# COLORING HASSE DIAGRAMS AND DISJOINTNESS GRAPHS OF CURVES

János Pach & István Tomon
(Budapest & Zürich)

# CHROMATIC NUMBER VS. CLIQUE NUMBER

$\chi(G)$ **chromatic number** - minimum number of colors needed to color $V(G)$ so that no edge is monochromatic

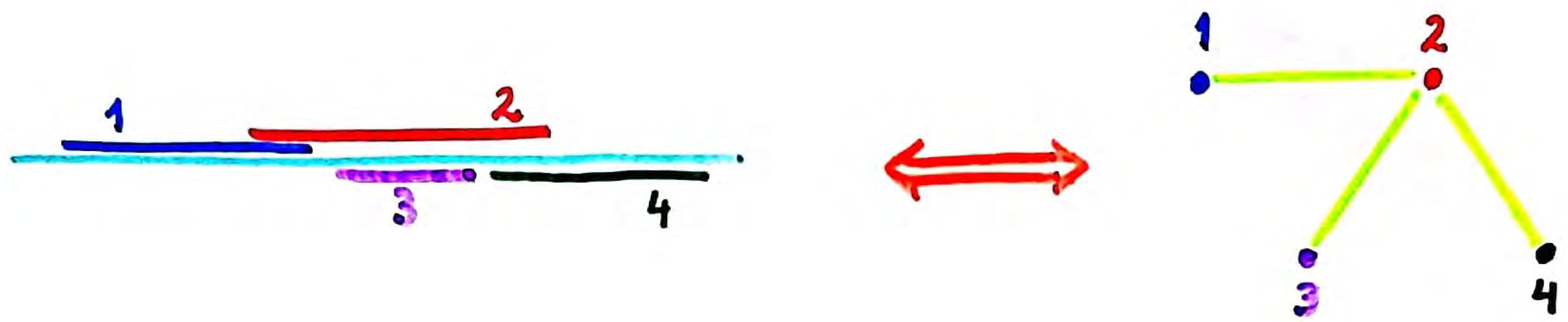
$\geq$

$\omega(G)$ **clique number** - maximum size of a complete subgraph of $G$

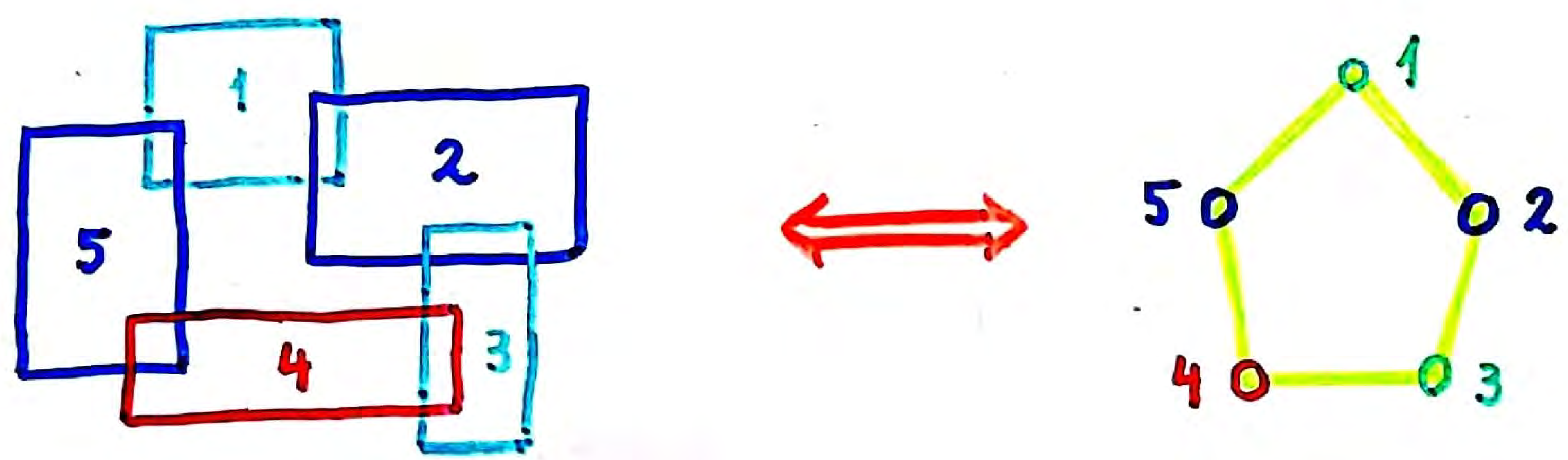
**Theorem** (Erdős 1959)

For every $k$ and $\ell$, the exists a graph $G = G(k,\ell)$ with $\chi(G) = k$ and with no cycle of length $\leq \ell$.

