On the Edge-Length Ratio of Planar Graphs

Manuel Borrazzo and Fabrizio Frati

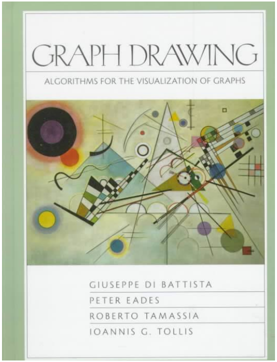
Roma Tre University

The 27^{th} International Symposium on Graph Drawing and Network Visualization

18^{th} September 2019

Introduction

The *edge-length ratio* of a drawing is a natural metric to guarantee the readability of a graph drawing.



Edge-length ratio

Definition

The *edge-length ratio* $\rho(\Gamma)$ of a straight-line drawing Γ of a graph $G = (V, E)$ is the ratio between the lengths of the longest and of the shortest edge in the drawing.

$$\rho(\Gamma) = \max_{e_1, e_2 \in E(G)} \frac{\ell_\Gamma(e_1)}{\ell_\Gamma(e_2)},$$

where $\ell_\Gamma(e)$ denotes the length of the segment representing an edge e in Γ.

Planar edge-length ratio

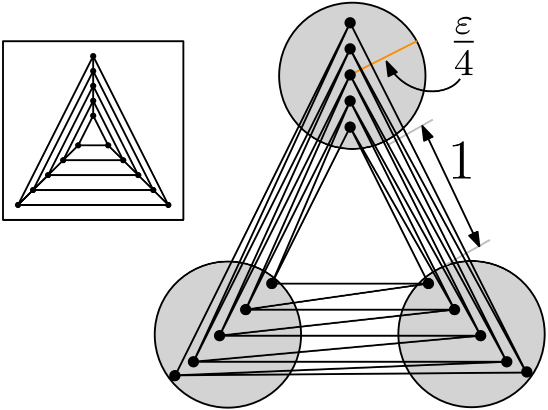
Definition

The *planar edge-length ratio* $\rho(G)$ of a graph G is the minimum edge-length ratio of any planar straight-line drawing Γ of G.

$$\rho(G) = min(\rho(\Gamma))$$

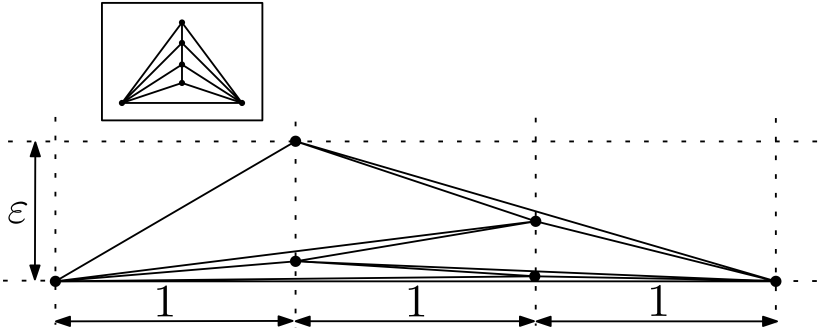
Examples of graphs admitting a good edge-length ratio

Example 1: The nested-triangle graph has planar edge-length ratio less than $1+\epsilon$.



Examples of graphs admitting a good edge-length ratio

Example 2: The plane 3-tree obtained as the join of a path with an edge has planar edge-length ratio less than 3.



State of the art (1)

Deciding whether a graph has planar edge-length ratio equal to 1 is an **NP-hard** problem.

- Eades et al.[1] for biconnected planar graphs;
- Cabello et al.[2] for triconnected planar graphs.

[1] *"Fixed edge-length graph drawing is NP-hard"*, Discrete Applied Mathematics 28(2), (1990)

[2] *"Planar embeddings of graphs with specified edge lengths"*, J. Graph Algorithms Appl. 11(1), (2007)

State of the art (2)

The study of combinatorial bounds for the planar edge-length ratio of planar graphs started with Lazard et al.[3].

1. Outerplanar graphs have planar edge-length ratio smaller than 2.
2. There exist outerplanar graphs whose planar edge-length ratio is larger then $2 - \epsilon$.

