Variants of the Segment Number of a Graph

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Alexander Wolff

Julius-Maximilians-Universität Würzburg, Germany

Measures of Visual Complexity

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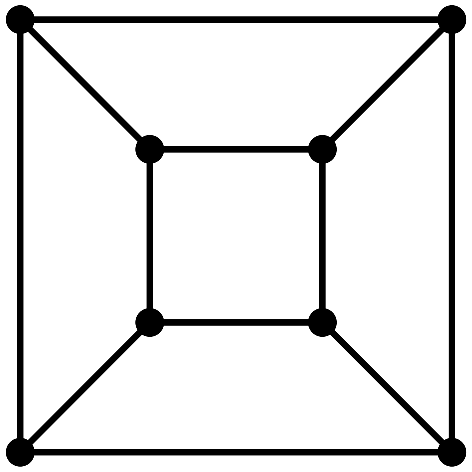
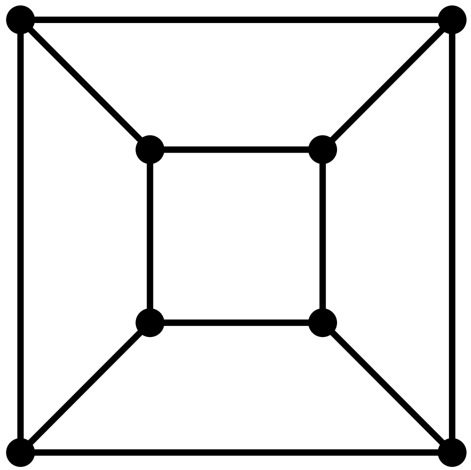


Slope number

[Wade & Chu 1994]

Measures of Visual Complexity



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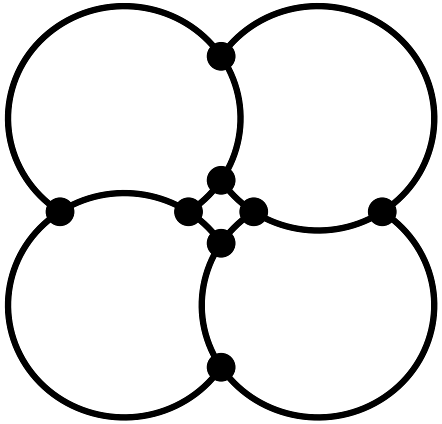

Arc number

[Schulz 2015]

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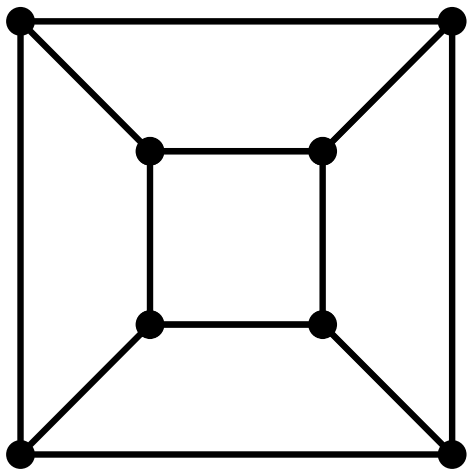

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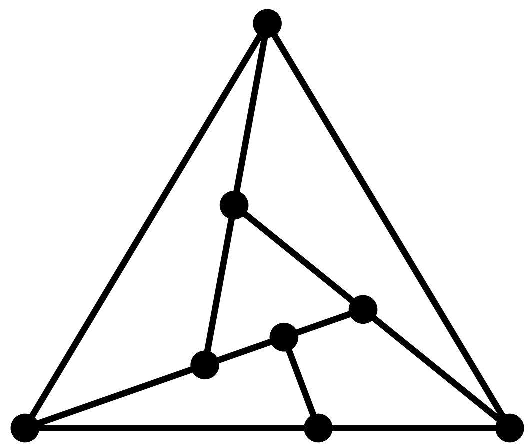
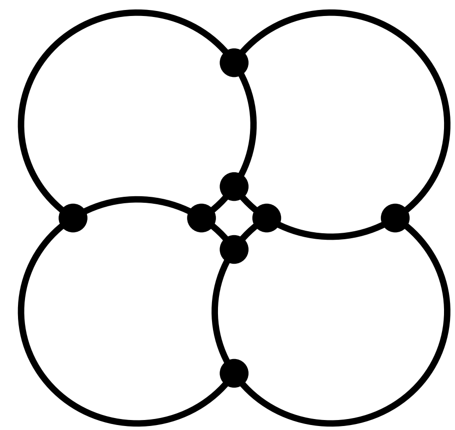


Segment number ($\text{seg}_2(G)$)
[Dujmović et al. 2007]



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Line cover number

[Chaplick et al. 2016]

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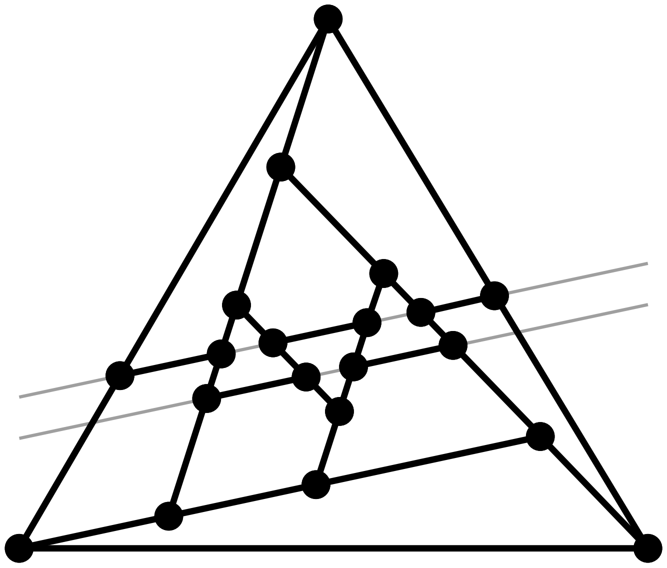

$\rho_2^1(G)$
[Scherm 2016]

Line cover number
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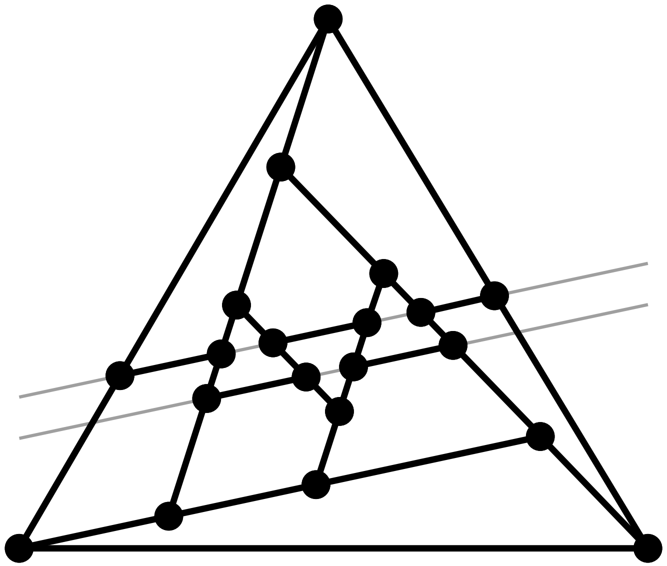

