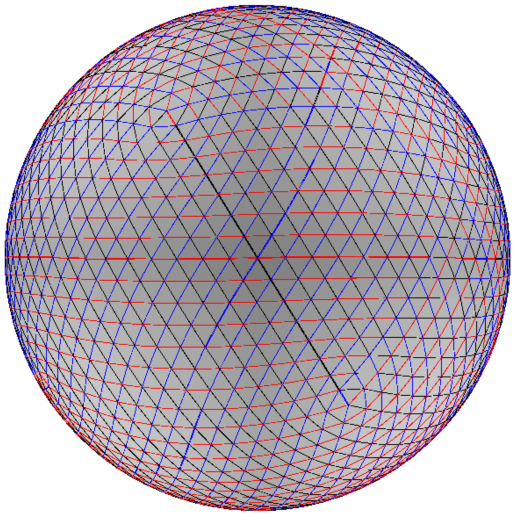
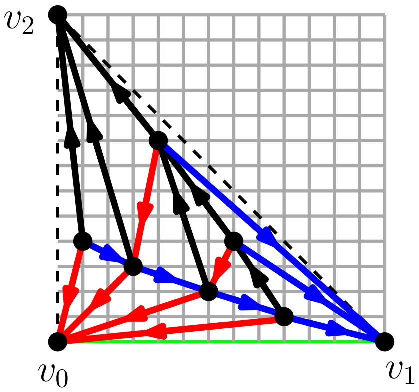
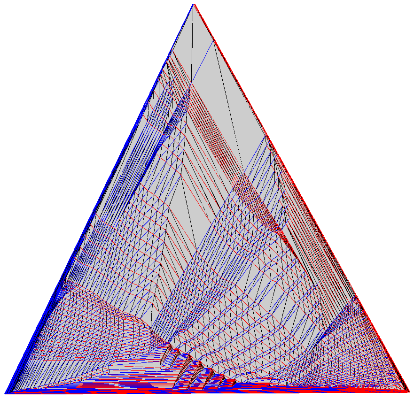
Balanced Schnyder woods for planar triangulations: an experimental study with applications to graph drawing and graph separators

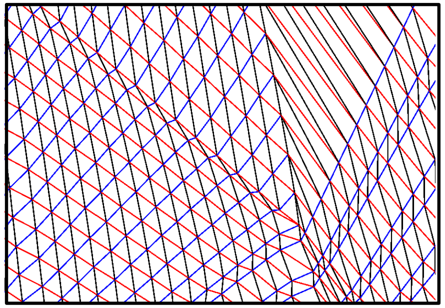


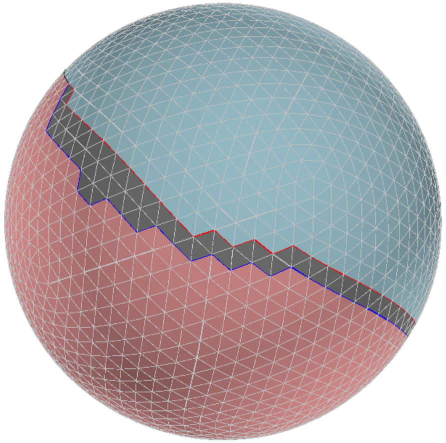
sept. 18th,
Graph Drawing 2019

Luca Castelli Aleardi



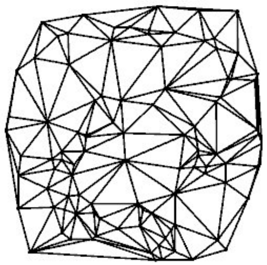




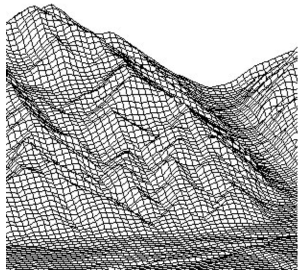


Planar graphs are ubiquitous

(from computational geometry to computer graphics, geometric processing, ...)

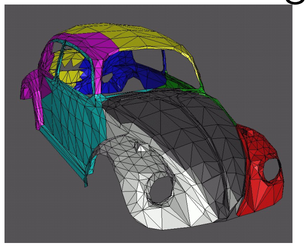


Delaunay triangulation



GIS Technology

geometric modeling



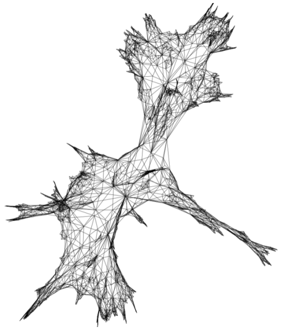
3D reconstruction

David statue (Stanford's Digital Michelangelo Project, 2000)

