Verification of zeros in underdetermined systems

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Introduction

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We assume that an interval function $I_f : \mathcal{B}_m \to \mathcal{B}_n$ is given such that

- For each $B \in \mathcal{B}_m$ it holds that $I_f(B) \supseteq f(B)$, and
- "If the diameter of B is small enough, then the diameter of $I_f(B)$ is small."

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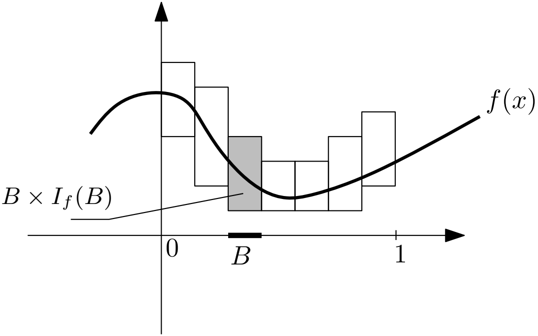

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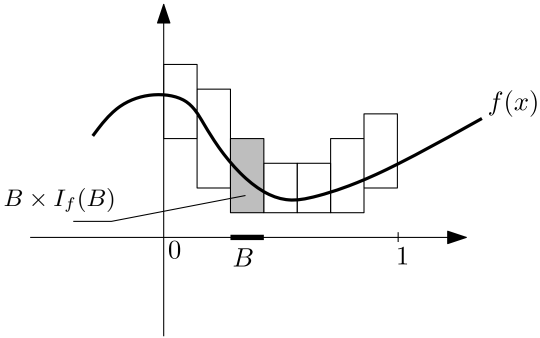

If $f(x) = 0$ has no solution, then the above test eventually succeeds.

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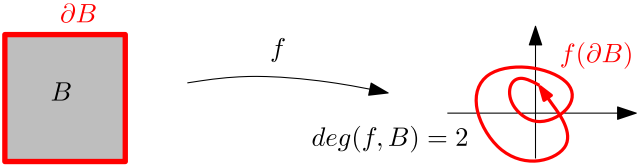

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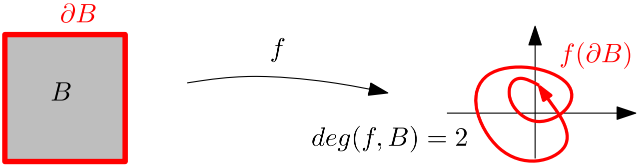

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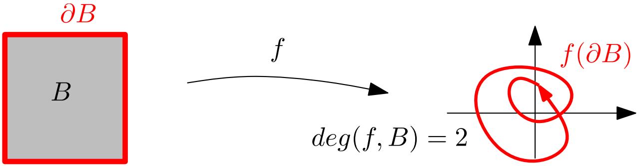

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  [Franek, Ratschan, *Effective Topological Degree Computation Based on Interval Arithmetic*, AMS Math of Compu, 2015]

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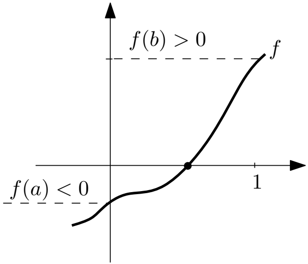

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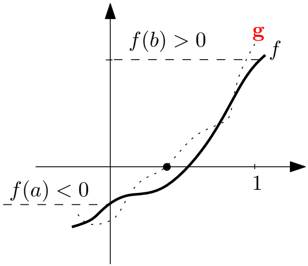

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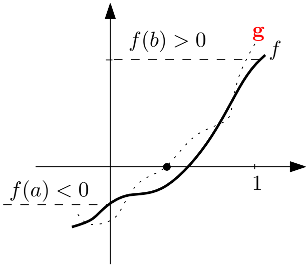

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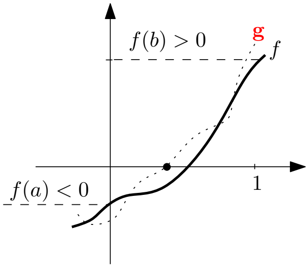

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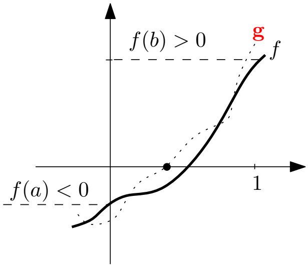

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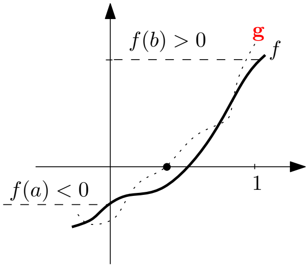

