Intervals of Sign Regular Matrices

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Outline

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Notations

$\mathbb{IR}$: set of the compact, nonempty real intervals $[a] = [\underline{a}, \overline{a}]$, $\underline{a} \leq \overline{a}$,
$\mathbb{IR}^n$: set of n-vectors with components from $\mathbb{IR}$, *interval vectors*
$\mathbb{IR}^{n \times n}$: set of n-by-n matrices with entries from $\mathbb{IR}$. *interval matrices*
Elements from $\mathbb{IR}^n$ and $\mathbb{IR}^{n \times n}$ may be regarded as vector intervals and matrix intervals, respectively, w.r.t. the usual entrywise partial ordering, e.g.,

$$\begin{aligned} [A] &= ([a_{ij}])_{i,j=1}^n = \left([\underline{a}_{ij}, \overline{a}_{ij}]\right)_{i,j=1}^n \\ &= [\underline{A}, \overline{A}], \quad \text{where } \underline{A} = \left(\underline{a}_{ij}\right)_{i,j=1}^n, \ \overline{A} = (\overline{a}_{ij})_{i,j=1}^n . \end{aligned}$$

A *vertex matrix* of $[A]$ is a matrix $A = (a_{ij})_{i,j=1}^n$ with $a_{ij} \in \{\underline{a}_{ij}, \overline{a}_{ij}\}$, $i, j = 1, \ldots, n$.

A suitable partial order for the special class of matrices is the *checkerboard order*. For $A, B \in \mathbb{R}^{n\times n}$ define

$$A \leq^* B := (-1)^{i+j} a_{ij} \leq (-1)^{i+j} b_{ij}, \quad i,j = 1,2,\ldots,n.$$

This partial order is related to the usual entry-wise partial order by

$$A \leq^* B \Leftrightarrow A^* \leq B^*, \text{ where } A^* := SAS, \; S := \mathrm{diag}(1,-1,\ldots,(-1)^{n+1}),$$

is the *checkerboard transformation*.

A matrix interval $[\underline{A}, \overline{A}]$ with respect to the usual entry-wise partial order can be represented as an interval $[\downarrow A, \uparrow A]^*$ with respect to the checkerboard order, where

$$(\downarrow A)_{ij} := \begin{cases} \underline{a}_{ij} & \text{if } i+j \text{ is even,} \\ \overline{a}_{ij} & \text{if } i+j \text{ is odd,} \end{cases}$$

$$(\uparrow A)_{ij} := \begin{cases} \overline{a}_{ij} & \text{if } i+j \text{ is even,} \\ \underline{a}_{ij} & \text{if } i+j \text{ is odd.} \end{cases}$$

Systems of linear interval equations $[A]x = [b]$

Solution set $\Sigma := \Sigma([A],[b]) \;:=\; \{x \in \mathbb{R}^n \mid Ax = b,\; A \in [A],\; b \in [b]\}$

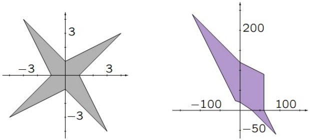
The matrix interval $[A]$ is called *regular* if A is nonsingular for all $A \in [A]$.

Properties of the solution set

- Σ is closed.
- If $[A]$ is regular, then Σ is compact, connected, and convex in each orthant.

$$[A]^H[b] \;:=\; \square\Sigma\left([A],[b]\right)$$

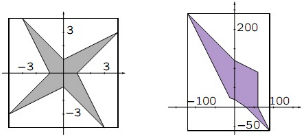
Examples



$$\begin{pmatrix} [2,4] & [-1,1] \\ [-1,2] & [2,4] \end{pmatrix} x = \begin{pmatrix} [-2,2] \\ [-2,2] \end{pmatrix} \qquad \begin{pmatrix} [2,3] & [0,1] \\ [1,2] & [2,3] \end{pmatrix} x = \begin{pmatrix} [0,120] \\ [60,240] \end{pmatrix}$$

Solution sets for Barth-Nuding and Hansen interval systems

Examples (cont'd)



Interval hulls for Barth-Nuding and Hansen interval systems

Important classes of matrices

An n-by-n matrix A is called

- *M-matrix* if A can be written as $A = \alpha I - B$ for some nonnegative matrix B and positive scalar $\alpha > \rho(B)$.
- *inverse M-matrix* if A^{-1} exists and A^{-1} is an M-matrix.
- *inverse nonnegative* if A^{-1} exists and $0 \leq A^{-1}$.
- *positive (semi)-definite* if A is symmetric and all principal minors of A are positive (nonnegative).