Real-world graphs are very regular and far from random or pathological cases

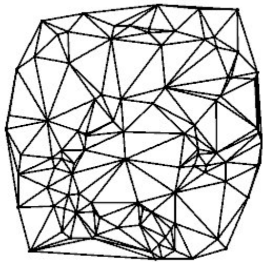
regularity measure: we use d_6, the proportion of degree 6 vertices

$d_6 \approx 0.11$



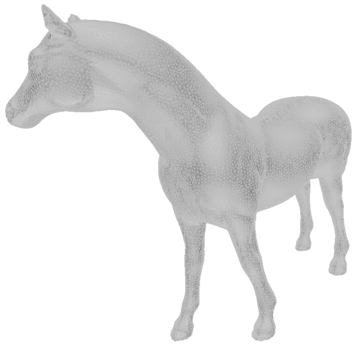
random planar triang.

$d_6 \approx 0.28$



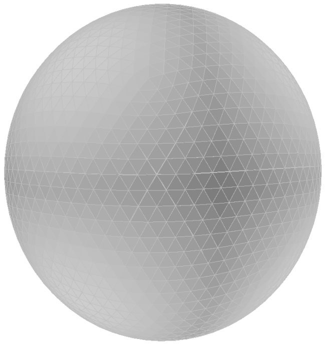
Delaunay triangulation

$d_6 \approx 0.50$



$d_6 \approx 0.82$

$d_6 \approx 0.99$

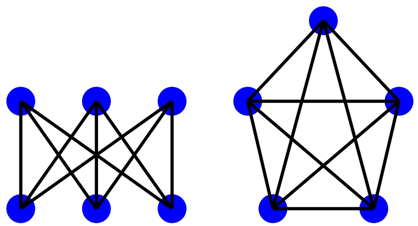


Some facts about planar graphs

("As I have known them")

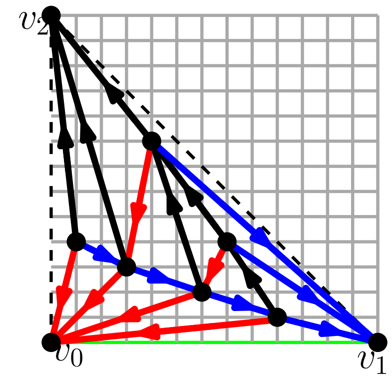
Kuratowski theorem (1930) (cfr Wagner's theorem, 1937)

- G contains neither K_5 nor $K_{3,3}$ as minors



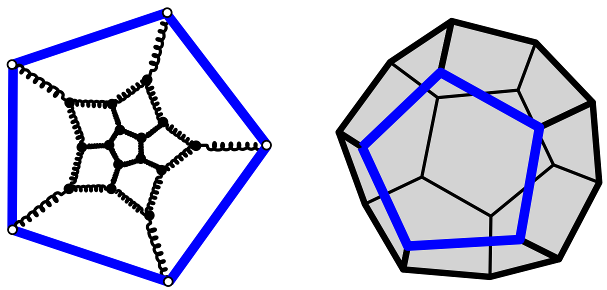
Schnyder woods ('89)

- planarity criterion via dimension of partial orders: $dim(G) \leq 3$
- linear-time grid drawing, with $O(n) \times O(n)$ resolution



Thm (Tutte barycentric method, 1963)

Every 3-connected planar graph G admits a convex representation in R^2.



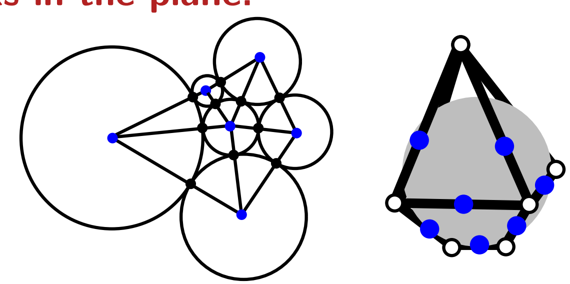
Thm (Colin de Verdière, 1990) Colin de Verdiere invariant (multiplicity of λ_2 eigenvalue of a generalized laplacian)

- $\mu(G) \leq 3$

$$\begin{bmatrix} 4 & -1 & \dots & \dots & 0 \\ -1 & 5 & \dots & & \\ \dots & & \dots & & \\ \dots & & & & \dots \\ 0 & \dots & & & 3 \end{bmatrix} \qquad L_G[i,k] = \begin{cases} deg(v_i) \\ -A_G[i,j] \end{cases}$$

Thm (Koebe-Andreev-Thurston)

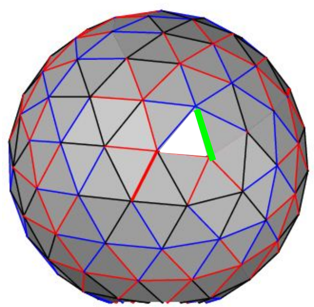
Every planar graph with n vertices is isomorphic to the intersection graph of n disks in the plane.

