Viability kernel computation based on interval methods

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Introduction

Since viability theory has been introduced by Jean-Pierre Aubin [1], almost exclusively discrete methods have been developed to approximate the viability kernel. We approached the computation of viability kernel with several methods based on interval analysis. Using guaranteed integration, we are able to compute a guaranteed kernel, which is not or hardly achievable with discrete methods.

Problem statement

Let us consider a dynamic system $\dot{\mathbf{x}} = f(\mathbf{x}, u)$, where x is the state vector, f the evolution function and u the control vector. We assume that $u \in \mathcal{U}$, where $\mathcal{U}$ is the set of possible control of the system. Let us define the flow map $\varphi(\mathbf{x_0}, u, t) = x(t)$, where $\mathbf{x}(t)$ is the solution to the evolution function with the initial condition $\mathbf{x}(0) = \mathbf{x_0}$. Given a subset $\mathcal{K}$ of the state space of the system, the viability kernel problem consists of characterizing the subset $\mathcal{C}$ of $\mathcal{K}$ such as for all state vector of $\mathcal{C}$ the system can stay in $\mathcal{K}$. Thus, we have

$$\mathcal{C} = \{\mathbf{x_0} \in K, \exists u \in \mathcal{U} | \forall t, \varphi(x_0, u, t) \in \mathcal{K}\} \quad (1)$$

Resolution Tools

We characterize the viability kernel using the following tools:

- $\mathcal{C}$ is approximated with an inner approximation $\mathcal{C}^-$ and an outer approximation $\mathcal{C}^+$, such as $\mathcal{C}^- \subset \mathcal{C} \subset \mathcal{C}^+$ [3].
- Lyapunov theory [2] is used to find ellipsoids of attraction of the system used to initialize $\mathcal{C}^-$.
- CSP programming is used to contract on the ellipsoids of attraction.
- Guaranteed integration of the evolution function is used to inflate $\mathcal{C}^-$ and reduce $\mathcal{C}^+$ such as the enclosure of $\mathcal{C}$ is thiner. If there exist a couple (u, t) such as $[\varphi]([\mathbf{x_0}], u, t) \subset \mathcal{C}^-$, with $[\mathbf{x_0}]$ a subset of $\mathcal{K} \backslash \mathcal{C}^-$, we add $[\mathbf{x_0}]$ to $\mathcal{C}^-$. If there exist t such as $[\varphi]([\mathbf{x_0}], \mathcal{U}, t) \cap \mathcal{K} = \emptyset$, which means no control exists such as the system can stay in $\mathcal{K}$ over the time, we remove $[\mathbf{x_0}]$ from $\mathcal{C}^+$.

Example

Let consider the "Car on the hill" two dimensional problem. The state equations are the following:

$$\begin{cases} \dot{x_1} = x_2 \\ \dot{x_2} = -9.81 sin(g(x_1)) - 0.7x_2 + u \end{cases} \tag{2}$$

where $g(s) = (1.1sin(1.2s) - 1.2sin(1.1s))/2$ represents the shape of the hill. Using Lyapunov theory, we are able to find ellipsoids of attraction. Then we contract on these ellipsoid as shown on figure 1.

We use the green area to initialize $\mathcal{C}^-$, and we initialize $\mathcal{C}^+$ with $\mathcal{K}$ which corresponds the green and blue areas. Then, we integrate elements of the blue area over the time, trying to remove them from $\mathcal{C}^+$ or to add them to $\mathcal{C}^-$.

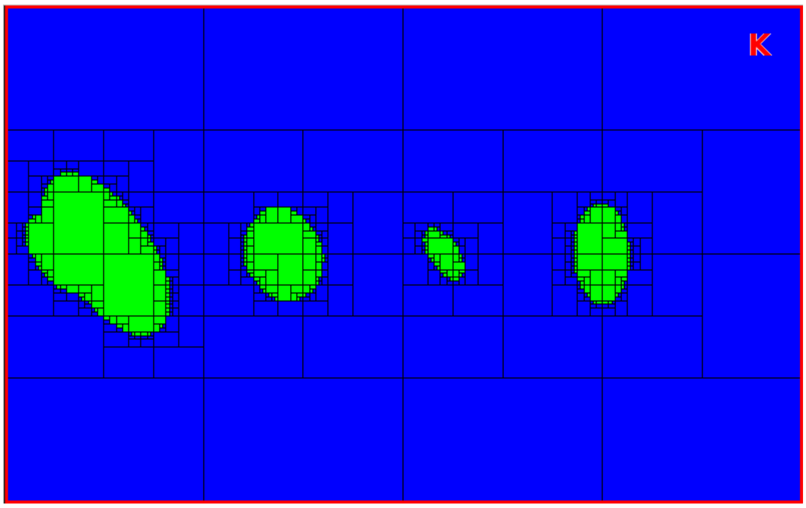

Figure 1: Results of contraction on the ellipsoids of stability. There is two ellipsoids overlapping on the left.

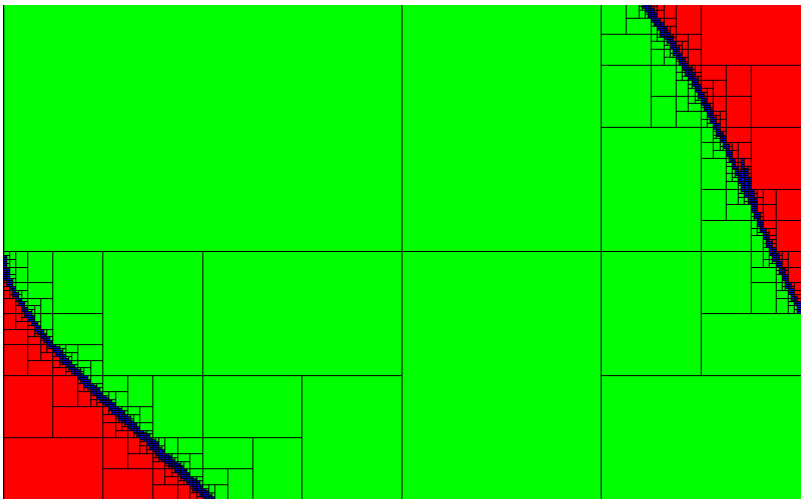

Figure 2: Computation of the viability kernel.

Figure 2 shows the result we obtain. $\mathcal{C}^-$ is shown in green, $\mathcal{C}^+$ corresponds to the green and blue areas. The red area contains the state vectors such that there is no feasible control such that the system always stays inside $\mathcal{K}$. The edge of the viability kernel is enclosed in the blue region, as $\mathcal{C}^- \subset \mathcal{C} \subset \mathcal{C}^+$. As the integration is guaranteed, the approximation of the viability kernel is also guaranteed.

References

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