# INTERSECTION GRAPHS




**Theorem** (Gallai, Hajós)
For the intersection graph of any system of intervals along a line, we have $\chi(G) = \omega(G)$.




$\omega = 2$

$\chi = 3$

# $\chi$ - BOUNDED FAMILIES OF GRAPHS

A family of graphs $\mathcal{G}$ is $\chi$-bounded if there exists a function $f$ such that $\chi(G) \leq f(\omega(G))$ for $\forall G \in \mathcal{G}$
(Gyárfás - Lehel 1985)

**Theorem** (Asplund - Grünbaum 1960)
The family of intersection graphs of axis-parallel rectangles in the plane is $\chi$-bounded.

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The family of intersection graphs of axis-parallel rectangles in the plane is $\chi$-bounded.

The disjointness graph of a family of objects is the complement of its intersection graph.

**Theorem**
The family of disjointness graphs of axis-parallel rectangles in the plane is $\chi$-bounded.

$\chi(G) \leq c \cdot \omega(G)$ ???

# NOT $\chi$-BOUNDED FAMILIES OF GRAPHS

**Theorem** (Pawlik-Kozik-Krawczyk-Lasoń-Micek-Trotter-Walczak 2014)

There exist triangle-free ($\omega = 2$) intersection graphs of curves in the plane with arbitrarily large chromatic numbers $\chi$.

[$\Omega(\log \log n)$]

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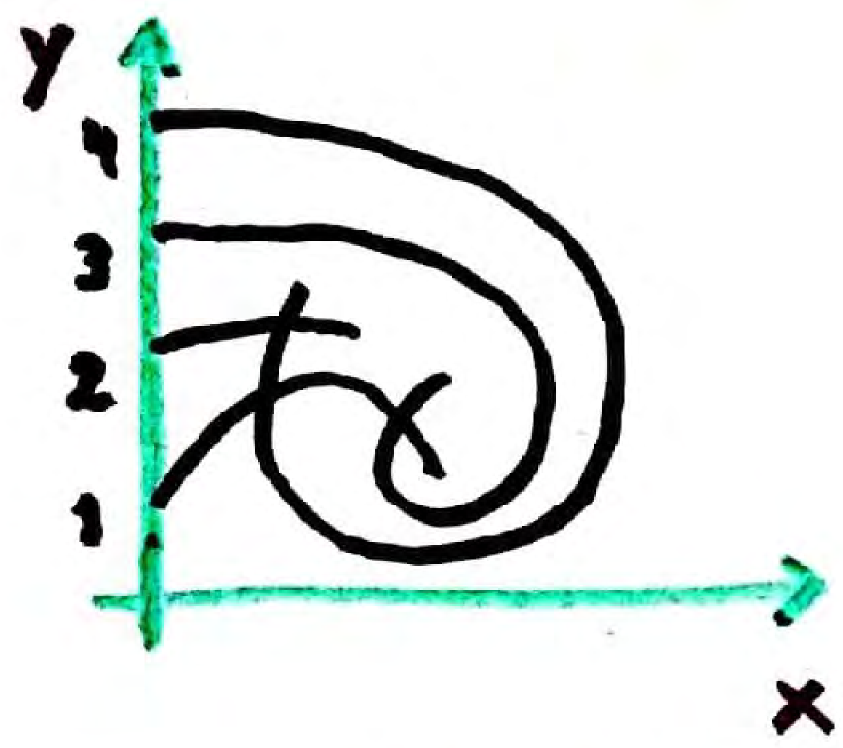
**Theorem** (P.-Tardos-Tóth 2017)
There exist triangle-free ($\omega = 2$) disjointness graphs of curves in the plane with arbitrarily large chromatic numbers $\chi$. [$\lceil \log_2 n \rceil$]

