Validated Simulation of Differential Algebraic Equations*

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Introduction

Our recent results on validated simulation of ordinary differential equations (ODE) with implicit Runge-Kutta schemes [1] lead us to go up in complexity of kind of differential equations. Indeed, we are able to simulate ODE with interval parameters which is one of the requirement for our solver of differential algebraic equation (DAE). We currently focus on the DAE in Hessenberg index 1 form, that is

$$\dot{\mathbf{y}} = f(t, \mathbf{x}, \mathbf{y}), \tag{1a}$$

$$0 = g(t, \mathbf{x}, \mathbf{y}) \ . \tag{1b}$$

In Equation (1) $\mathbf{y}$ is the state variable and $\mathbf{x}$ is the algebraic variable (without an expression for its derivative) and $\dot{\mathbf{y}}$ stands for the time derivative of $\mathbf{y}$. This kind of *DAE* is common and used by a majority of simulation tools as Simulink and Modelica-like software.

A simulation procedure for ODE consists in two phases repeated at each simulation step k, starting from a given guaranteed initial value $[\mathbf{y}_k]$ at time instant t_k then compute an enclosure $[\mathbf{y}_{k+1}] \ni \mathbf{y}(t_{k+1})$ in function of $[\mathbf{y}_k] \ni \mathbf{y}(t_k)$ with the step-size $h_k = t_{k+1} - t_k$.

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The major issue in the validated integration of DAE is the consistency of the initial values [4]. The additional constraints, Equation (1b), to be satisfied by the differential and the algebraic values are generally obtained by the Pantelides algorithm [3]. One attempt was made in order to solve DAE with guarantee by using an approximation of the solution and by adding a "post" consistency verification [2]. This approach produced mitigated results, and we propose a new approach in the following.

Main idea

Our method is based on the ability of the interval representation to enclose a set of solutions and use two contractors to reduce this enclosure around the solution.

An enclosure

Firstly, we do need the guaranteed enclosures of the solution of the differential equations (Equation (1a)), denoted by $[\tilde{\mathbf{y}}_k]$, and of the solution of the algebraic constraints (Equation (1b)), denoted by $[\tilde{\mathbf{x}}_k]$, at each step k of integration process. These enclosures are obtained with a novel operator mixing a classical Picard-Lindelöf operator with $[\tilde{\mathbf{x}}_k]$ as a $\forall$-parameter and a parametric Krawczyk operator with $[\tilde{\mathbf{y}}_k]$ as a $\forall$-parameter. The goal is then to find a post fixpoint simultaneously satisfying $[\tilde{\mathbf{y}}_k]$ and $[\tilde{\mathbf{x}}_k]$. This operator prove the existence and the unicity of the solution for the dynamical part for all values of the algebraic variable and by the way the fulfillment of the constraints, whatever the state variable.

Two contractors

After obtaining these enclosures, we have to reduce $[\tilde{\mathbf{y}}_k]$ around the solution $\mathbf{y}(t_{k+1})$. It is done with the help of our powerful validated implicit Runge-Kutta schemes [1]. Essentially, we used a validated Radau quadrature IIA, known for its efficiency and stability on DAE.

This scheme is able to manage with an interval parameters, such as $[\tilde{\mathbf{x}}_k]$. After that, the second contractor is used to reduce $[\tilde{\mathbf{x}}_k]$ around the solution $\mathbf{x}(t_{k+1})$. We combine for this purpose the Krawczyk used in the first step and a forward/backward contractor.

Main results

We solve the pendulum problem in index 1 form whose dynamics is given by

$$f: \quad \begin{cases} \dot{p} = u \\ \dot{q} = v \\ m\dot{u} = -p\lambda \\ m\dot{v} = -q\lambda - g \end{cases}$$

associated with constraint

$$g: \quad 0 = m(u^2 + v^2) - gq - \ell^2\lambda \ .$$

In functions f and g, m is the mass of the pendulum, ℓ is the length of the rod, u and v are Cartesian coordinates of the mass while p and q stand for the angular speed, g is the gravity force and λ stands for the Lagrange multiplier.

The simulation time is set to 1.6 seconds and with the initial conditions given by $p(0) = 1, q(0) = 0, u(0) = 0, v(0) = 0$, and $\lambda(0) \in [-0.01, 0.01]$. The consistency is verified with Krawczyk which gives $\lambda(0) \in [-0, 0]$. The trajectory computes by our method is given in Figure 1. This simulation takes about ten minutes with a maximal diameter of 0.02 for the final solution.

Conclusion

We presented in this abstract the first solid approach for the validated integration of the DAE under the Hessenberg-index 1 form. The first results are already interesting even if many issues have been opened to obtain an efficient tool.

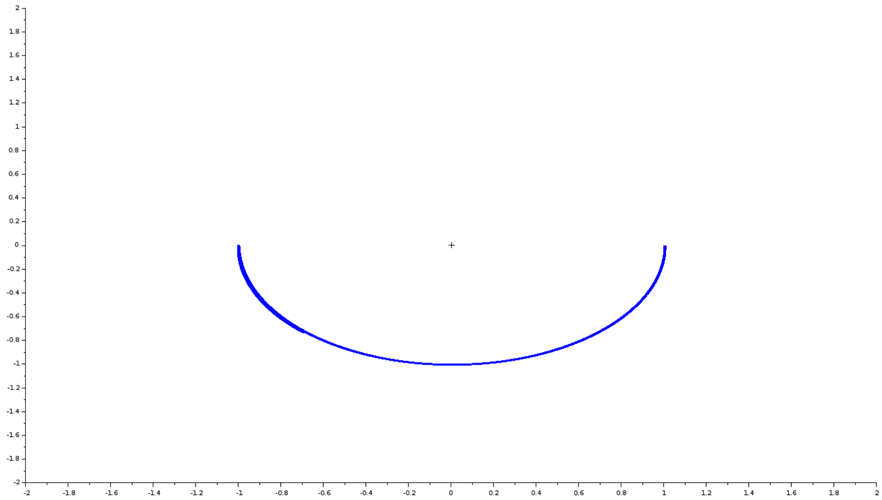

Figure 1: Trajectory of the pendulum.

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