Computing exact bounds for the solution set of parametric interval linear systems

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Introduction

Consider linear algebraic systems involving linear dependencies between a number of interval parameters $p = (p_1, \ldots, p_k) \in (\mathbf{p}_1, \ldots, \mathbf{p}_k)$

$$\begin{aligned} A(p)x &= b(p) \\ A(p) := A_0 + \sum_{i=1}^{k} p_i A_i, & \qquad b(p) := b_0 + \sum_{i=1}^{k} p_i b_i, \end{aligned} \tag{1}$$

where $A_i \in \mathbb{R}^{n \times n}$, $b_i \in \mathbb{R}^n$, $i = 0, \ldots, k$. Performing worst-case analysis of uncertain systems one is interested in the parametric (united) solution set of the system (1)

$$\Sigma^p = \Sigma(A(p), b(p), \mathbf{p}) \;:=\; \{x \in \mathbb{R}^n \mid (\exists p \in \mathbf{p})(A(p)x = b(p))\}\,. \tag{2}$$

A parametric solution set (2), in general, has a complicated structure. It is nonconvex even in a single orthant. The boundary of Σ^p consists of parts of polynomials that may have arbitrary high degree. This causes difficulties in computing bounds of the solution set.

There are several methods for calculating lower and upper bounds for each component of the solution set, if the latter is bounded. Most of these methods require that the parametric matrix $A(p)$ be strongly regular on $\mathbf{p}$, which restricts the scope of their applicability. The obtained bounds often overestimate the solution set considerably, especially for large parameter intervals.

We are interested in computing the exact interval hull $\square\Sigma^p$ of the solution set. For a bounded set Σ^p, $\square\Sigma^p := \bigcap\{u \in \mathbb{IR}^n \mid \Sigma^p \subseteq [u]\}$. This is an NP-hard problem even in the special case when Σ^p has linear boundary.

Proposed methodology

In [1] the boundary of a nonempty solution set Σ^p is described by parts of parametric hypersurfaces. The latter are defined by n parametric coordinate functions $x_i(q) := \{A^{-1}(q)b(q)\}_i$ depending on $n-1$ parameters $q \in \mathbb{R}^{n-1}$, $q \subseteq p$, $q \in \mathbf{q}$. Furthermore, following the same methodology, the projection of Σ^p on a 2-dimensional coordinate space can be represented by parts of parametric hypersurfaces depending on only one parameter.

Basing on the above representation of the boundary of a parametric solution set, computing the interval hull of $\{\Sigma^p\}_i$, $i = 1, \ldots, n$, is reduced to solving $\binom{k}{m}2^{k-m+1}$ constrained global optimization problems with corresponding objective functions $x_i(q)$ depending on m interval parameters, where m can be chosen to be any number $1 \le m \le n-1$. For $m = 1$, the corresponding constrained global optimization problems can be solved exactly in exact arithmetic and software tools which provide that are available.

We discuss various aspects (pros and cons) of the proposed methodology and its applicability along with some numerical examples.

References

[1] E. D. Popova and W. Krämer, Visualizing parametric solution sets, *BIT Numerical Mathematics* 48:95–115, 2008.