[3] *"On the edge-length ratio of outerplanar graphs"*, Theor. Comput. Sci. 770, (2019)

The questions we look at

1. What is the edge-length ratio for planar graphs?
2. What is the edge-length ratio for notable classes of graphs like series-parallel or bipartite graphs?

Our results

1. **Theorem 1**: planar graphs have planar edge-length ratio in $\Theta(n)$
2. **Theorem 2**: planar 3-trees with depth k have planar edge-length ratio in $O(k)$
3. **Theorem 3**: 2-trees have planar edge-length ratio in $O(n^{0.695})$
4. **Theorem 4**: for any fixed $\epsilon > 0$, bipartite planar graphs have planar edge-length ratio smaller than $1 + \epsilon$

Theorem 1: edge-length ratio of planar graphs (1)

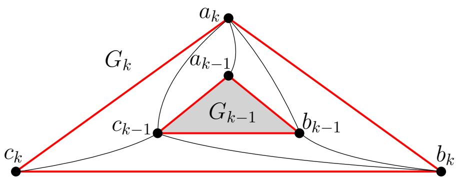
Theorem

For arbitrarily large values of n, there exists an n-vertex planar graph whose planar edge-length ratio is in $\Omega(n)$.

Proof:

- Consider any planar straight-line drawing Γ of G
- Assume that the length of the shortest edge of G in Γ is 1
- Let $T_k = a_k b_k c_k$ and $T_{k-1} = a_{k-1} b_{k-1} c_{k-1}$. We prove that: $P(T_k) \geq P(T_{k-1}) + c$, for a constant c

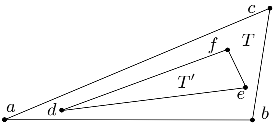
This implies that the edge-length ratio of Γ is $\Omega(n)$.



Theorem 1: edge-length ratio of planar graphs (2)

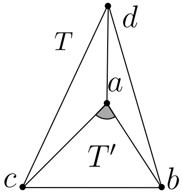
Lemma

Let T and T' be triangles such that T' is contained into T, then $P(T) > P(T')$

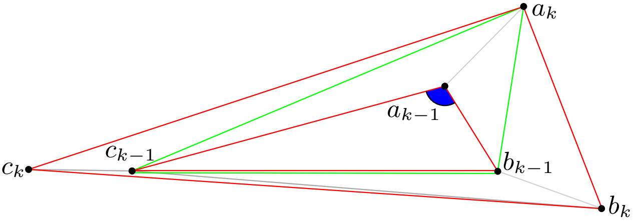


Lemma

If $||\overline{ad}|| \geq 1$ and $b\widehat{a}c \leq 90^\circ$, then $P(T) > P(T') + 1$

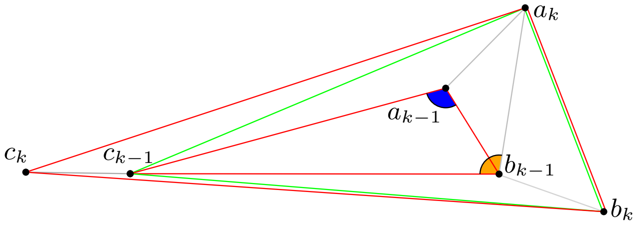


Theorem 1: edge-length ratio of planar graphs (3)



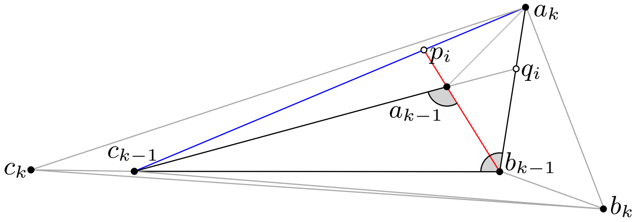
- If $b_{k-1}\widehat{a_{k-1}}c_{k-1} \leq 90^\circ$, then $P(T_k) > P(T_{k-1}) + 1$

Theorem 1: edge-length ratio of planar graphs (4)