# GROUNDED CURVES AND COVER GRAPHS




grounded curves
on the y-axis

$\Longrightarrow$


disjointness graph

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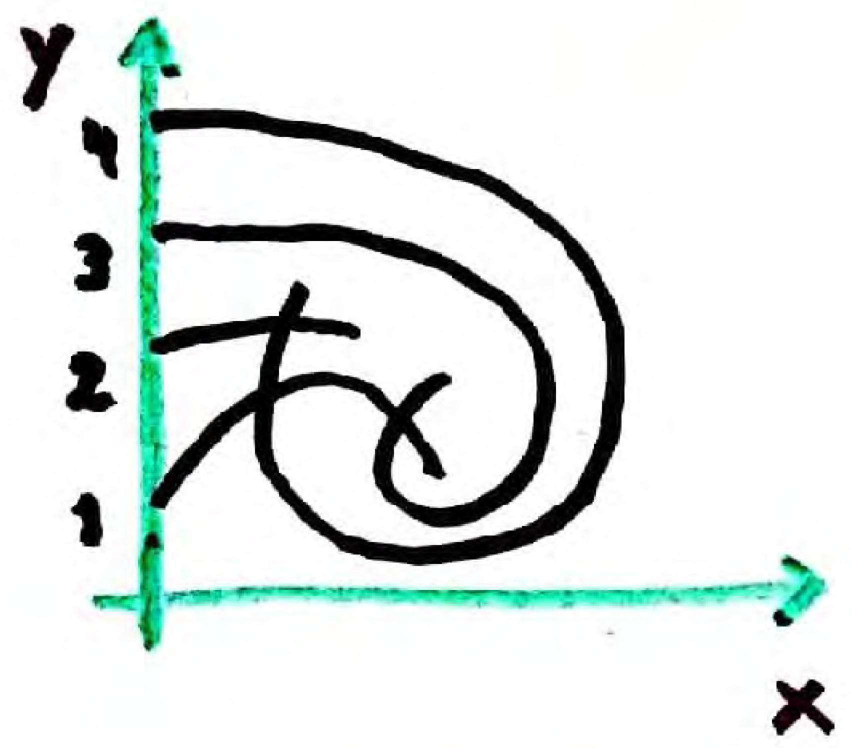


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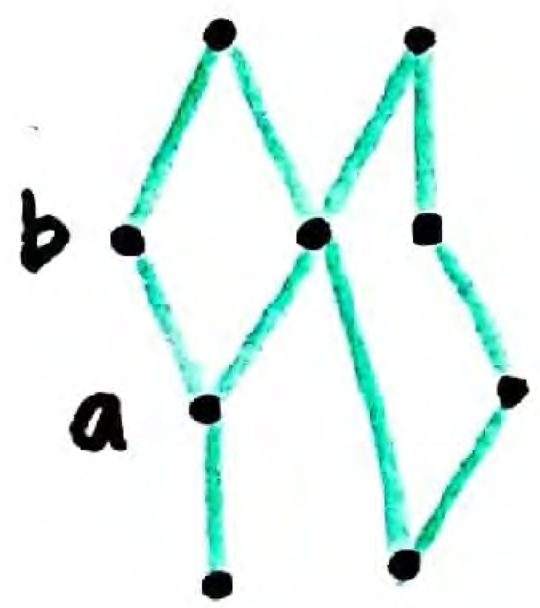