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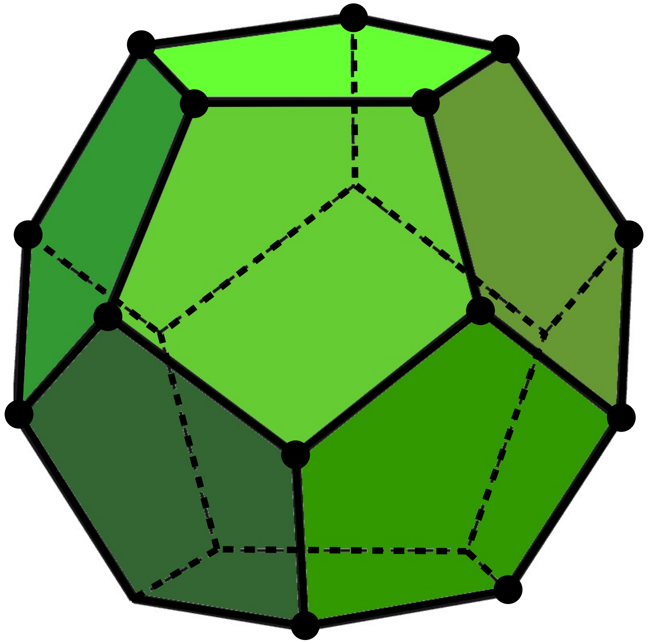

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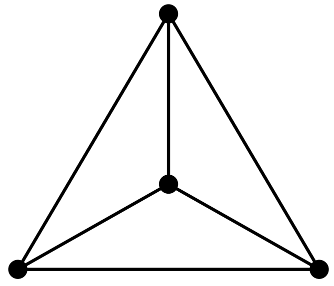

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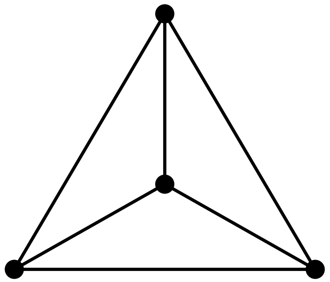

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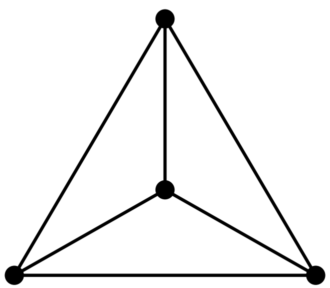

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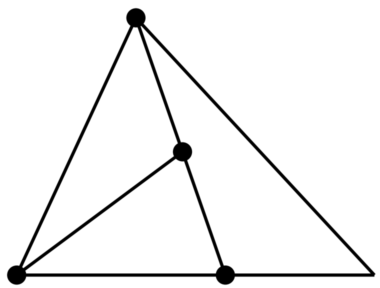

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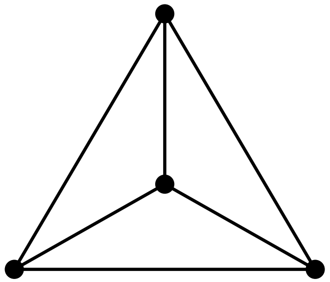

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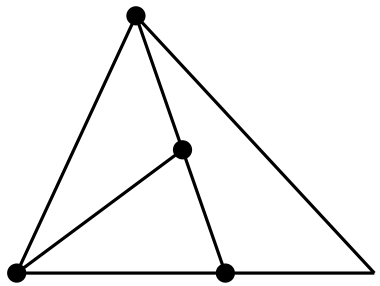

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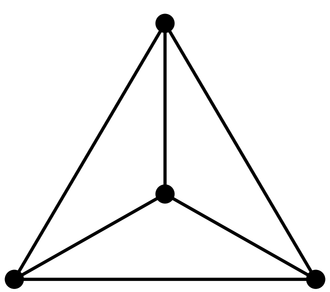

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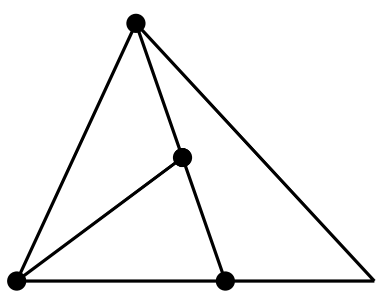

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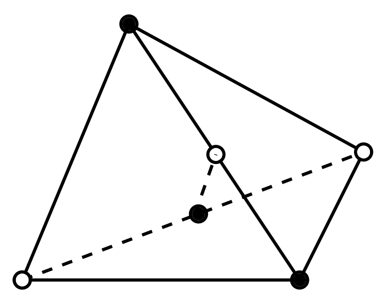

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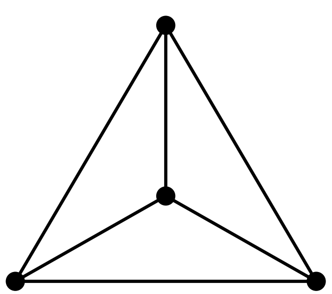
$\text{seg}_\times(G)$, where drawings are 2D, crossings are OK, but no bends and no overlaps.



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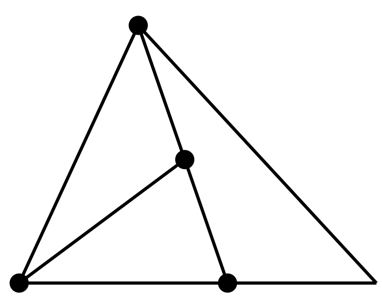

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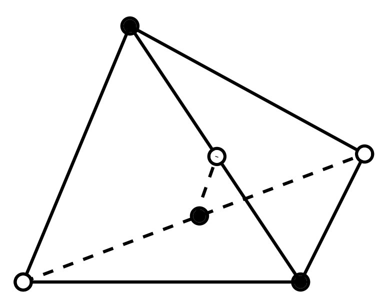

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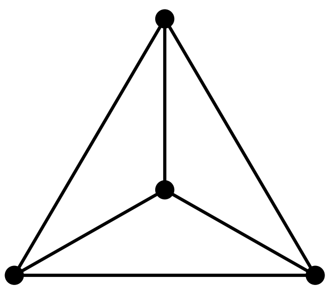

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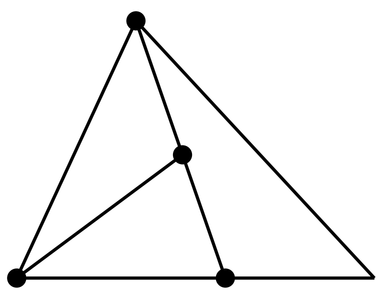

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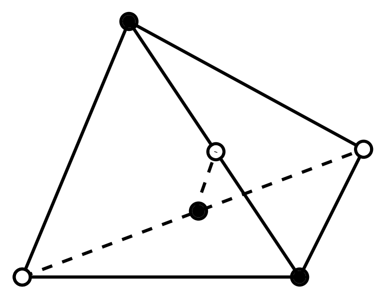

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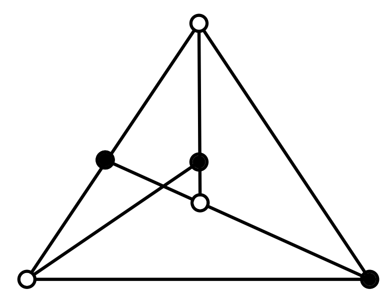

$\mathrm{seg}_\times(K_{3,3}) = 6$

Relations Between Segment Number Variants

$\mathrm{seg}_{\times}(G) \leq \mathrm{seg}_3(G)$

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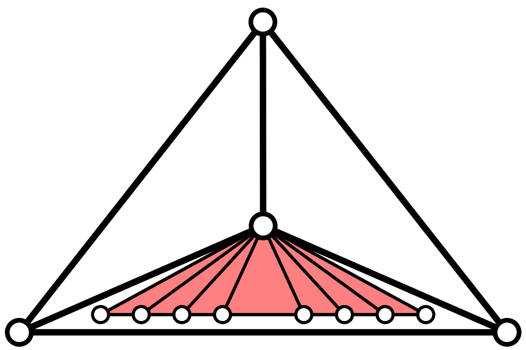

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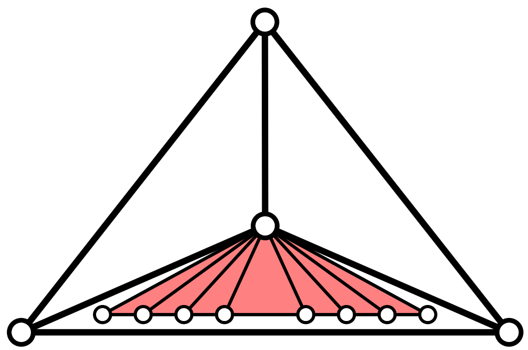