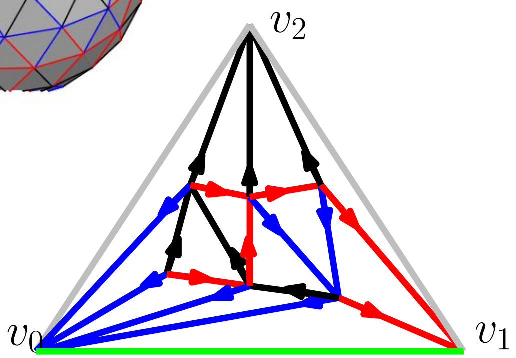


Schnyder woods

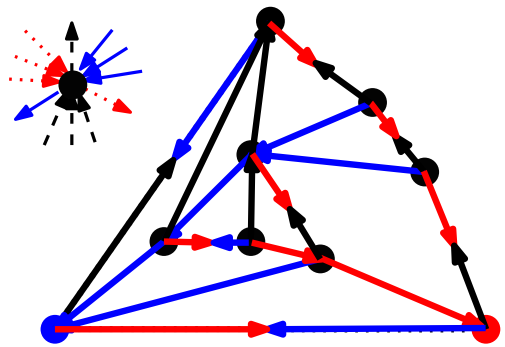
(quick overview)

Planar triangulations
[Schnyder '90]

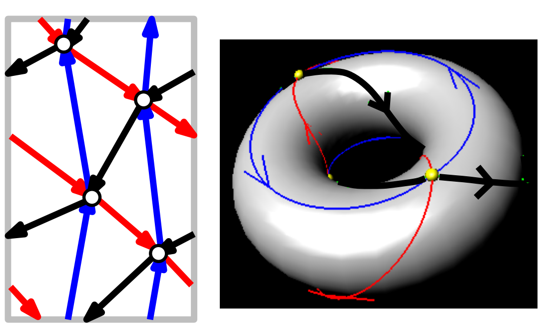




3-connected planar graphs
[Felsner '01]