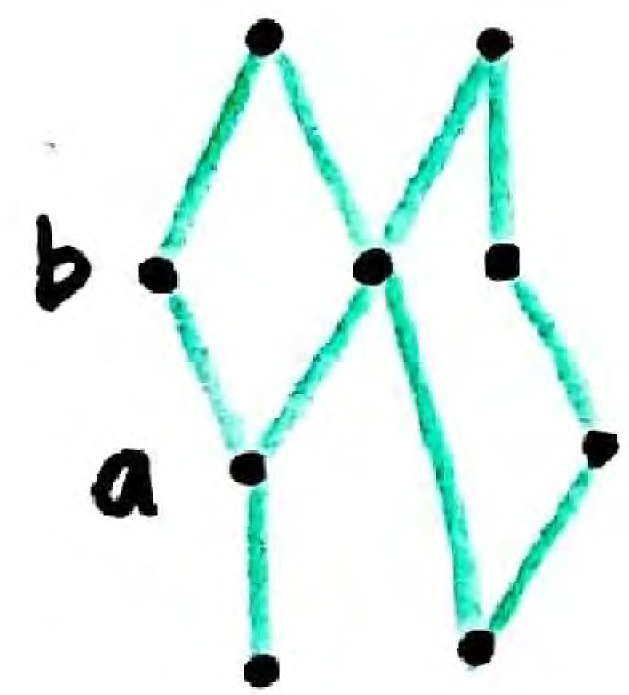

$(P, <)$ partial order

b covers a if $b > a$ and there is no $c \in P$ with $b > c > a$

ba is an edge of the cover graph (= undirected Hasse diagram)

Theorem (Sinden 1966 ←, Middendorf-Pfeiffer 1993 →)

G is a cover graph ⟺
- G is triangle-free and
- G is the disjointness graph of a family of grounded curves.

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For every $r$ and $n$, there exists a partially ordered set of $n$ elements whose cover graph has girth $\geq r$ and chromatic number $\geq \Omega\left(\frac{1}{r}\log n\right)$.

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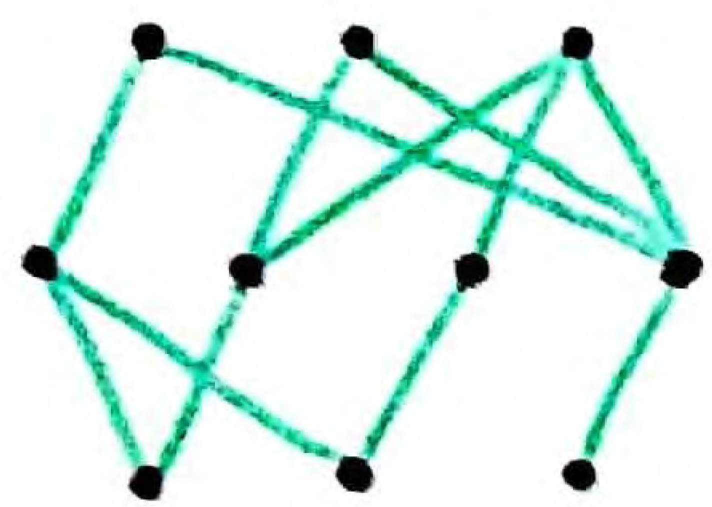
For every r and n, there exists a partially ordered set of n elements whose cover graph has girth $\geq r$ and chromatic number $\geq \Omega\left(\frac{1}{r}\log n\right)$.

**Corollary**

For every r and n, there exists a family of n curves whose disjointness graph has girth $\geq r$ and chromatic number $\geq \Omega\left(\frac{1}{r}\log n\right)$. [$\longrightarrow$ $r=4$ case]

# UNIQUELY GENERATED POSETS

$P$



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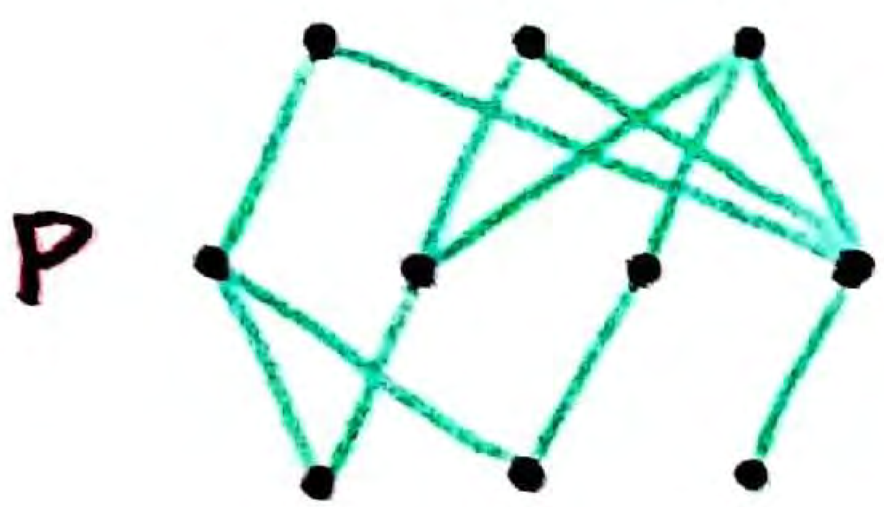


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**Theorem** (P.-Tomon 2019)

(i) If $P$ is a uniquely generated poset with $n$ vertices, then for its cover graph $G$, we have $\chi(G) \le \lfloor \log_2 n \rfloor + 1$.

