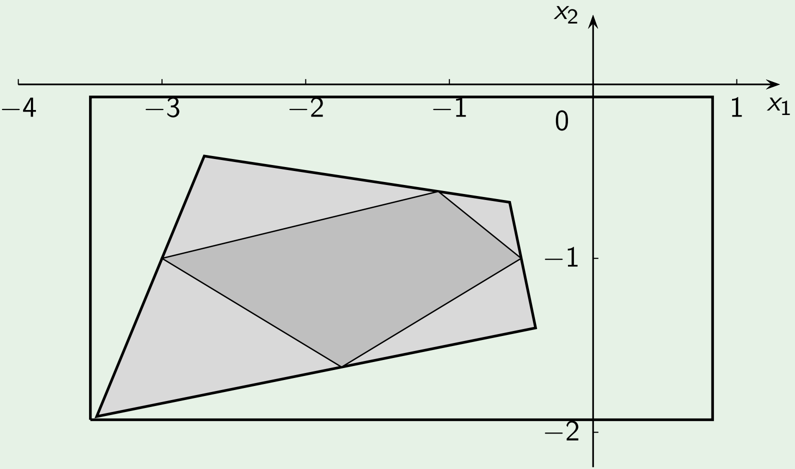
$$u = \operatorname{mag}(\boldsymbol{\Sigma}) = \operatorname{mag}(\mathbf{x}^{\mathrm{GS}}) = \operatorname{mag}(\mathbf{x}^{\mathrm{K}}).$$

Corollary

We have $\Sigma \in [-u, u]$*.*

Example (Typical case)

The solution set, the preconditioned solution set and its enclosure.



New interval operator[1]

Theorem (Hladík, 2014)

Let $\Sigma \subseteq \mathbf{x} \in \mathbb{IR}^n$. *Then*

$$\mathbf{\Sigma}_i \subseteq \frac{\mathbf{b}_i - \sum_{j\neq i} \mathbf{a}_{ij}\mathbf{x}_j + [\gamma_i, -\gamma_i]u_i}{\mathbf{a}_{ii} + \gamma_i[-1,1]}$$

for every $\gamma_i \in [0, \alpha_i]$, *and* $i = 1, \dots, n$, *where*

$$d_i := (\langle \mathbf{A} \rangle^{-1})_{ii}, \quad i = 1, \dots, n,$$
$$\alpha_i := \langle \mathbf{a}_{ii} \rangle - 1/d_i, \quad i = 1, \dots, n.$$

[1] M. Hladík. New operator and method for solving real preconditioned interval linear equations. SIAM J. Numer. Anal., 52(1):194–206, 2014.

New interval operator[1]

Theorem (Hladík, 2014)

Let $\Sigma \subseteq \mathbf{x} \in \mathbb{IR}^n$. *Then*

$$\mathbf{\Sigma}_i \subseteq \frac{\mathbf{b}_i - \sum_{j\neq i} \mathbf{a}_{ij}\mathbf{x}_j + [\gamma_i, -\gamma_i]u_i}{\mathbf{a}_{ii} + \gamma_i[-1, 1]}$$

for every $\gamma_i \in [0, \alpha_i]$, *and* $i = 1, \dots, n$, *where*

$$d_i := (\langle \mathbf{A} \rangle^{-1})_{ii}, \quad i = 1, \dots, n,$$
$$\alpha_i := \langle \mathbf{a}_{ii} \rangle - 1/d_i, \quad i = 1, \dots, n.$$

Remarks

- Generalization of the interval Gauss–Seidel operator (let $\gamma := 0$).
- Its performance depends on computation of u and d.
  Tight lower bounds are sufficient.

[1] M. Hladík. New operator and method for solving real preconditioned interval linear equations. SIAM J. Numer. Anal., 52(1):194–206, 2014.