- *sign regular (SR)* with signature $\epsilon = (\epsilon_1, \ldots, \epsilon_n)$ if all its minors of order k have sign ϵ_k or are allowed also to vanish for all $k = 1, \ldots, n$.
- *strictly sign regular (SSR)* with signature $\epsilon = (\epsilon_1, \ldots, \epsilon_n)$ if all its minors of order k are nonzero and have sign ϵ_k for all $k = 1, \ldots, n$.
- *almost strictly sign regular (ASSR)* with signature $\epsilon = (\epsilon_1, \ldots, \epsilon_n)$ if A is SR with signature $\epsilon = (\epsilon_1, \ldots, \epsilon_n)$ and any minor is nonzero if and only if the entries on the main diagonal of the corresponding submatrix are nonzero.
- *totally nonnegative (TN)* and *totally positive (TP)* if A is SR and SSR with signature $\epsilon = (1, \ldots, 1)$, respectively.
- *totally nonpositive (t.n.p.)* and *totally negative (t.n.)* if A is SR and SSR with signature $\epsilon = (-1, \ldots, -1)$, respectively.

Inverse nonnegative matrices

Examples of inverse nonnegative matrices

- M-matrices.
- Let $S = \text{diag}\,(1, -1, \ldots, (-1)^{n-1})$. Then for any nonsingular SR matrix A with signature $\epsilon = (\epsilon_1, \ldots, \epsilon_n)$ such that $\epsilon_{n-1} \cdot \epsilon_n = 1$, SAS is inverse nonnegative.
- Let $S = \text{diag}\,(1, -1, \ldots, (-1)^{n-1})$. Then for any nonsingular SR matrix A with signature $\epsilon = (\epsilon_1, \ldots, \epsilon_n)$ such that $\epsilon_{n-1} \cdot \epsilon_n = -1$, $-SAS$ is inverse nonnegative.

Proposition [Kuttler, 1971]

Let $[A] = [\underline{A}, \overline{A}]$ be a matrix interval and $\underline{A}$ and $\overline{A}$ be inverse nonnegative. Then $[A]$ is inverse nonnegative and $\overline{A}^{-1} \leq \underline{A}^{-1}$.

Theorem [Beeck, 1974]

If $[A] \in \mathbb{IR}^{n\times n}$ is inverse nonnegative, then

$$A^H b = \begin{cases} [\overline{A}^{-1}\underline{b},\ \underline{A}^{-1}\overline{b}] & \text{if} \quad 0 \leq \underline{b}, \\ [\underline{A}^{-1}\underline{b},\ \underline{A}^{-1}\overline{b}] & \text{if} \quad 0 \in [b], \\ [\underline{A}^{-1}\underline{b},\ \overline{A}^{-1}\overline{b}] & \text{if} \quad \overline{b} \leq 0. \end{cases}$$

In the general case, one has to solve at most $2n$ linear systems to find $\inf(A^H b)$ and similarly $\sup(A^H b)$.

Matrix Intervals

Interval Property

We say that a class $\mathcal{C}$ of n-by-n matrices possesses the *interval property* if for any n-by-n interval matrix $[A] = [\underline{A}, \overline{A}] = ([\underline{a}_{ij}, \overline{a}_{ij}])_{i,j=1,\ldots,n}$ the membership $[A] \subseteq \mathcal{C}$ can be inferred from the membership to $\mathcal{C}$ of a specified set of its vertex matrices.

Classes of matrices possessing the interval property

- M-matrices or, more generally, inverse-nonnegative matrices [Kuttler, 1971], where only the bound matrices $\underline{A}$ and $\overline{A}$ are required to be in the class;
- inverse M-matrices [Johnson and Smith, 2002], where all vertex matrices are needed;
- positive definite matrices [Bialas and Garloff, 1984], [Rohn, 1994], where a subset of cardinality 2^{n-1} is required (here only symmetric matrices in $[A]$ are considered).

In the following classes of matrices only $\downarrow A$ and $\uparrow A$ are needed:

- *SSR* matrices [Garloff, 1982], [Adm and Garloff].
- The following classes of matrices [Adm and Garloff, 2013], [Adm and Garloff]:
  - nonsingular *ASSR* matrices,
  - nonsingular tridiagonal *SR* matrices,
  - nonsingular totally nonnegative,
  - tridiagonal *TN* matrices,
  - nonsingular totally nonpositive.

Garloff's Conjecture [Garloff, 1982]

[Garloff, 1982]

If $\downarrow A$ and $\uparrow A$ are non-singular and totally nonnegative then the whole matrix interval $[\downarrow A, \uparrow A]^*$ is non-singular and totally nonnegative.