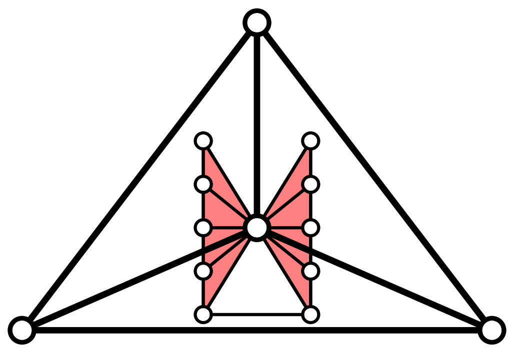

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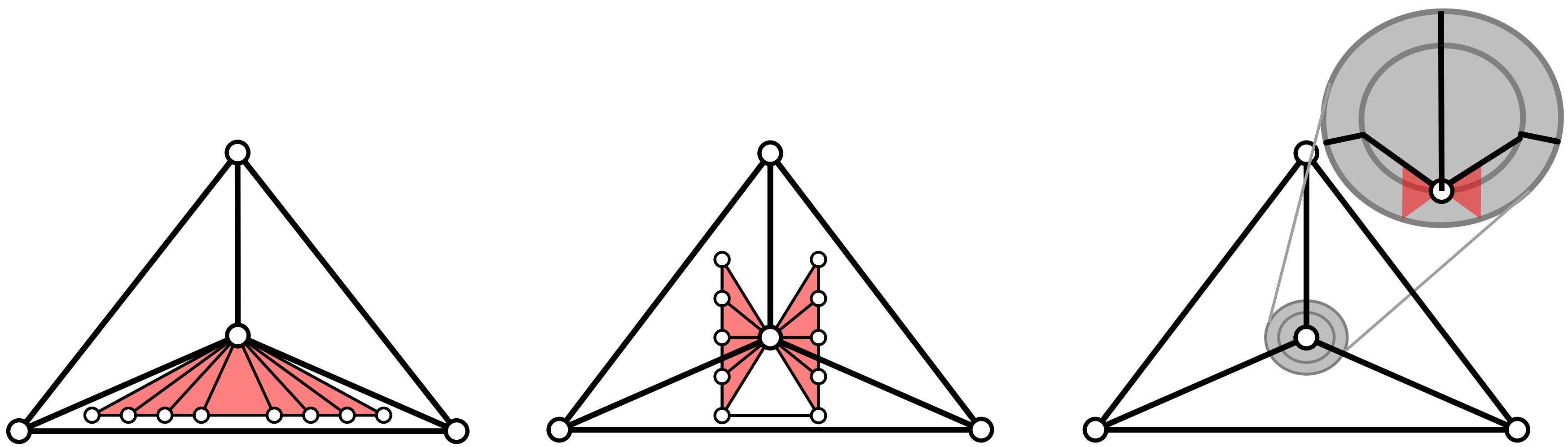

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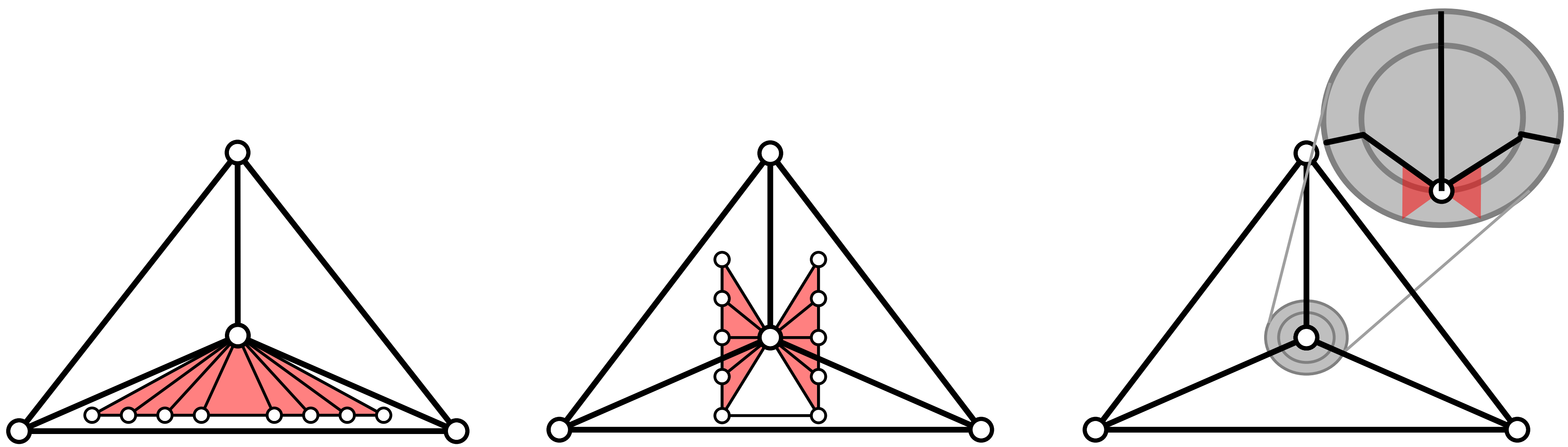

Open Problem. Find upper bounds for $\mathrm{seg}_2(G)/\mathrm{seg}_{3,\times,\angle}(G)$ for planar G.

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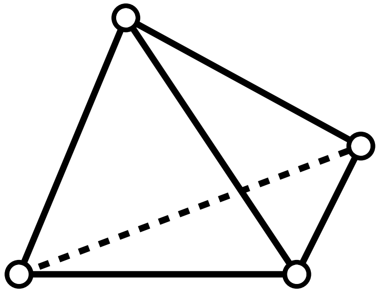
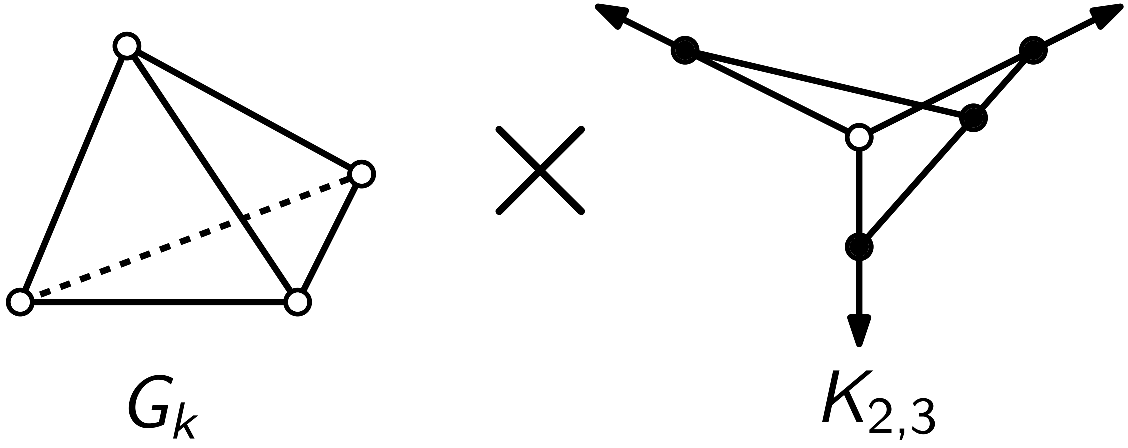


