Recent Results on Cooperative Interval Games

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Definition of an interval game

Classical cooperative game

Classical cooperative game is an ordered pair (N, v), where $N = \{1, 2, \ldots, n\}$ is a set of players and $v : 2^N \to \mathbb{R}$ is a characteristic function. We further assume $v(\emptyset) = 0$.

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Cooperative interval game

Cooperative interval game is an ordered pair (N, w), where $N = \{1, 2, \dots, n\}$ is a set of players and $w : 2^N \to \mathbb{IR}$ (set of all closed real intervals) is a characteristic function. We further assume $w(\emptyset) = [0, 0]$.

Example of an interval game

Consider a game (N, w) with $N = \{1, 2, 3\}$ and w defined as:

X	$w(X)$
$\emptyset$	$[0, 0]$
$\{1\}$	$[1, 2]$
$\{2\}$	$[2, 7]$
$\{3\}$	$[3, 6]$
$\{1, 2\}$	$[4, 4]$
$\{1, 3\}$	$[4, 9]$
$\{2, 3\}$	$[4, 9]$
$\{1, 2, 3\}$	$[5, 7]$

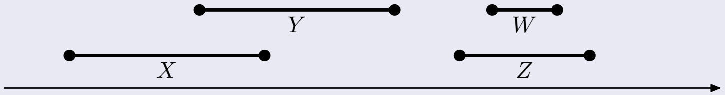
$\preceq$ relation approach

With this approach, payoff distributions and properties of games are formulated in terms $\preceq$ relation.

"Weekly better" partial order on intervals

Interval B is "weakly better" than A ($A \preceq B$) if $\underline{A} \leq \underline{B}$ and $\overline{A} \leq \overline{B}$.

Y is weakly better than X, as is W. On the other hand, W and Z are imcomparable.



Selection approach

Selection

Classical game (H, v_H) is a selection of interval game (G, w_G) if $v_H(S) \in w_G(S)$ for every possible coalition S.
Selections can be interpreted as a possible outcomes.

Consider a previous game (N, w) with player set $N = \{1, 2, 3\}$. (N, z) is a selection of (N, w).

X	$w(X)$	$z(X)$
$\{1\}$	$[1, 2]$	1
$\{2\}$	$[2, 7]$	3
$\{3\}$	$[3, 6]$	4
$\{1, 2\}$	$[4, 4]$	4
$\{1, 3\}$	$[4, 9]$	6
$\{2, 3\}$	$[4, 9]$	6
$\{1, 2, 3\}$	$[5, 7]$	7

Our goal

Surely, selection approach is more natural but does it yield a meaningful concepts and is better over $\prec$ relation approach?

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Our following results show that it does...

Classical core

Classical core

A core of classical game $(N, w) \in IG^N$ is defined as

$$\mathcal{C}(w) := \left\{(x_1, x_2, \ldots, x_n) \mid \sum_{i \in S} x_i \geq w(S),\ \forall S \in 2^N \backslash \{\emptyset\}, \sum_{i \in N} x_i = w(N)\right\}.$$

We have payoff distribution of grand coalition N value in which every coalition gets at least as much as it can achieve on its own so no one has an incentive to split off and not accept this payoff distribution – hence it is stable output - an equilibrium for cooperative game.

Definitions of cores for interval games

There are two definitions of core for interval games.

Interval core

An interval core of $(N, w) \in IG^N$ is defined as

$$\mathcal{C}(w) := \left\{(I_1, I_2, \ldots, I_n) \mid \sum_{i \in S} I_i \succeq w(S),\ \forall S \in 2^N \setminus \{\emptyset\}, \sum_{i \in N} x_i = w(N)\right\}.$$

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Selection core

A selection core of $(N, w) \in IG^N$ is defined as

$$\mathcal{SC}(w) := \bigcup \{ C(v) \mid v \text{ selection of } w \}.$$

"Union of cores of all possible outcomes"- a most natural, but unexplored.

Comparing two definitions of core

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To be able to compare these two sets, we consider a set of all real vectors generated by interval vectors.