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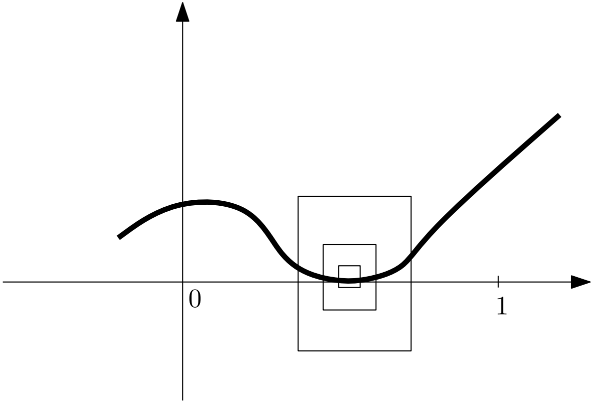
If $m = n$ and $f(x) = 0$ is robust, then the degree test eventually succeeds. [Franek, Ratschan, Zgliczynski, *Quasi-decidability of a Fragment of the First-order Theory of Real Numbers*]

Robustness

If $f(x) = 0$ has a "non-robust" zero, then it may be indetectable via the I_f oracle.



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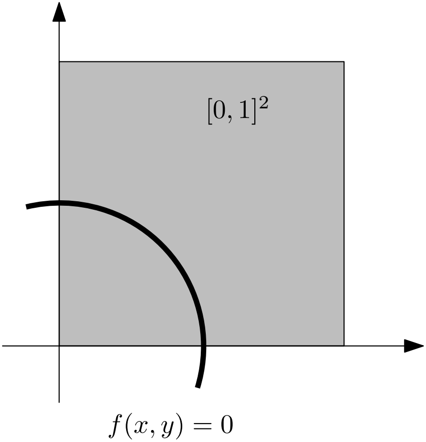

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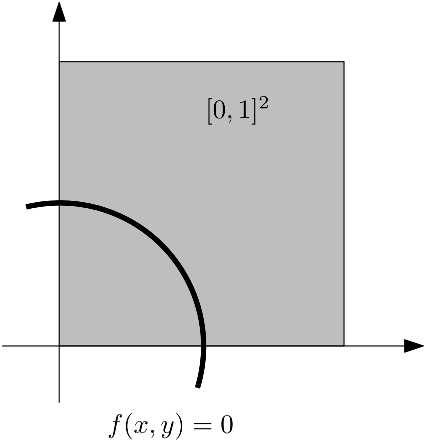
One way how to verify a zero is to fix certain $m - n$ coordinates to be α and analyze $f(\alpha, \cdot) = 0$.



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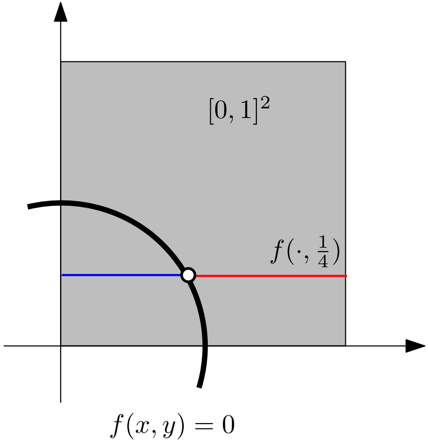

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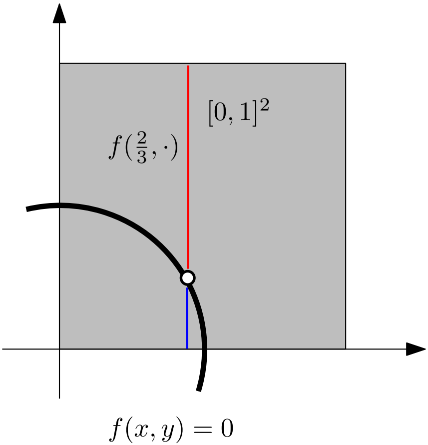

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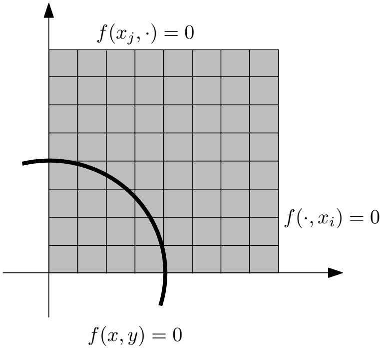

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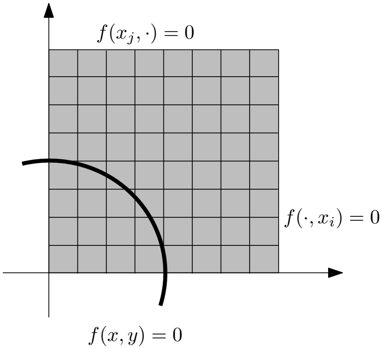

If $df(x)$ is regular in each $x \in f^{-1}(0)$, then the section test eventually succeeds.