G_k

Relations Between Segment Number Variants



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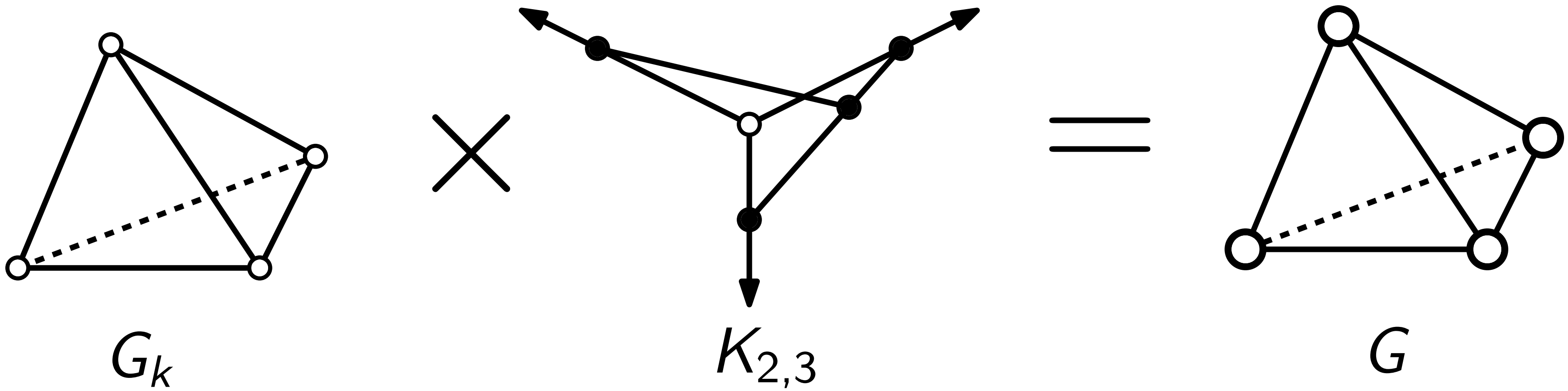

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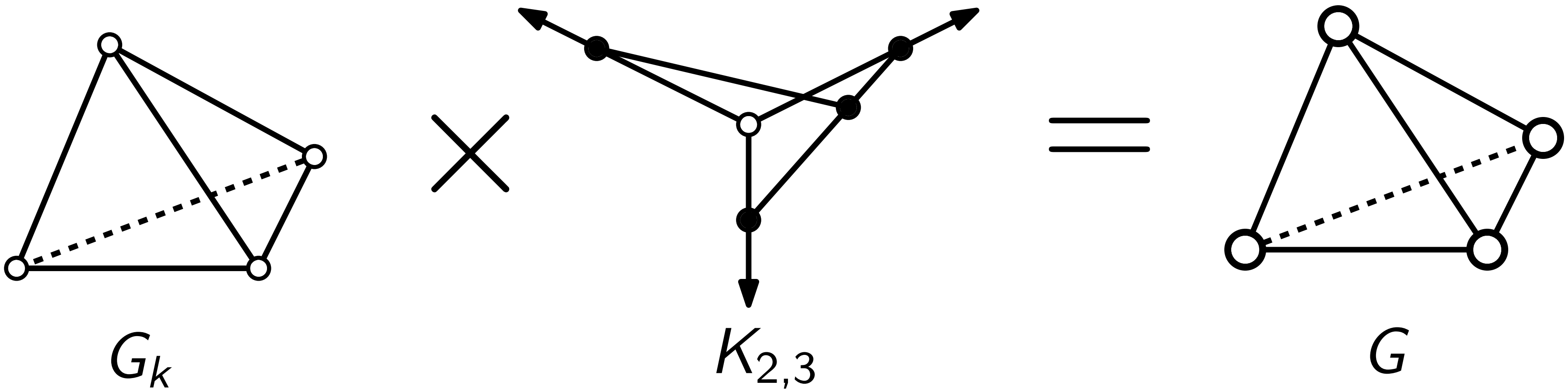

$$\frac{\mathrm{seg}_3(G)}{\mathrm{seg}_\times(G)} = \frac{7k/2}{5k/2+3} \to \frac{7}{5}.$$

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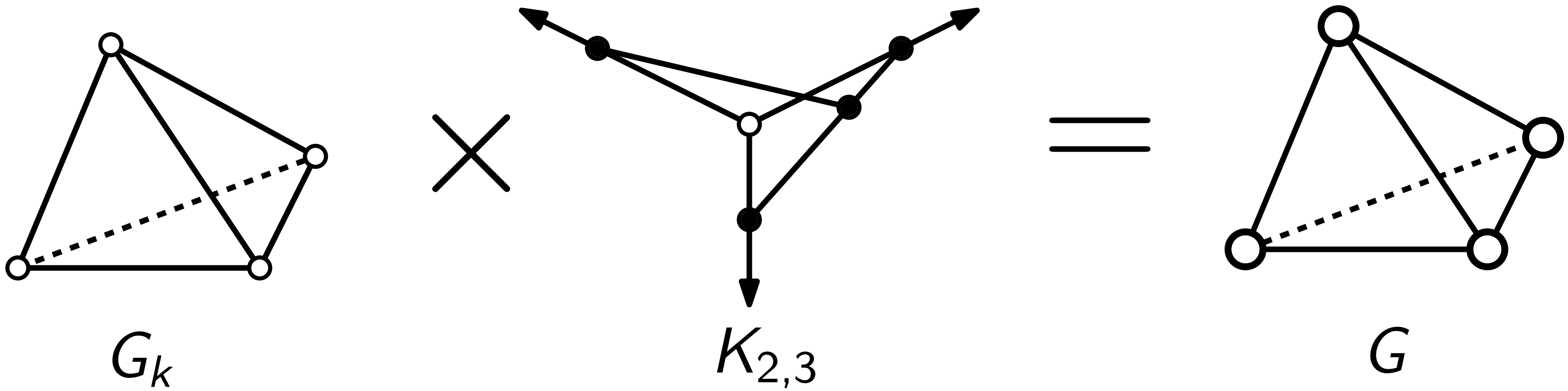

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Open Problem. Can you do better?

Bounds on segment numbers of cubic graphs

G is a cubic graph with $n \geq 6$ vertices.
$n/2 \leq \text{seg}_{2,3,\angle,\times}(G) \leq 3n/2$ and $\text{seg}_{2,3,\angle,\times}(\sqcup K_4) = 3n/2$.

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γ	$\text{seg}_2(G)^*$	$\text{seg}_3(G)$	$\text{seg}_\angle(G)^*$	$\text{seg}_\times(G)$
1	$5n/6..3n/2$	$5n/6^*..7n/5$	$5n/6..3n/2$	$5n/6^*..7n/5$
2	$3n/4..3n/2$	$5n/6..7n/5$	$3n/4..n+1$	$3n/4^*..n+2$
3	$n/2+3^{**}$	$7n/10..7n/5$	$n/2+3$	$n/2..n+2$
H	$3n/4..3n/2$	$5n/6..n+1$	$3n/4..n+1$	$3n/4^*..n+2$

* For planar G.