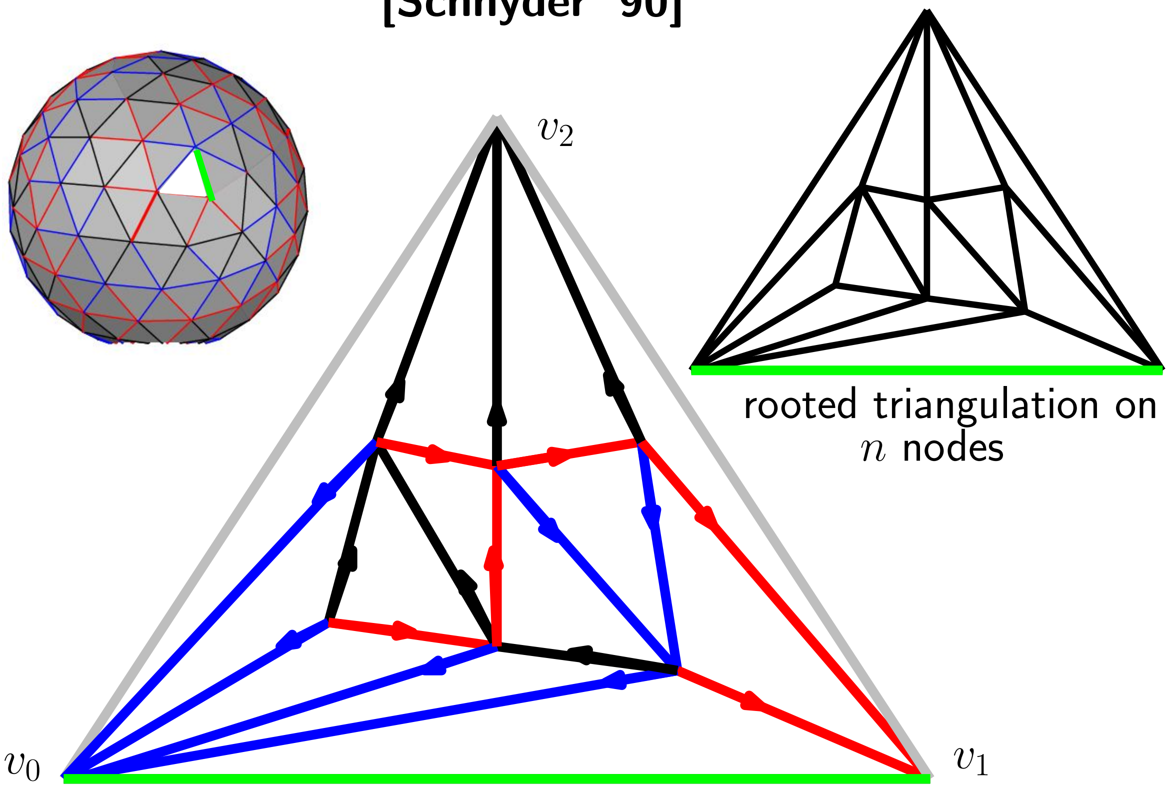


toroidal triangulations
[Goncalves Lévêque, '14]
genus g triangulations
[Castelli Aleardi Fusy Lewiner, '08]

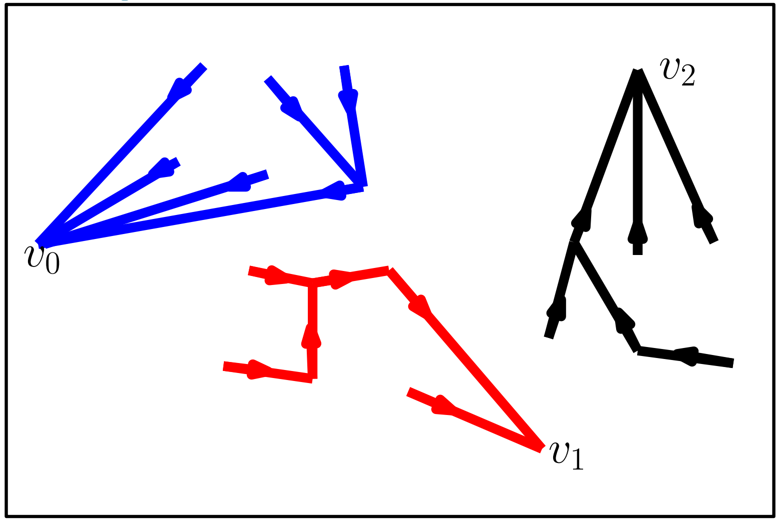


Schnyder woods: (planar) definition

[Schnyder '90]



rooted triangulation on n nodes



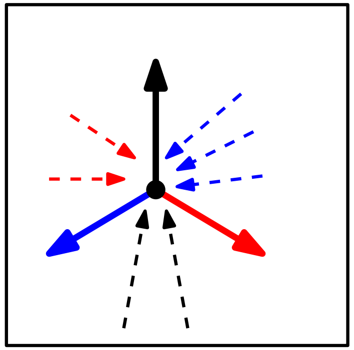
T_0, T_1 and T_2 are vertex spanning trees

Def

A Schnyder wood of a (rooted) planar triangulation is partition of all inner edges into three sets T_0, T_1 and T_2, s.t.

i) edge are colored and oriented in such a way that each inner nodes has exaclty one outgoing edge of each color

ii) colors and orientations around each inner node must respect the local Schnyder condition



Looking for "nice" Schnyder woods

Counting Schnyder woods: (there are an exponential number)

[Bonichon '05]

Schnyder woods of triangulations of size n: $\approx 16^n$

planar triangulations of size n: $|\mathcal{T}_n| \approx 2^{3.2451}$

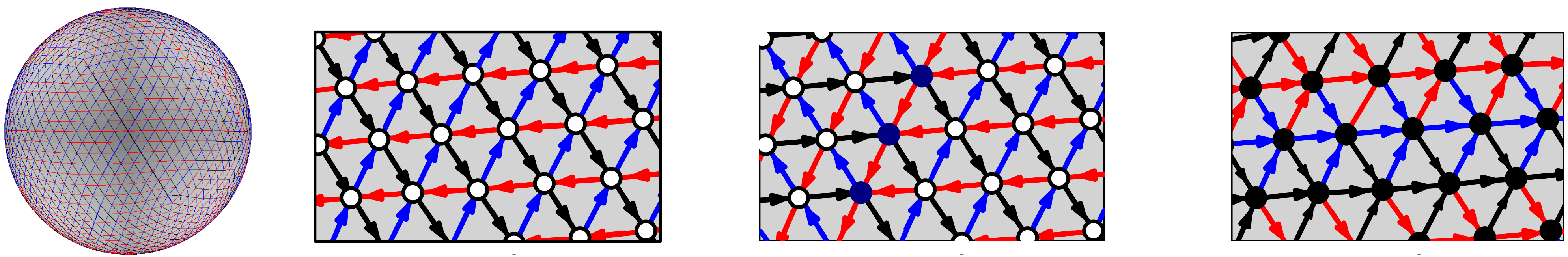
[Felsner Zickfeld '08]