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The following function $H : [-1,1]^4 \to \mathbb{R}^3$ has a zero in the origin:

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No analogy in smaller dimensions m, n.

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Theorem

If X is compact, then the above robust satisfiability problem is equivalent to the non-existence of the following extension:

$$\begin{tikzcd} X \arrow[dr, dashed] & \\ f^{-1}(S^{n-1}(r)) \arrow[u, "\cup|"] \arrow[r, "f"] & S^{n-1}(r) \end{tikzcd}$$

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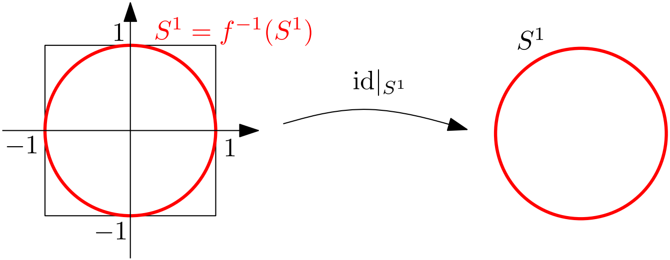

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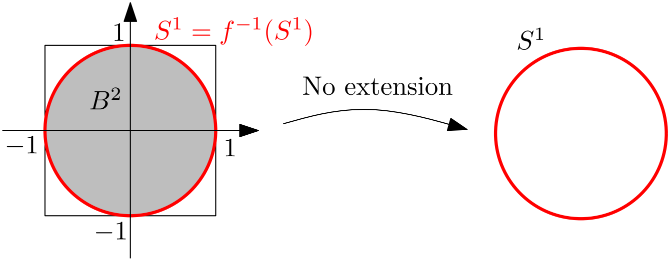

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Theorem

The problem of deciding, for m, n, X, f and $r > 0$ whether or not each continuous g, $\|g - f\| \leq r$, has a zero,

is decidable if $m \leq 2n - 3$ or n is even.

If n is fixed and $m \leq 2n - 3$, then the decision procedure is polynomial.

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If f is given via an interval function I_f, we can algorithmically construct an arbitrary close piecewise linear approximation.

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- Natural instances of zero detection problems with incomplete information? An approximation of f can be given as a multidimensional bitmap, for example.
- Effective implementation? Realistic only in low dimensions.
- If f is given via a formula, does the "Section method" succeeds in most natural cases?

Partial extensions..

The extension problem is solved hierarchically:

$$
\begin{tikzcd}
X \arrow[d, phantom, "\rotatebox{-90}{$\supseteq$}"] \arrow[ddddddr, dotted, bend left] & \\
\color{red}{A \cup X^{(2n-2)}} \arrow[d, phantom, "\rotatebox{-90}{$\supseteq$}"] \arrow[dddddr, dotted, bend left] & \\
A \cup X^{(2n-3)} \arrow[d, phantom, "\vdots"] \arrow[ddddr, dotted] & \\
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A \cup X^{(n)} \arrow[d, phantom, "\rotatebox{-90}{$\supseteq$}"] \arrow[ddr, dotted] & \\
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If an extension $f : A \cup X^{(k)}$ is given and $k < 2n - 3$, we can compute the obstruction to extendability to $A \cup X^{(k+1)}$: the obstruction is an element of $H^{k+1}(X, A, \pi_k(S^{n-1}))$.

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First obstruction $\neq 0$ iff Section method succeeds

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References

[Franek, Krčál. *Robust Satisfiability of Systems of Equations*, to appear in JACM]
(Reductions: robust satisfiability $\longleftrightarrow$ extension problem)

[Franek, Ratschan, Zgliczynski, *Quasi-decidability of a Fragment of the First-order Theory of Real Numbers*, submitted]
(Exploiting topological degree in case $m = n$)

[Cadek, Krcal, Matousek, Vokrinek, Wagner: *Extending Continuous Maps: Polynomiality and Undecidability*, STOC 13]
(Algorithmization of the extension problem)