New interval operator

Theorem

We have

$$u \geq \operatorname{mag}(\mathbf{b}) + \mathbf{A}_\Delta(\operatorname{mag}(\mathbf{b}) + \mathbf{A}_\Delta \operatorname{mag}(\mathbf{b})),$$
$$d_i \geq \underline{d}_i := \overline{a}_{ii}/(1 - ((\mathbf{A}_\Delta)^2)_{ii}), \quad i = 1, \ldots, n.$$

New interval operator

Theorem

We have

$$u \geq \mathrm{mag}\,(\mathbf{b}) + \mathbf{A}_{\Delta}(\mathrm{mag}\,(\mathbf{b}) + \mathbf{A}_{\Delta}\,\mathrm{mag}\,(\mathbf{b}))),$$

$$d_i \geq \underline{d}_i := \overline{a}_{ii}/(1 - ((\mathbf{A}_{\Delta})^2)_{ii}), \quad i = 1, \ldots, n.$$

Remarks

- Both bounds computable in time $\mathcal{O}(n^2)$.
- For $\gamma_i > 0$, it outperforms the interval Gauss–Seidel operator if $\mathbf{x}$ is sufficiently tight.

New interval operator

Theorem

We have

$$u \geq \operatorname{mag}(\mathbf{b}) + \mathbf{A}_\Delta(\operatorname{mag}(\mathbf{b}) + \mathbf{A}_\Delta \operatorname{mag}(\mathbf{b}))),$$
$$d_i \geq \underline{d}_i := \overline{a}_{ii}/(1 - ((\mathbf{A}_\Delta)^2)_{ii}), \quad i = 1, \ldots, n.$$

Remarks

- Both bounds computable in time $\mathcal{O}(n^2)$.
- For $\gamma_i > 0$, it outperforms the interval Gauss–Seidel operator if $\mathbf{x}$ is sufficiently tight.

Efficient implementation of the new operator

Call one iteration of the operator on the initial box $[-u, u]$.

New enclosing method

Algorithm (Magnitude method)

1. Compute $\mathbf{u}$, an enclosure to the solution of $\langle \mathbf{A} \rangle u = \mathrm{mag}(\mathbf{b})$.
2. Calculate $\underline{d}$, a lower bound on d (e.g., by the above theorem).
3. Evaluate

$$\mathbf{x}_i^* := \frac{\mathbf{b}_i + (\sum_{j \neq i} \mathbf{a}_{ij}{}_{\Delta} \overline{u}_j - \gamma_i \underline{u}_i)[-1, 1]}{\mathbf{a}_{ii} + \gamma_i [-1, 1]}, \quad i = 1, \ldots, n,$$

where $\gamma_i := \langle \mathbf{a}_{ii} \rangle - 1/\underline{d}_i$.

New enclosing method

Algorithm (Magnitude method)

1. Compute $\mathbf{u}$, an enclosure to the solution of $\langle \mathbf{A} \rangle u = \mathrm{mag}(\mathbf{b})$.
2. Calculate $\underline{d}$, a lower bound on d (e.g., by the above theorem).
3. Evaluate

$$\mathbf{x}_i^* := \frac{\mathbf{b}_i + (\sum_{j \neq i} \mathbf{a}_{ij\,\Delta} \overline{u}_j - \gamma_i \underline{u}_i)[-1, 1]}{\mathbf{a}_{ii} + \gamma_i [-1, 1]}, \quad i = 1, \ldots, n,$$

where $\gamma_i := \langle \mathbf{a}_{ii} \rangle - 1/\underline{d}_i$.

Theorem

If u and d are calculated exactly, then $\mathbf{x}^* = \mathbf{\Sigma}$.

New enclosing method

Algorithm (Magnitude method)

1. Compute $\mathbf{u}$, an enclosure to the solution of $\langle \mathbf{A} \rangle u = \mathrm{mag}(\mathbf{b})$.
2. Calculate $\underline{d}$, a lower bound on d (e.g., by the above theorem).
3. Evaluate

$$\mathbf{x}_i^* := \frac{\mathbf{b}_i + (\sum_{j \neq i} \mathbf{a}_{ij}{}_{\Delta} \overline{u}_j - \gamma_i \underline{u}_i)[-1,1]}{\mathbf{a}_{ii} + \gamma_i [-1,1]}, \quad i = 1, \ldots, n,$$

where $\gamma_i := \langle \mathbf{a}_{ii} \rangle - 1/\underline{d}_i$.

Theorem

If u and d are calculated exactly, then $\mathbf{x}^* = \mathbf{\Sigma}$.