(count of Schnyder woods of a fixed triangulation)

$$2.37^n \leq \max_{T \in \mathcal{T}_n} |SW(T)| \leq 3.56^n$$

$\mathcal{T}_n :=$ class of planar triangulations of size n

$SW(T) :=$ set of all Schnyder woods of the triangulation T



Egalitarian orientations: (only for unconstrained orientations)

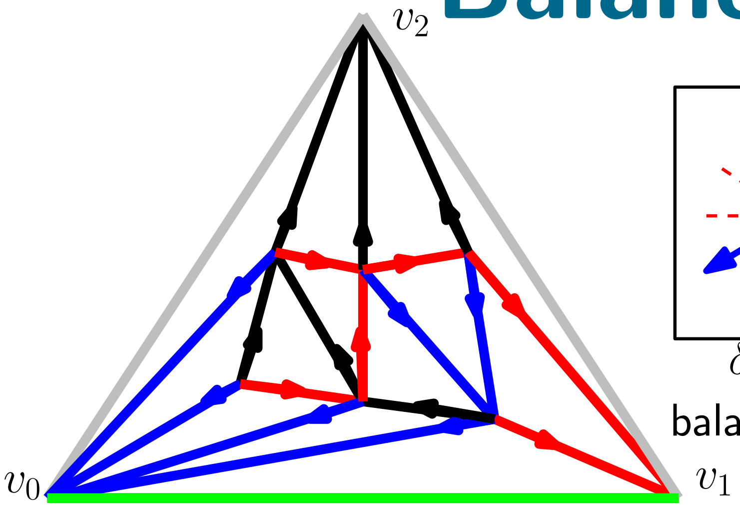
[Borradaile et al. '17]

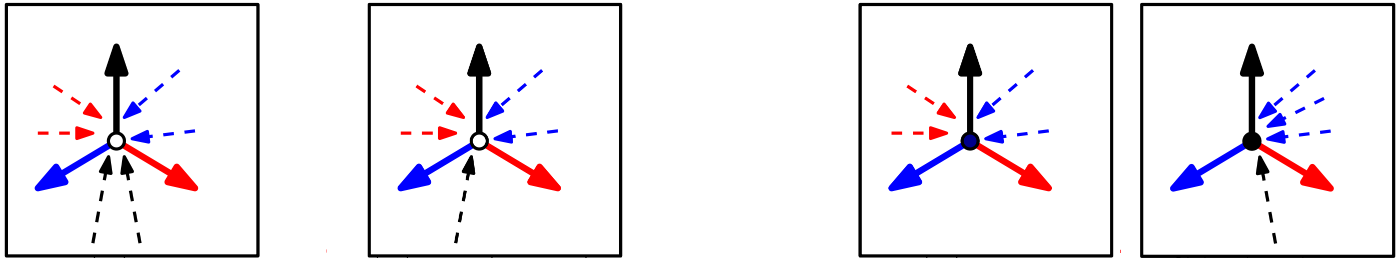
"find an orientation s. t. no vertex is unfairly hit with too many arcs directed into it"

Goal: find an edge orientation that minimizes the lexicographic order of indegrees

(or minimize maximum indegree)