** by [Durocher et al. 2013; Igamberdiev et al. 2017]

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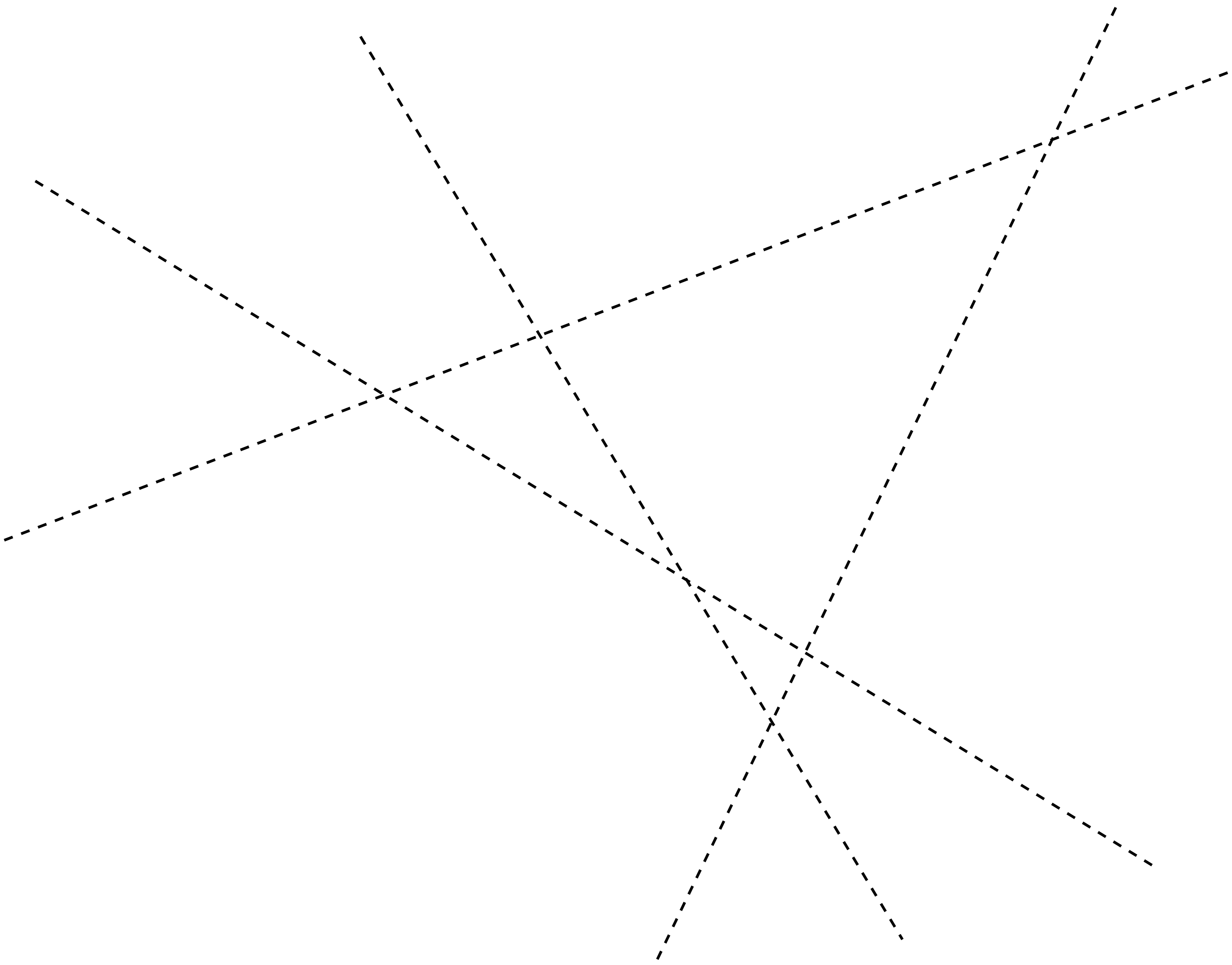
- $\text{seg}_3(G) \leq k$,
- $\text{seg}_\angle(G) \leq k$,
- $\text{seg}_\times(G) \leq k$.

Arrangement Graphs



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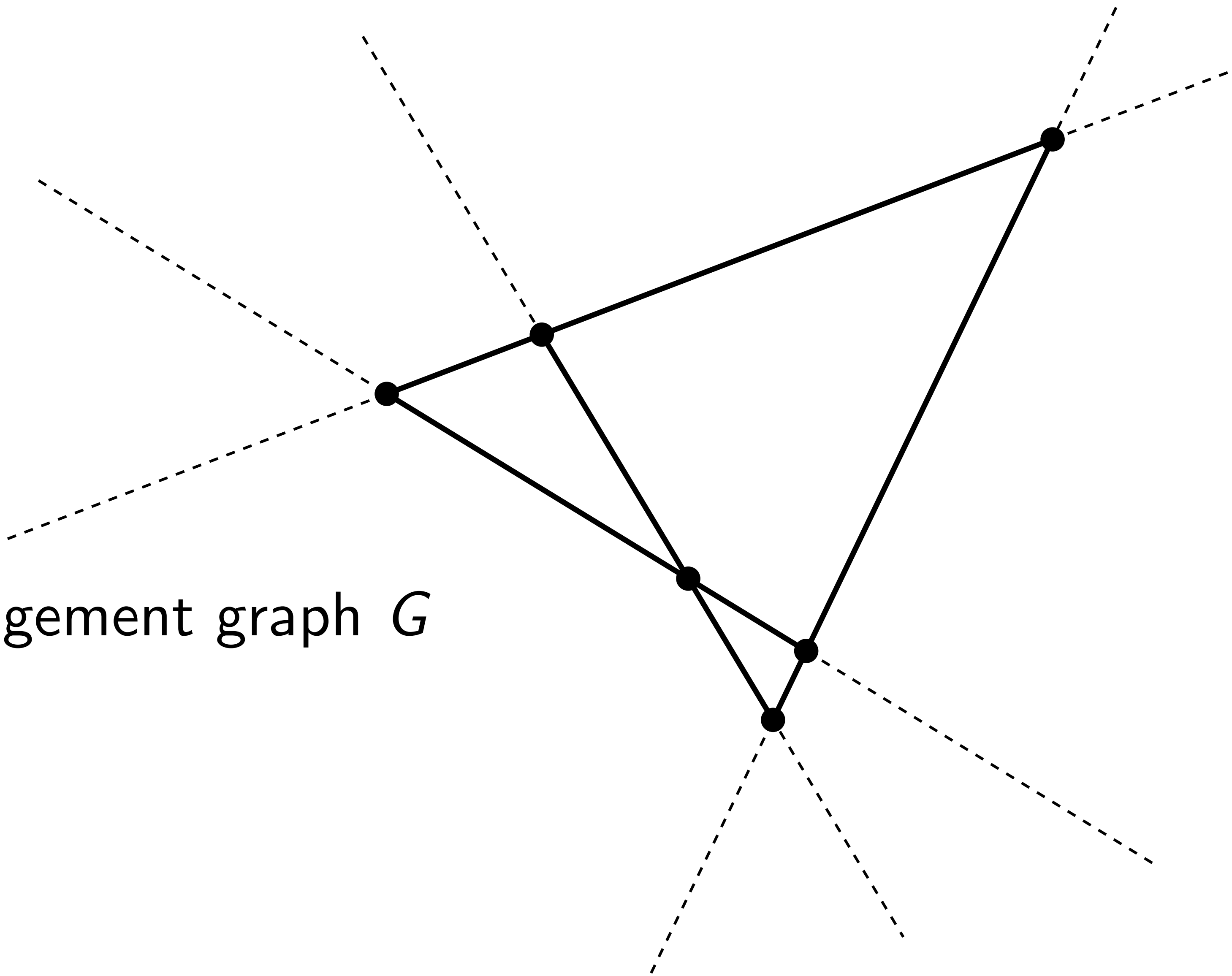
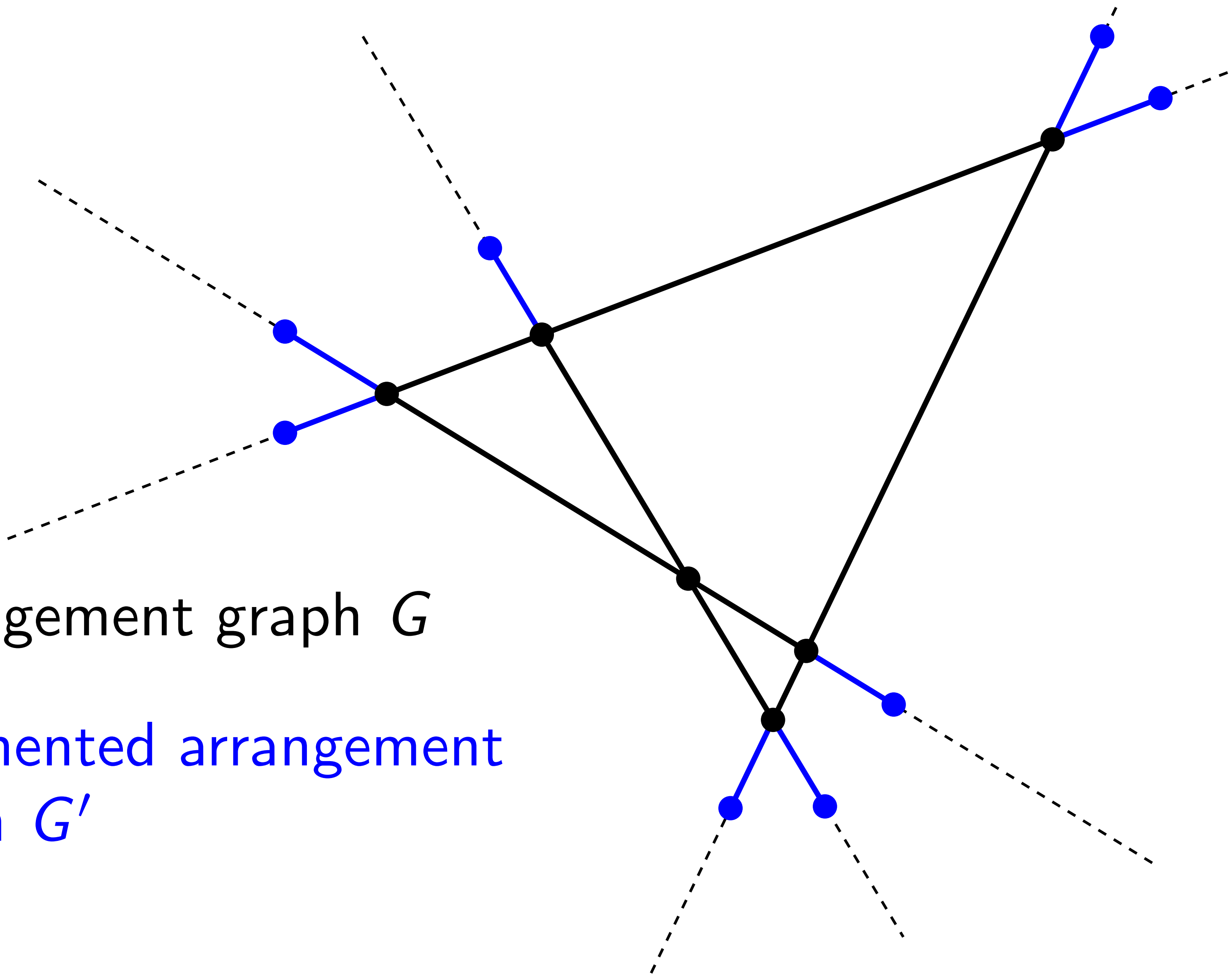