(ii) For every $r > 3$ and $n > n_0(r)$, there exists a uniquely generated poset with $n$ vertices whose cover graph $G$ has girth $\ge r$ and $\chi(G) \ge \Omega(\frac{1}{r} \log_2 n)$.

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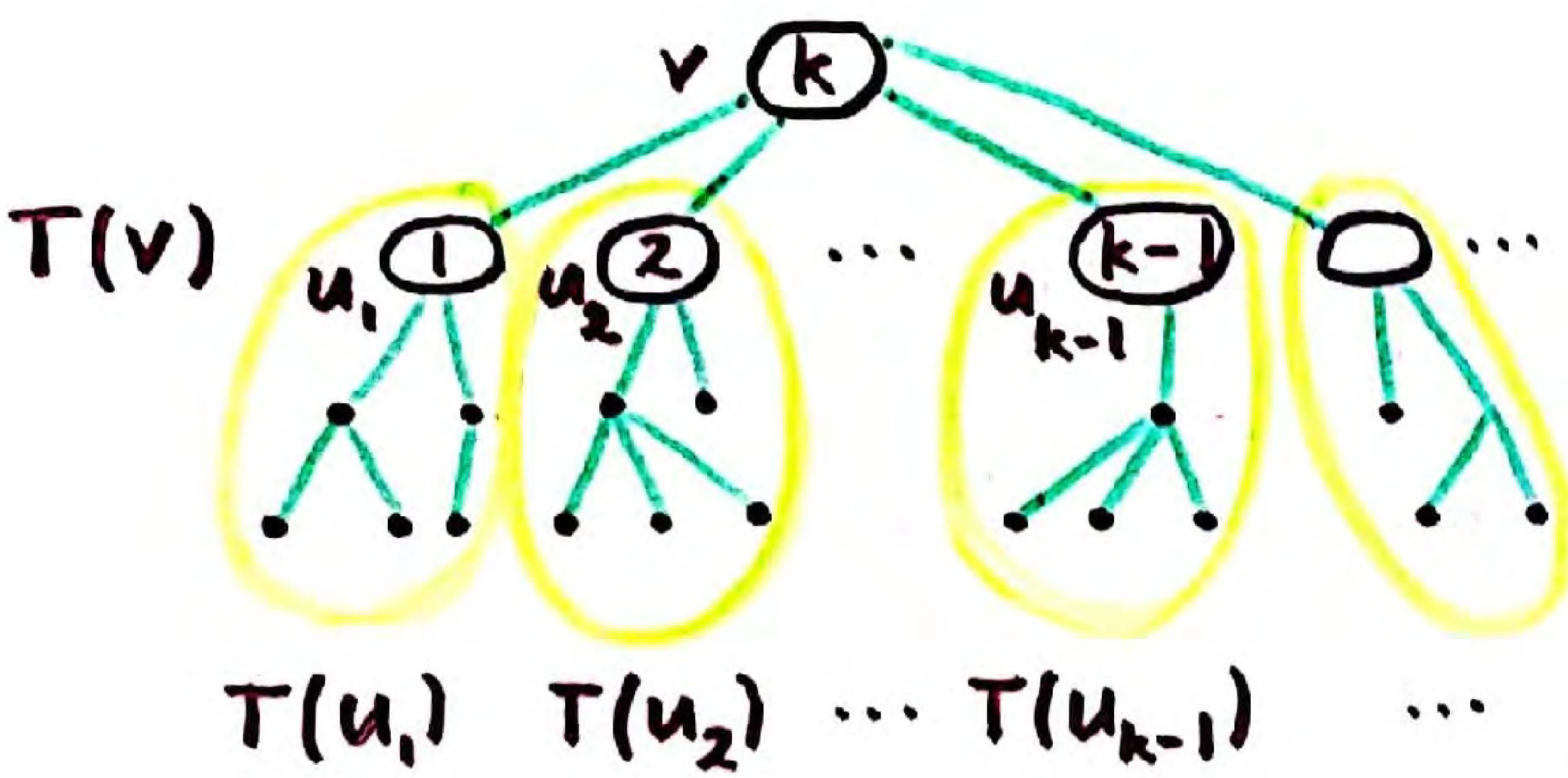
(i) If $P$ is a uniquely generated poset with $n$ vertices, then for its cover graph $G$, we have $\chi(G) \le \lfloor \log_2 n \rfloor + 1$.