Balanced Schnyder woods





$\delta(v) = 0$ balanced vertex

$\delta(v) = (2-1)-1 = 0$ balanced vertex

$\delta(v) = 1$ $\delta(v) = (3-0)-1 = 2$ unbalanced vertices

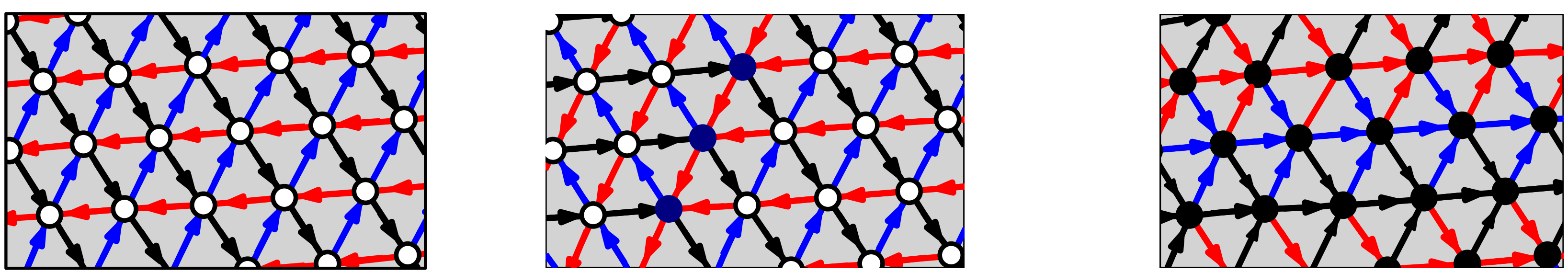
Def

$$\delta(v) := \begin{cases} \max\limits_{i \in \{0,1,2\}} indeg_i(v) - \min\limits_{i \in \{0,1,2\}} indeg_i(v) & \text{if } degree(v) = 3k \\ \max\limits_{i \in \{0,1,2\}} indeg_i(v) - \min\limits_{i \in \{0,1,2\}} indeg_i(v) - 1 & \text{otherwwise} \end{cases}$$

vertex **defect**

$$indeg_i(v) := \#\text{incoming edges of color i}$$

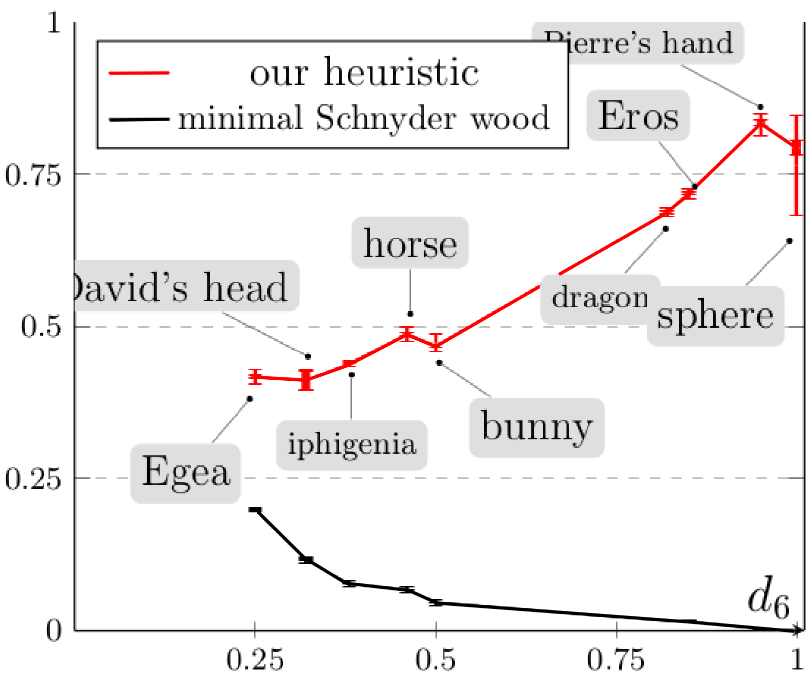
A Schnyder wood is **balanced** if most vertices have a small **defect**



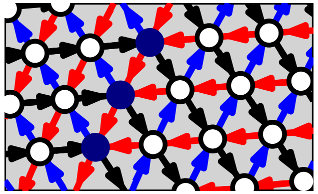
perfectly balanced — well balanced — strongly unbalanced