Arrangement graph G

Arrangement Graphs



Arrangement graph G

Augmented arrangement graph G'

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The ARRANGEMENT GRAPH RECOGNITION problem is to decide whether a given graph is the arrangement graph of some set of lines. It is $\exists\mathbb{R}$-complete. [Eppstein 2014]

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$\Uparrow$

Euclidean Pseudoline Stretchability is $\exists\mathbb{R}$-hard. [Matoušek 2014, Schaefer 2009]

A planar graph G is an arrangement graph on k lines
$\Leftrightarrow \rho^1_2(G') \leq k$ [Chaplick et al. 2017]
$\Leftrightarrow \mathrm{seg}_2(G') \leq k$
$\Leftrightarrow \mathrm{seg}_\angle(G') \leq k$
$\Leftrightarrow \mathrm{seg}_\times(G') \leq k$.

Open problem. Is any variant of segment number FPT?

Lower Bounds for Cubic Graphs

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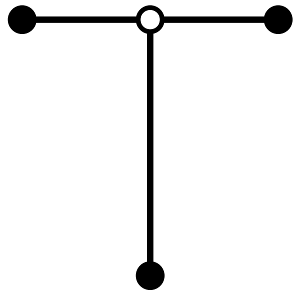

Flat vertex (f)

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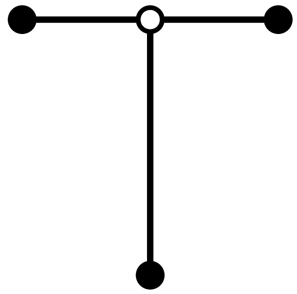

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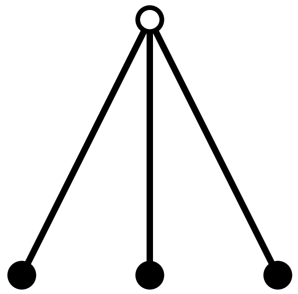

Tripod vertex (t)

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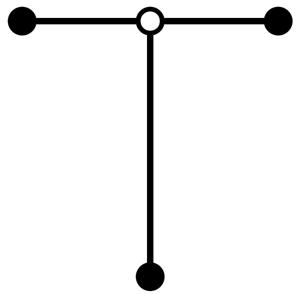

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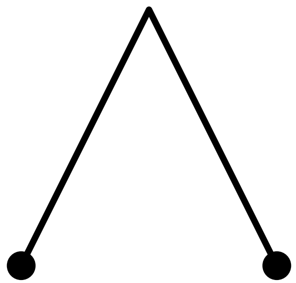
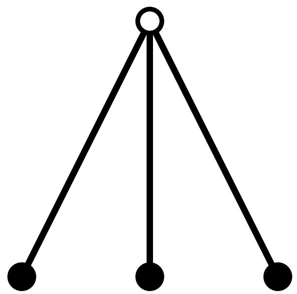


Bend (b)



Tripod vertex (t)

Lemma.

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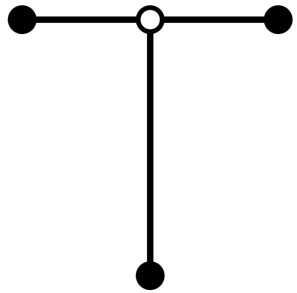
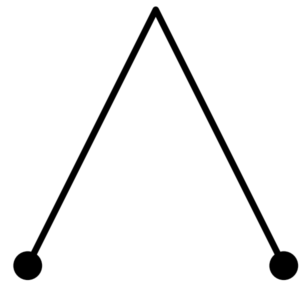
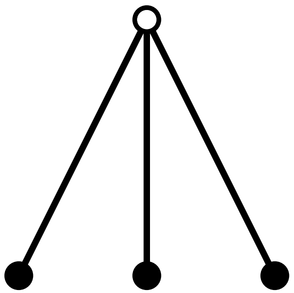


Flat vertex (f)



Bend (b)



Tripod vertex (t)

Lemma. For any straight-line drawing δ of a cubic graph with n vertices, $\text{seg}(\delta) = n/2 + t(\delta) + b(\delta)$.

Connected Cubic Graphs

For any cubic connected graph G with $n \geq 6$ vertices, $\text{seg}_3(G) \leq 7n/5$.

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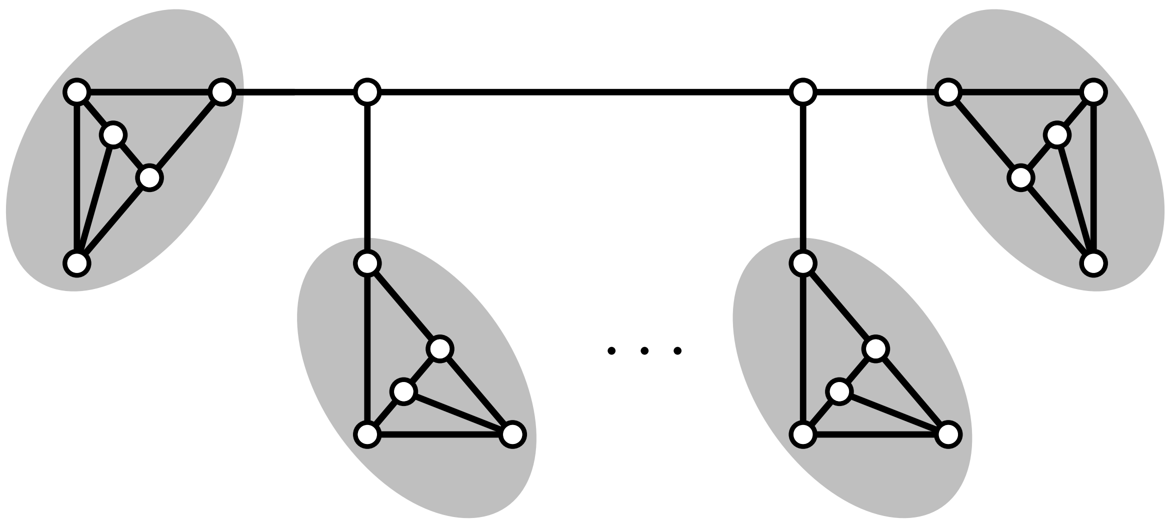