**Proof.** Use greedy coloring with $1, 2, 3, \dots$

Let $T(v) = \{ u \in P : u \prec v \}$ (tree)




$$
\begin{aligned}
|T(v)| &\ge 1 + \sum_{i=1}^{k-1} |T(u_i)| \\
&\ge 1 + \sum_{i=1}^{k-1} 2^{i-1} \\
&= 2^{k-1}
\end{aligned}
$$

**Theorem.**

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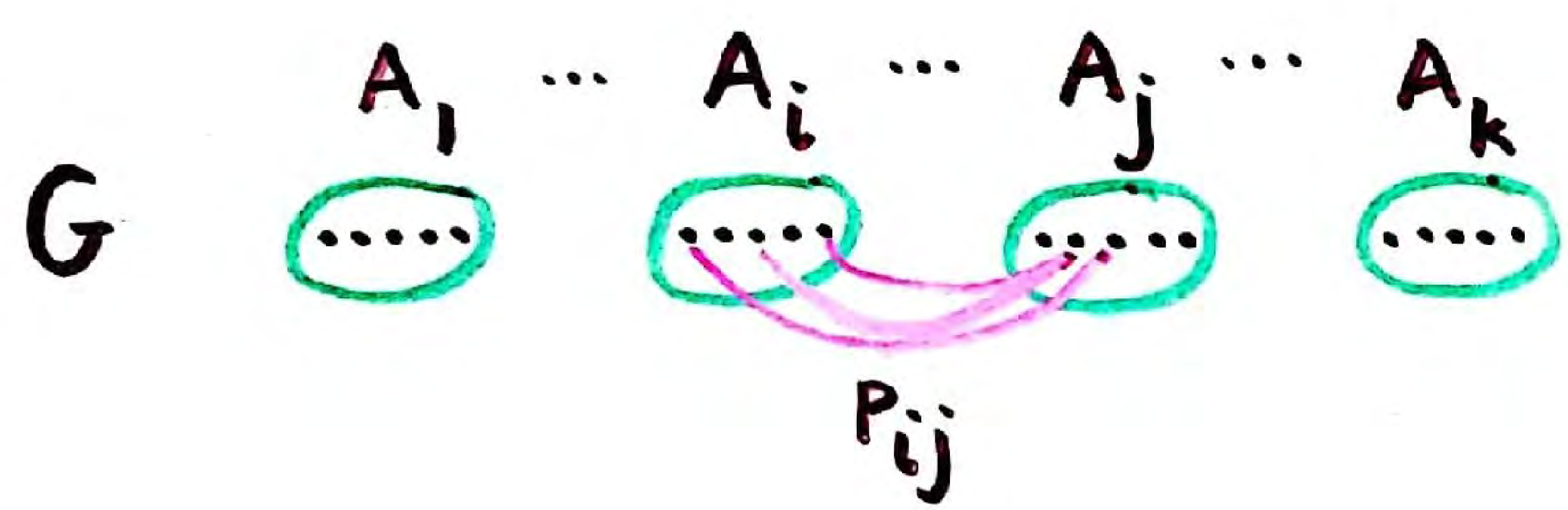
**Proof.** Let $k = \frac{\log n}{10r}$, $m = \frac{n}{k} = 10r\frac{n}{\log n}$