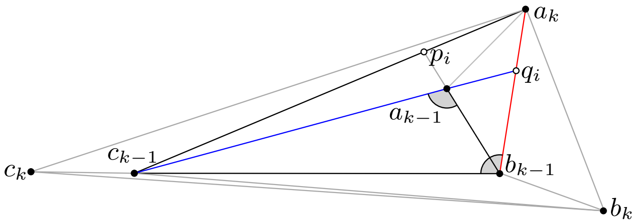
- If $b_{k-1}\widehat{a_{k-1}}c_{k-1} > 90^\circ$ and $c_{k-1}\widehat{b_{k-1}}a_k \leq 90^\circ$, then $P(T_k) > P(T_{k-1}) + 1$

Theorem 1: edge-length ratio of planar graphs (5)



Let p_i be the intersection point between the straight line $\overline{a_{k-1}b_{k-1}}$ with $\overline{c_{k-1}a_k}$.

Theorem 1: edge-length ratio of planar graphs (6)

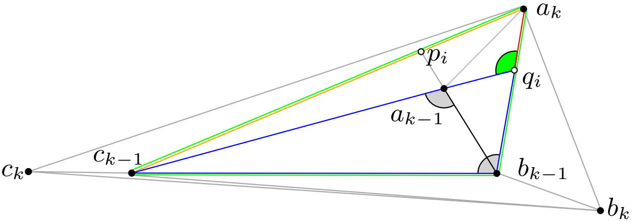


Let q_i be the intersection point between the straight line $\overline{a_{k-1}c_{k-1}}$ with $\overline{b_{k-1}a_k}$.

We distinguish two cases:

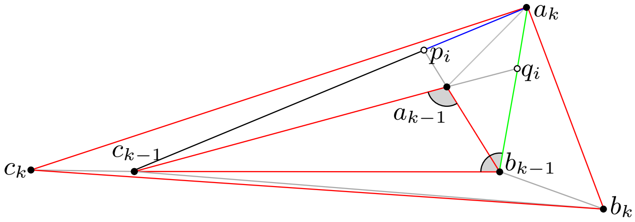
1. $|\overline{a_k q_i}| \geq 0.4$
2. $|\overline{a_k q_i}| \leq 0.4$

Theorem 1: edge-length ratio of planar graphs (7)



- If $|\overline{a_k q_i}| \geq 0.4$, then $P(b_{k-1} c_{k-1} q_i) > P(T_{k-1})$ and since $c_{k-1} \widehat{q_i} a_k > 90^\circ$ we have $|\overline{c_{k-1} a_k}| > |\overline{c_{k-1} q_i}|$, and hence $P(T_k) > P(T_{k-1}) + 0.4$

Theorem 1: edge-length ratio of planar graphs (8)



- If $|\overline{a_k q_i}| \leq 0.4$, then $|\overline{a_k p_i}| \geq 0.4$, and hence $P(T_k) - P(T_{k-1})$ will assume its minimum value when $|\overline{b_{k-1} a_k}| = 1$ and $|\overline{a_k p_i}| = 0.4$, then $P(T_k) > P(T_{k-1}) + 0.32$

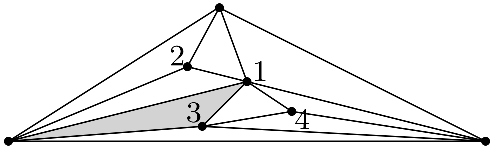
Theorem 2: edge-length ratio of plane 3-trees

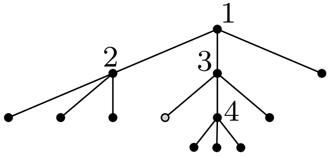
Theorem

Every plane 3-tree with depth k has planar edge-length ratio in $O(k)$.

A plane 3-tree G is naturally associated with a rooted ternary tree T_G, whose internal nodes represent the internal vertices of G and whose leaves represent the internal faces of G.

The proof is by induction. Let $depth(G) := depth(T_G) = k$, then the planar edge-length ratio of G is in $O(k)$.