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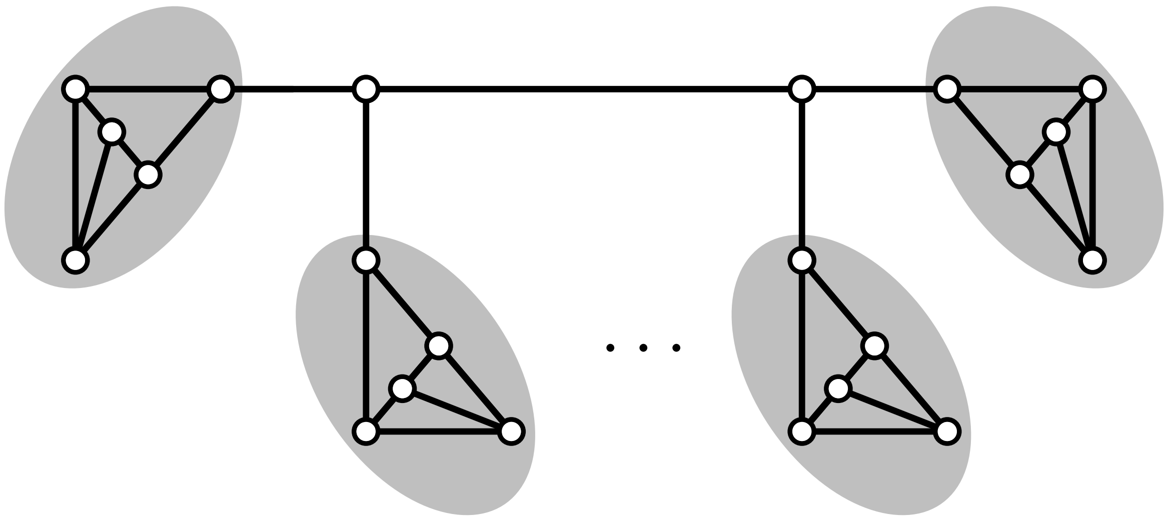

$$n = 6k - 2$$
$$\text{seg}_{2,3,\angle,\times}(G) = 5k - 1 > 5n/6$$

Biconnected Cubic Graphs

For any cubic biconnected planar graph G with n vertices, $\mathrm{seg}_\angle(G) \leq n + 1$. A corresponding drawing can be found in linear time.

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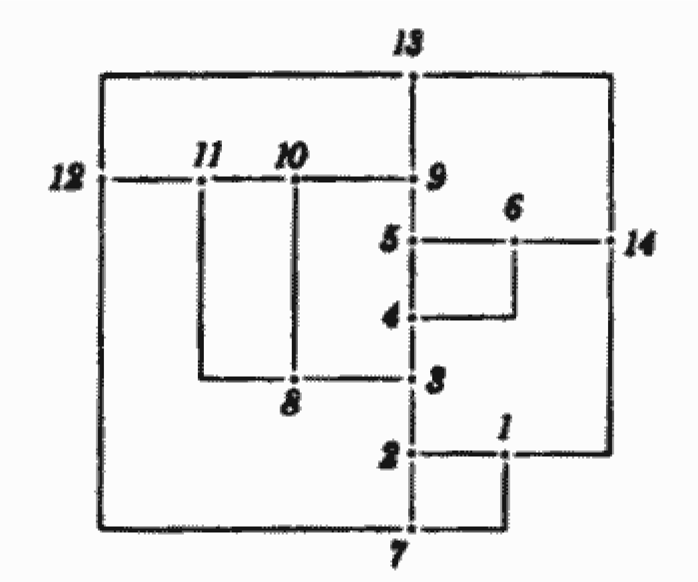

[Liu et al. 1994]

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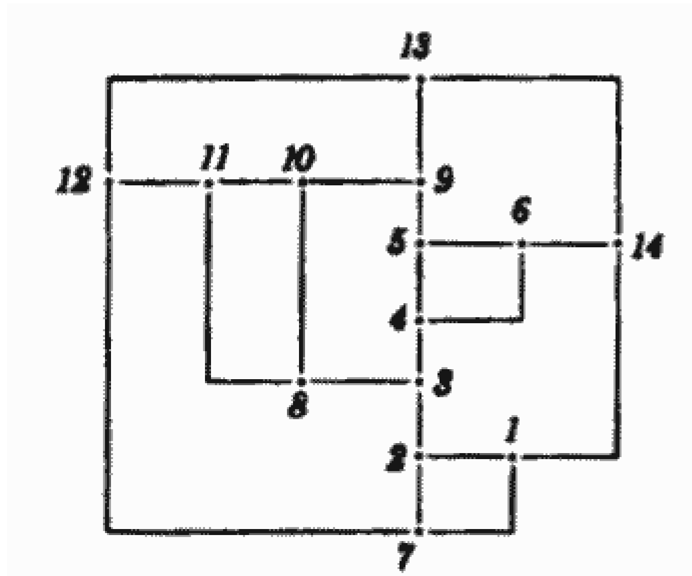

[Liu et al. 1994]

Open Problem. What about 4-regular graphs? They have $2n$ edges. If we bend every edge once, we already need $2n$ segments – and not all 4-regular graphs can be drawn with at most one bend per edge.

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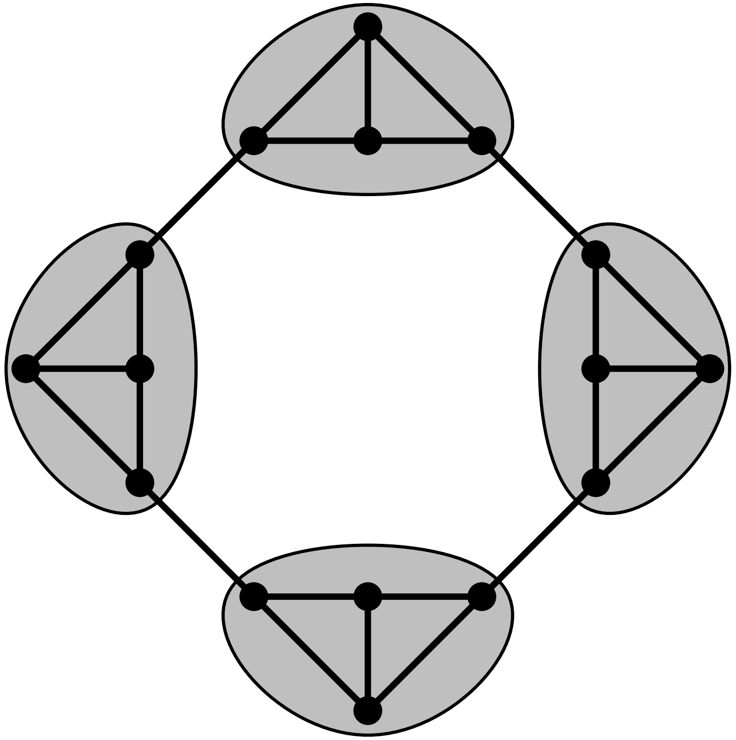