$$|A_1| = \dots = |A_k| = m$$

$$P_{ij} = \frac{2^{j-i}}{m} < \frac{1}{10r}\frac{\log n}{n^{1-1/10r}}$$

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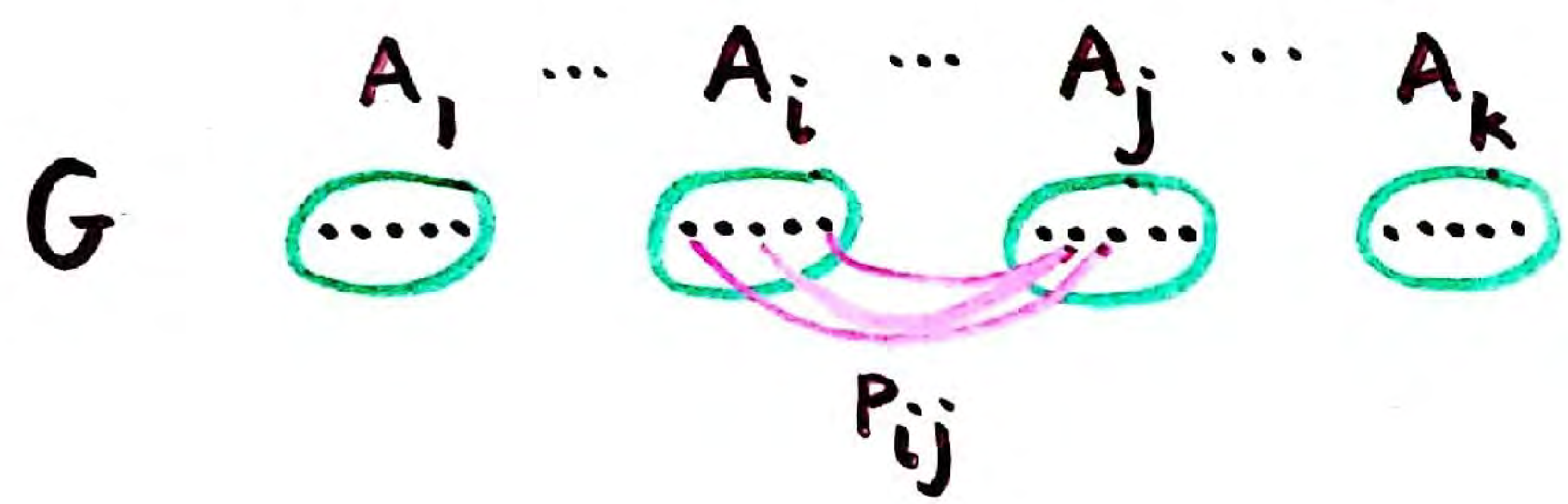


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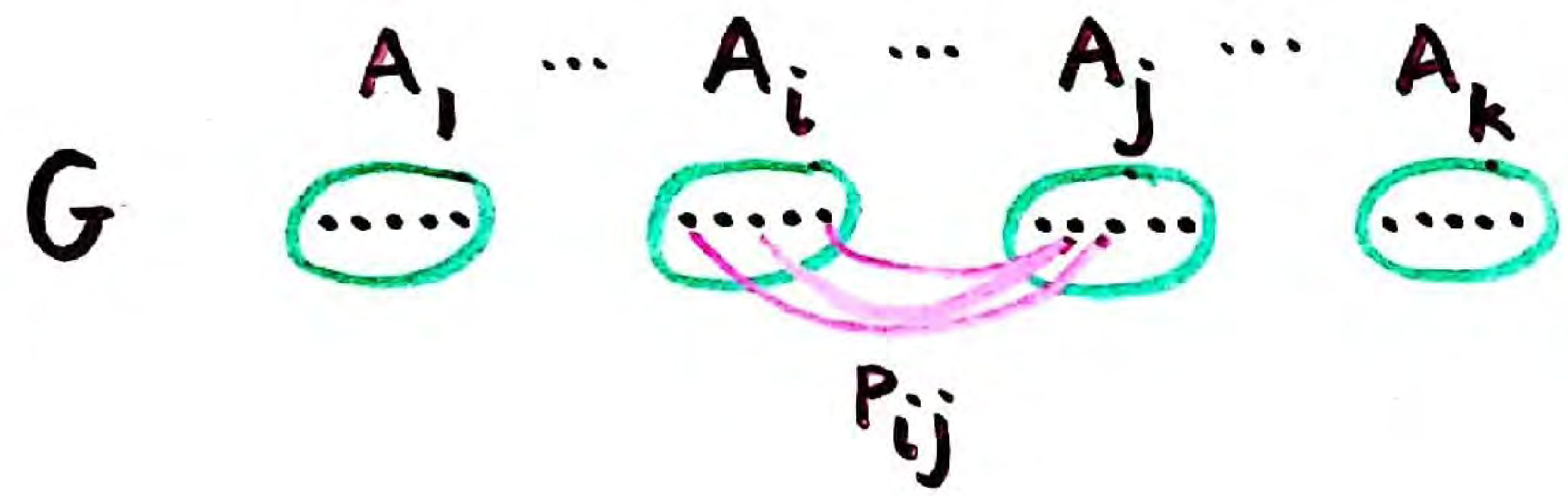