Cauchon Algorithm [Launois and Lenagan, 2014]

We denote by $\leq$ the lexicographic order on $\mathbb{N}^2$, i.e.,

$$(g,h) \leq (i,j) :\Leftrightarrow \ (g < i) \text{ or } (g = i \text{ and } h \leq j).$$

Set $E^\circ := \{1, ..., n\}^2 \setminus \{(1,1)\},\ E := E^\circ \cup \{(n+1,2)\}$.
Let $(s,t) \in E^\circ$. Then
$(s,t)^+ := min\{(i,j) \in E \mid (s,t) \leq (i,j),\ (s,t) \neq (i,j)\}$.

Algorithm

Let $A \in \mathbb{R}^{n,n}$. As r runs in decreasing order over the set E, we define matrices $A^{(r)} = (a_{ij}^{(r)}) \in \mathbb{R}^{n,n}$ as follows.

1. Set $A^{(n+1,2)} := A$.

2. For $r = (s,t) \in E^{\circ}$:

(a) if $a_{st}^{(r^+)} = 0$ then put $A^{(r)} := A^{(r^+)}$.

(b) if $a_{st}^{(r^+)} \neq 0$ then put

$$a_{ij}^{(r)} := \begin{cases} a_{ij}^{(r^+)} - \dfrac{a_{it}^{(r^+)} a_{sj}^{(r^+)}}{a_{st}^{(r^+)}} & \text{for } i < s \text{ and } j < t, \\ a_{ij}^{(r^+)} & \text{otherwise.} \end{cases}$$

3. Set $\tilde{A} := A^{(1,2)}$ is called the matrix obtained from A (by the Cauchon Algorithm).

Example

If $n = 5$ and A is totally positive, then

$$\tilde{A} = \begin{bmatrix} \frac{[12345]}{[2345]} & \frac{[1234|2345]}{[234|345]} & \frac{[123|345]}{[23|45]} & \frac{[12|45]}{[2|5]} & a_{15} \\ \frac{[2345|1234]}{[345|234]} & \frac{[2345]}{[345]} & \frac{[234|345]}{[34|45]} & \frac{[23|45]}{[3|5]} & a_{25} \\ \frac{[345|123]}{[45|23]} & \frac{[345|234]}{[45|34]} & \frac{[345]}{[45]} & \frac{[34|45]}{[4|5]} & a_{35} \\ \frac{[45|12]}{[5|2]} & \frac{[45|23]}{[5|3]} & \frac{[45|34]}{[5|4]} & \frac{[45]}{[5]} & a_{45} \\ a_{51} & a_{52} & a_{53} & a_{54} & a_{55} \end{bmatrix}$$

Theorem [Goodearl, Launois and Lenagan, 2011], [Adm and Garloff, 2013]

- A is totally nonnegative iff $0 \leq \tilde{A}$ and for all $i, j = 1, \ldots, n$
  $\tilde{a}_{ij} = 0 \;\Rightarrow\; \tilde{a}_{ik} = 0 \;\; k = 1, \ldots, j-1, \;$ or $\; \tilde{a}_{kj} = 0 \;\; k = 1, \ldots, i-1.$

$$\tilde{A} = \left[\begin{array}{ccccc} & & & 0 & \\ & \text{or} & \rightarrow & \vdots & \\ & \downarrow & & 0 & \\ & & & & \\ 0 & \ldots & 0 & \boxed{0} & \\ & & & & \\ & & & & \end{array}\right]$$

- If A is totally nonnegative matrix then A is nonsingular iff $0 < \operatorname{diag}(\tilde{A})$.

Theorem [Adm and Garloff, 2013]

Let A, B be nonsingular and totally nonnegative matrices and let $A \leq^* Z \leq^* B$. Then

1. $\tilde{A} \leq^* \tilde{Z} \leq^* \tilde{B}$;
2. Z is nonsingular and totally nonnegative;
3. if A, B possess the same pattern of zero minors then Z has this pattern, too.

The assumption of nonsingularity of certain principal minors cannot be relaxed:

$$A := \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \leq^* Z := \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \leq^* B := \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

totally nonnegative   has a negative minor   totally nonnegative

Corollary [Adm and Garloff, 2013]

Let $A, B, Z \in \mathbb{R}^{n,n}$ with $A \leq^* Z \leq^* B$. If A, B are totally nonnegative and

$$A[2, \ldots, n] \text{ and } B[2, \ldots, n]$$

or

$$A[1, \ldots, n-1] \text{ and } B[1, \ldots, n-1]$$

are nonsingular, then Z is totally nonnegative, too.

Open Problem

Conjecture [Adm and Garloff]

Assume that $\downarrow A$ and $\uparrow A$ are nonsingular and SR matrices, then $[\downarrow A, \uparrow A]^*$ is nonsingular and SR?

A partial result

It was shown in [Garloff, 1996] that the conclusion is true if we consider instead of the two bound matrices a set of vertex matrices with the cardinality of at most 2^{2n-1} (n being the order of the matrices).

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