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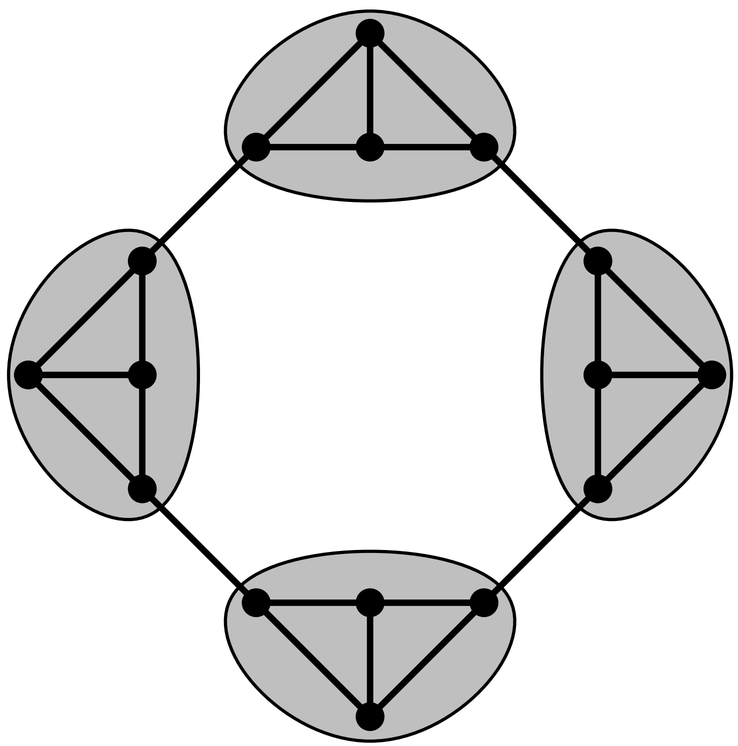

$$n = 4k \quad \text{seg}_{2,\angle,3,\times}(G) = 3n/4.$$

Each subgraph K' has an extreme point of its convex hull not connected to $G - V(K')$. It is a tripod or a bend, so $t(\delta) + b(\delta) \geq k$ and, by Lemma, $\text{seg}_{2,3,\angle,\times}(G) \geq 2k + t(\delta) + b(\delta) \geq 3k$.

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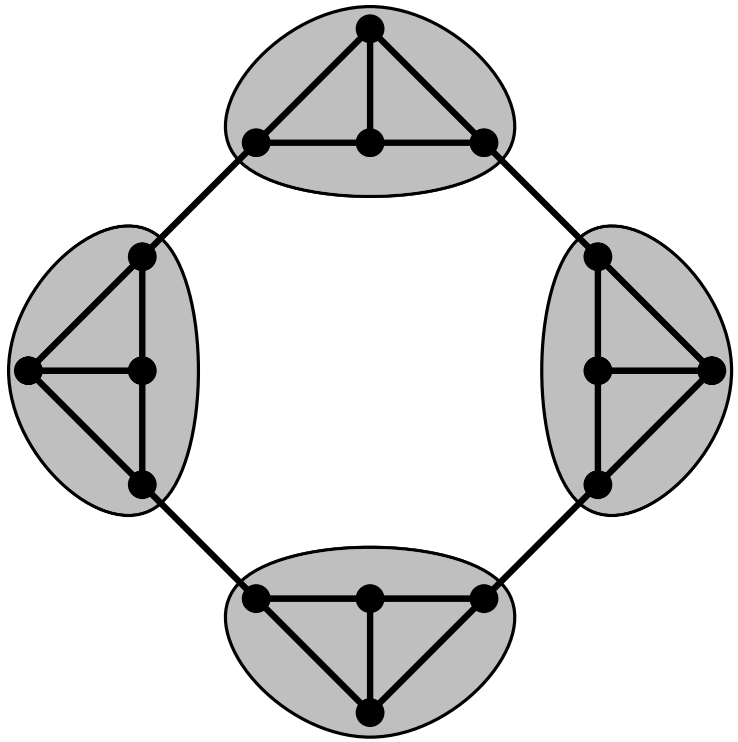

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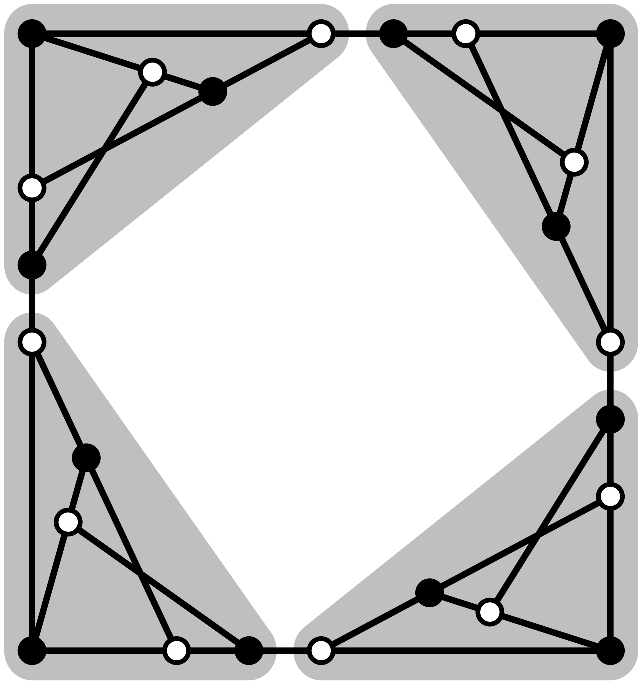
Hamiltonian Cubic Graphs

$$k \geq 3,\ n = 6k,\ \text{seg}_3(G) = 5n/6,\ \text{seg}_\times(G) = 2n/3$$



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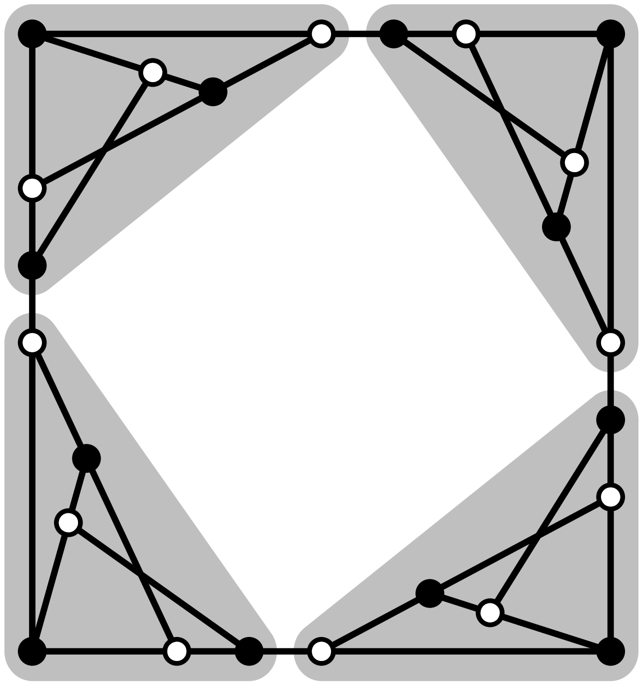

Open Problems: Improve Non-tight Bounds!

G is a cubic graph with $n \geq 6$ vertices.
$n/2 \leq \text{seg}_{2,3,\angle,\times}(G) \leq 3n/2$ and $\text{seg}_{2,3,\angle,\times}(\sqcup K_4) = 3n/2$.

γ	$\text{seg}_2(G)^*$	$\text{seg}_3(G)$	$\text{seg}_\angle(G)^*$	$\text{seg}_\times(G)$
1	$5n/6..3n/2$	$5n/6^*..7n/5$	$5n/6..3n/2$	$5n/6^*..7n/5$
2	$3n/4..3n/2$	$5n/6..7n/5$	$3n/4..n+1$	$3n/4^*..n+2$
3	$n/2+3^{**}$	$7n/10..7n/5$	$n/2+3$	$n/2..n+2$
H	$3n/4..3n/2$	$5n/6..n+1$	$3n/4..n+1$	$3n/4^*..n+2$

* For planar G.

** by [Durocher et al. 2013; Igamberdiev et al. 2017]

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