Computing balanced Schnyder woods

Proportion of balanced vertices

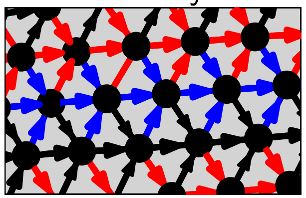


with our heuristic



well balanced

minimal Schnyder wood



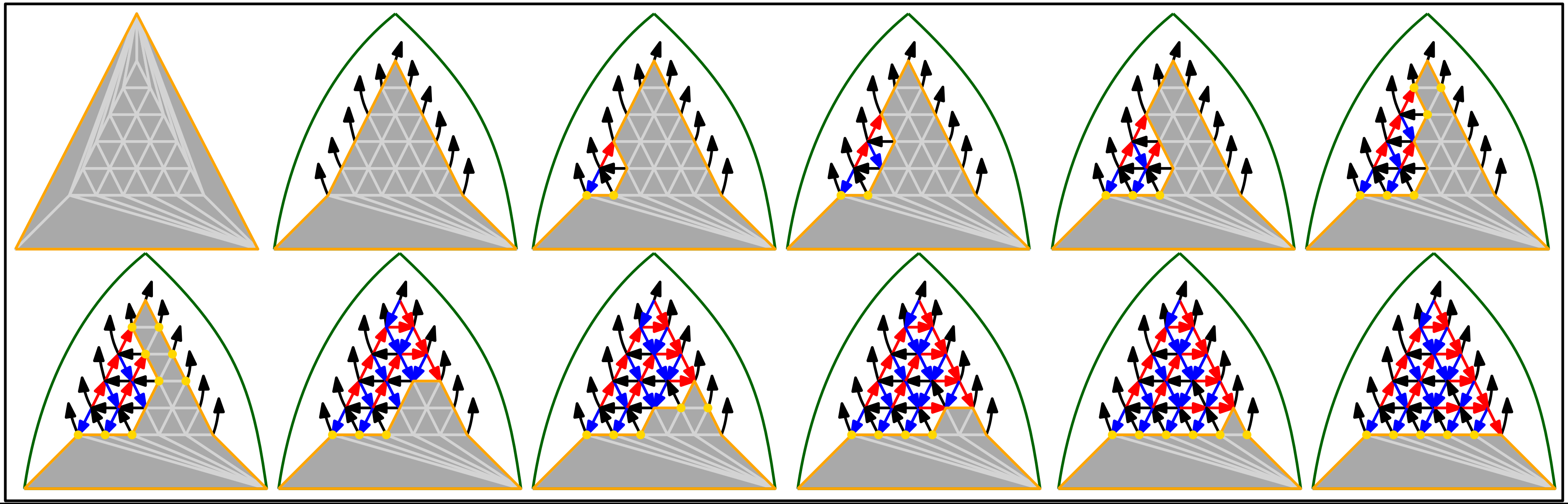
strongly unbalanced

(d_6 := proportion of degree 6 vertices)

```
balancedSchnyderWood(T, (v0, v1, v2), k)
B = {v0, v1, v2} // initialization
T = new int[n] // priority array
Q0 = ∅, Q1 = ∅, ... Q_{k-1} = ∅ // queue initialization
Q0.addLast(v2)
while(|B| ≠ {v0, v1}) {
  let M be the largest index s.t. Q_M ≠ ∅
  let v = Q_M.poll()
  if(v ∈ B and v is free) {
      let {v_l, v_{j1}, ... v_{jt}, v_r} be the neighbors of v on B
      colorOrient(v)
      conquer(v) // remove v from B
      T[v_l]++, T[v_r]++ // increase priority
      Q_{max(k-1,T[v_l])}.addLast(v_l)
      Q_{max(k-1,T[v_r])}.addLast(v_r)
      Q0.addLast(v_{j1}), ..., Q0.addLast(v_{jt})
  }
}
```

Incremental vertex shelling (Brehm's diploma thesis)

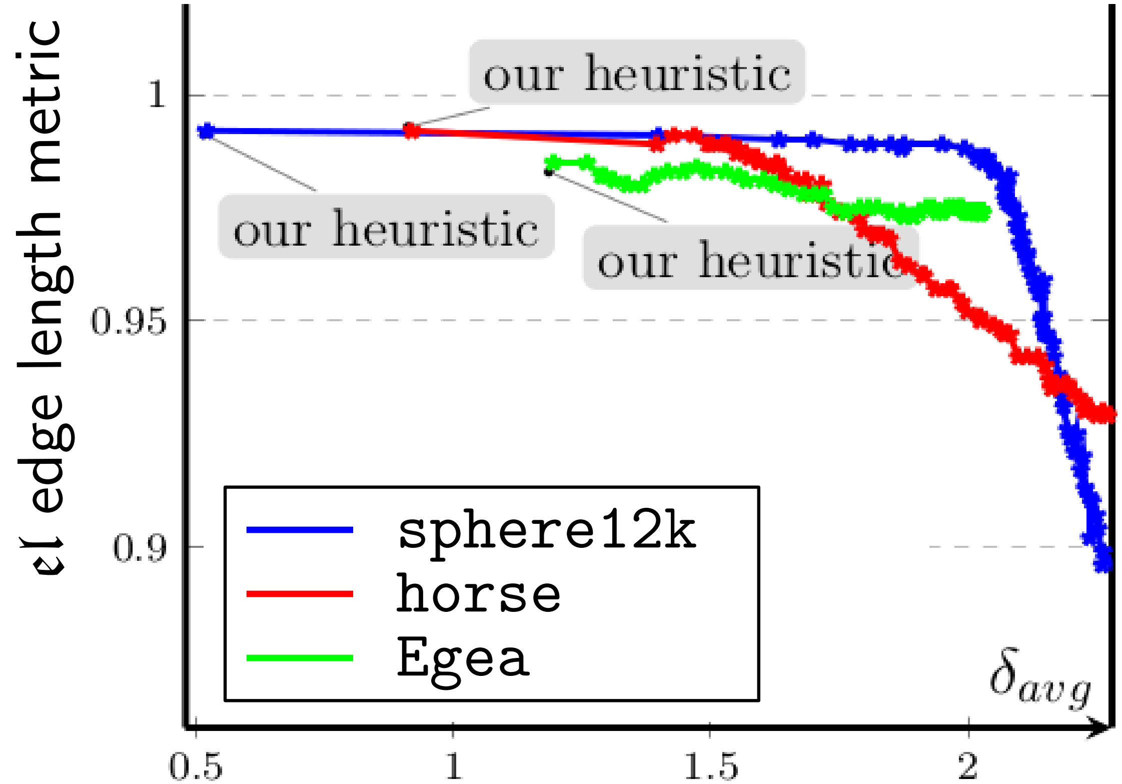
priority driven vertex conquest: remove first boundary vertices with higher number of ingoing edges