Theorem 3: edge-length ratio of 2-trees (1)

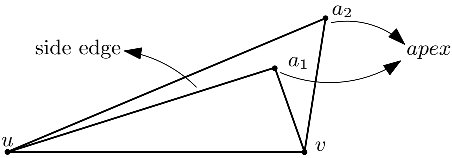
Theorem

Every n-vertex 2-tree has planar edge-length ratio in $O(n^{\log_2 \phi}) \subseteq O(n^{0.695})$, *where* $\phi = \frac{1+\sqrt{5}}{2}$ *is the golden ratio.*

Lazard et al.[4] asked whether the planar edge-length ratio of 2-trees is bounded by a constant; recently, at the 14^{th} Bertinoro Workshop on Graph Drawing, Fiala announced a negative answer to the above question.

[4] *"On the edge-length ratio of outerplanar graphs"*, Theor. Comput. Sci., (2019)

Theorem 3: edge-length ratio of 2-trees (2)



Definition

An *apex vertex* of the edge (u, v) is a vertex that is connected to u and v.

Definition

The *side edges* of (u, v) are all the edges with a vertex u or v and apex vertex of (u, v).

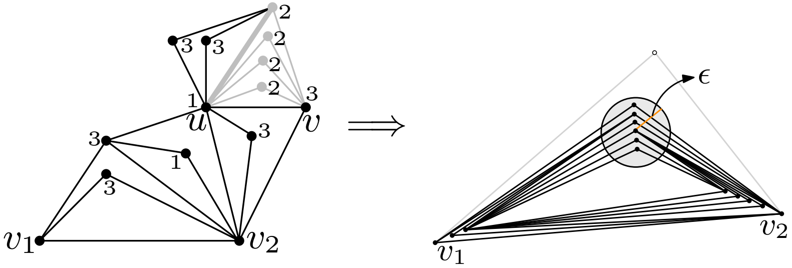
Definition

An edge (u, v) is *trivial* if it has no apex, otherwise it is *non-trivial*.

Theorem 3: L2T-drawer algorithm (3)

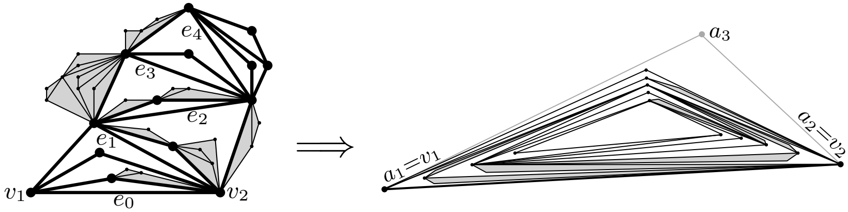
Definition

A *linear 2-tree* is a 2-tree such that every edge has at most one non-trivial side edge.



Our $L2T$-drawer algorithm constructs a planar straight-line drawing Γ of a linear 2-tree H.

Theorem 3: edge-length ratio of 2-trees (4)



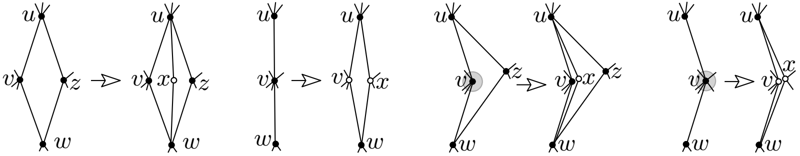
Proof:

1. Find a subgraph H of G that is a linear 2-tree, and such that every H-component of G has "few" internal vertices.
2. Construct a planar straight-line drawing Γ of H by the alogoritthm $L2T$-drawer.
3. Recursively draw each H-component independently, plugging such drawings into Γ, thus obtaining a drawing of G.

Theorem 4: edge-length ratio of bipartite planar graphs

Theorem

For every $\epsilon > 0$, every n-vertex bipartite planar graph has planar edge-length ratio smaller than $1 + \epsilon$.



(a) (b) (c) (d)

Proof:

The proof is based on the work of Brinkman et al.[5] and is by induction on *n*. The figure shows the *expansion* and *contraction* operations we use in order to perform induction.

5 *"Generation of simple quadrangulations of the spher"*, Discrete Mathematics 305(1 − 3), (2005)

Open problems

- What is the asymptotic behavior of the planar edge-length ratio of 2-trees?
- Is the planar edge-length ratio of cubic planar graphs sub-linear?
- Is the planar edge-length ratio of k-outerplanar graphs bounded by some function of k?

Thank you for your attention!