Theorem

We have $\mathbf{x}^* \subseteq \mathbf{x}^{\mathrm{GS}}$. If $\gamma = 0$, then equality holds.

Numerical experiments

Example

- Randomly generated examples for various dimensions and interval radii.
- The entries of $\mathbf{A}_c$ and $\mathbf{b}_c$ were generated randomly in $[-10, 10]$ with uniform distribution.
- All radii of $\mathbf{A}$ and $\mathbf{b}$ were equal to the parameter $\delta > 0$.
- The computations were carried out in MATLAB with INTLAB.
- Tightness of the computed enclosure $\mathbf{x}$ was measured by

$$\frac{\sum_{i=1}^{n} \mathbf{x}_{i\Delta}}{\sum_{i=1}^{n} \mathbf{\Sigma}_{i\Delta}}.$$

(Thus, the closer to 1, the sharper enclosure.)

Numerical experiments

Example (Tightness of enclosures for randomly generated data)

n	δ	`verifylss`	`Gauss-Seidel`	`magnitude`	`magnitude` ($\gamma = 0$)
5	1	1.1520	1.1510	**1.09548**	1.1196
5	0.1	1.08302	1.01645	**1.00591**	1.0164
5	0.01	1.01755	1.00148	**1.00037**	1.00148
10	0.1	1.07756	1.02495	**1.01107**	1.02474
10	0.01	1.02362	1.00378	**1.00132**	1.00378
15	0.1	1.06994	1.03121	**1.01755**	1.03074
15	0.01	1.02125	1.00217	**1.00047**	1.00216
20	0.1	1.05524	1.03076	**1.02007**	1.02989
20	0.01	1.02643	1.00348	**1.00097**	1.00348
30	0.01	1.02539	1.00402	**1.00129**	1.00401
30	0.001	1.00574	1.00026	**1.000039**	1.000256
50	0.01	1.02688	1.00533	**1.00226**	1.00531
50	0.001	1.00902	1.00051	**1.00011**	1.00051
100	0.001	1.01303	1.00057	**1.00013**	1.00057
100	0.0001	1.0024988	1.0000274	**1.0000022**	1.0000274

Numerical experiments

Example (Computational time in sec. for randomly generated data)

n	δ	`verifylss`	`Gauss-Seidel`	`magnitude`	`magnitude` ($\gamma = 0$)
5	1	3.2903	0.10987	0.004466	**0.003429**
5	0.1	0.004234	0.02937	0.004513	**0.003502**
5	0.01	**0.002342**	0.02500	0.004473	0.003456
10	0.1	0.018845	0.08370	0.004877	**0.003777**
10	0.01	**0.003161**	0.05305	0.004821	0.003799
15	0.1	0.246779	0.21868	0.005212	**0.004162**
15	0.01	0.005403	0.09163	0.005260	**0.004172**
20	0.1	16.9678	0.95238	0.005554	**0.004251**
20	0.01	0.008950	0.15602	0.005736	**0.004622**
30	0.01	0.019111	0.32294	0.006457	**0.005289**
30	0.001	**0.004488**	0.19544	0.006460	0.005260
50	0.01	0.210430	1.01155	0.008483	**0.007062**
50	0.001	0.010190	0.54813	0.008343	**0.006879**
100	0.001	0.044463	2.42025	0.016706	**0.014645**
100	0.0001	**0.013940**	1.48693	0.017089	0.014847

Conclusion

Performance

- The magnitude method overcomes the Gauss–Seidel iteration method with respect to both computational time and sharpness of enclosures.
- Compared to the INTLAB function `verifylss`, the magnitude method produces always tighter enclosures. Unless the input interval data are very narrow, it also overcomes `verifylss` with respect to computational time.

Conclusion

Performance

- The magnitude method overcomes the Gauss–Seidel iteration method with respect to both computational time and sharpness of enclosures.
- Compared to the INTLAB function `verifylss`, the magnitude method produces always tighter enclosures. Unless the input interval data are very narrow, it also overcomes `verifylss` with respect to computational time.

Open problems

- Extension our approach to parametric interval systems,
- Overcoming the assumption $\mathbf{A}_c = I_n$.