Generating function

The function $\mathrm{gen} : 2^{\mathbb{IR}^N} \to 2^{\mathbb{R}^N}$ maps to every set of interval vectors a set of real vectors. It is defined as

$$\mathrm{gen}(S) = \bigcup_{s \in S} \{(x_1, x_2, \ldots, x_n) \mid x_i \in s_i,\ s \in \mathbb{IR}^N\}.$$

Main tool

Core Coincidence Characterization

For every interval game (N, w) we have $\mathrm{gen}(\mathcal{C}(w)) = \mathcal{SC}(w)$ if and only if for every $x \in \mathcal{SC}(w)$ there exist nonnegative vectors $l^{(x)}$ and $u^{(x)}$ such that

$$\sum_{i \in N}(x_i - l_i^{(x)}) \;= \underline{w}(N), \tag{1}$$

$$\sum_{i \in N}(x_i + u_i^{(x)}) = \overline{w}(N), \tag{2}$$

$$\sum_{i \in S}(x_i - l_i^{(x)}) \;\geq \underline{w}(S),\ \forall S \in 2^N \setminus \{\emptyset\}, \tag{3}$$

$$\sum_{i \in S})x_i + u_i^{(x)}) \geq \overline{w}(S),\ \forall S \in 2^N \setminus \{\emptyset\}. \tag{4}$$

Corollaries of Core Coincidence Characterization

General relation of cores

For every interval game (N, w) we have $\mathrm{gen}(\mathcal{C}(w)) \subseteq \mathcal{SC}(w)$.

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Cores coincide if and only if player set has cardinality one or all characteristic function interval are degenerate intervals.

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...and results on *strong core*

Strong core definition

For a game $(N, w) \in IG^N$ the strong core is the union of vectors $x \in \mathbb{R}^n$ such that x is an element of core of every selection of (N, w).

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Characterization

An interval game (N, w) has a nonempty strong core if and only if $w(N)$ is a degenerate interval and the upper game $\overline{w}$ has a nonempty core. Moreover, strong core equals to $C(\overline{w})$.

Main special classes

We introduced **new classes**.

In classical cooperative GT	Interval games analogies	
monotonic	int. monotonic MIG^N	**sel. monotonic** SeMIG^N
superadditive	int. superadditive SIG^N	**sel. superadditive** SeSIG^N
convex	int. convex CIG^N	**sel. convex** SeCIG^N

Every new class is defined as a class for which *every* selection has a corresponding classical game property.

Characterizations

An interval game (N, w) is selection monotonic if and only if for every $S, T \in 2^N$, $S \subsetneq T$, holds $\overline{w}(S) \leq \underline{w}(T)$.

An interval game (N, w) is selection superadditive if and only if for every $S, T \in 2^N$ such that $S \cap T = \emptyset,\ S \neq \emptyset,\ T \neq \emptyset$ holds $\overline{w}(S) + \overline{w}(T) \leq \underline{w}(S \cup T)$.

An interval game (N, w) is selection convex if and only if for every $S, T \in 2^N$ such that $S \not\subseteq T,\ T \not\subseteq S,\ S \neq \emptyset,\ T \neq \emptyset$ holds $\overline{w}(S) + \overline{w}(T) \leq \underline{w}(S \cup T) + \underline{w}(S \cap T)$.

Relations

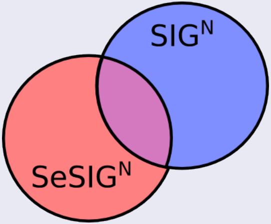
Superadditive Classes Relation Theorem

For every player set N with more than one player, the following assertions hold:

(i) $\mathrm{SeSIG}^N \not\subseteq \mathrm{SIG}^N$,

(ii) $\mathrm{SIG}^N \not\subseteq \mathrm{SeSIG}^N$,

(iii) $\mathrm{SeSIG}^N \cap \mathrm{SIG}^N \neq \emptyset$.



Selection monotonic/convex are incomparable with interval monotonic/convex respectively as well.

Our approach and results have the following advantages:

- Selection core does not omit any stable realizable payoff opposed to interval core.
- Strong core is universal since no matter what happens, we have stable output.
- Our classes have desired properties for all selections. This is in contrast with {superadditive, monotonic, convex} interval games which may contain a selection that does not have a corresponding property at all.

Thank you for your attention.

Questions?