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**Claim.** With positive probability $G$ satisfies

(1) there is no independent set of size $m$,

(2) there are $\leq \frac{n}{3}$ cycles of length $< r$,

(3) there are $\leq \frac{n}{3}$ pairs $u, v \in V(G)$ with 2 edge-disjoint monotone paths connecting them.

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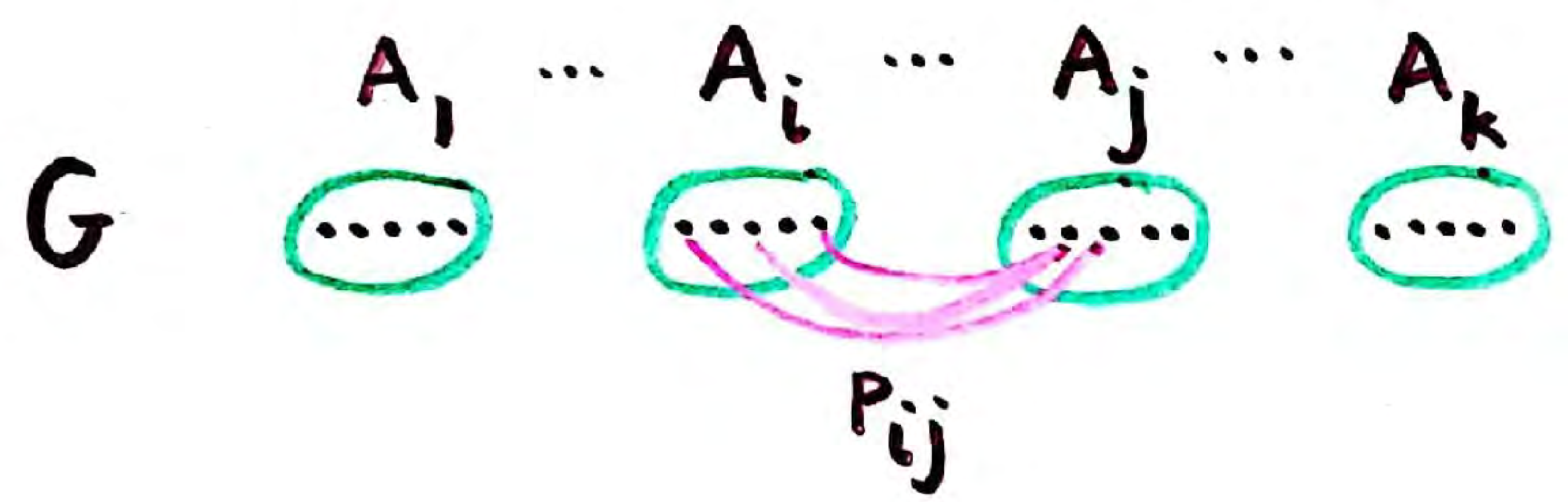


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Then
- delete a vertex from each "bad" cycle + pair,
- define $u \prec v$ if there is a monotone increasing path from $u$ to $v$.

Problem. Determine or estimate $k(n)$, the maximum chromatic number of the cover graph of an $n$-element partially ordered set.

$k(n) = O(\sqrt{n})$

$k(n) = O(\sqrt{n/\log n})$ Ajtai-Komlós-Szemerédi 1980

$k(n) = \Omega(\log n/\log\log\log n)$ Brightwell-Nešetřil 1991

$k(n) = \Omega(\log n)$ P.-Tomon 2019