Layout quality for Schnyder drawings

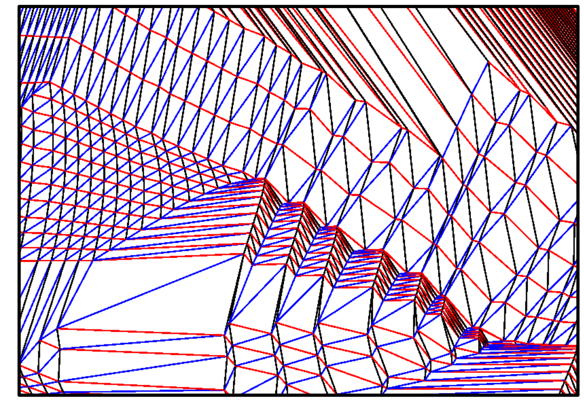
(higher values are better)

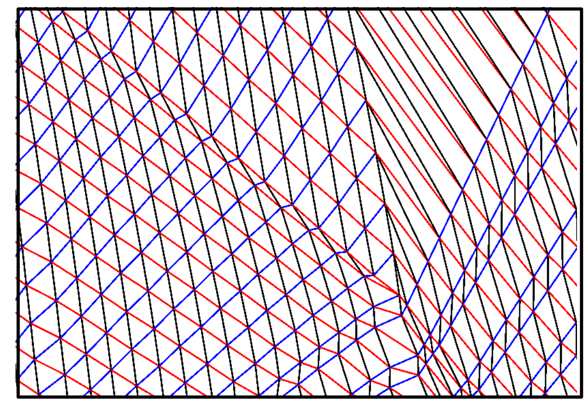
high values indicates more uniform edge length



$\delta_{avg} := \frac{1}{n}\sum_v \delta(v)$ (average vertex defect)

unbalanced

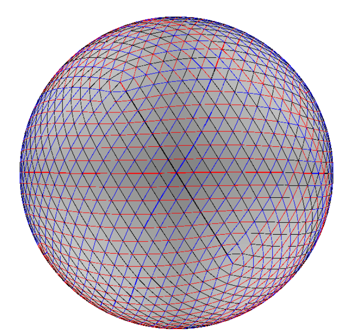




well balanced (our heuristic)

(Fowler and Kobourov, 2012)
average percent deviation of edge length

$$\boxed{\mathfrak{el} := 1 - \left(\frac{1}{|E|}\sum_{e\in E}\frac{|l(e)-l_{avg}|}{\max(l_{avg}, l_{max}-l_{avg})}\right)}$$



$l(e) :=$ edge length of e

From Schnyder woods to cycle separators

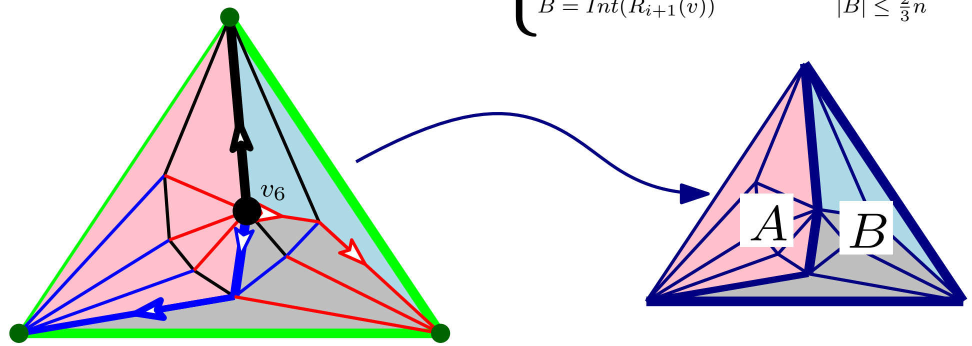
(Fox-Epstein et al. 2016, Holzer et al. 2009)

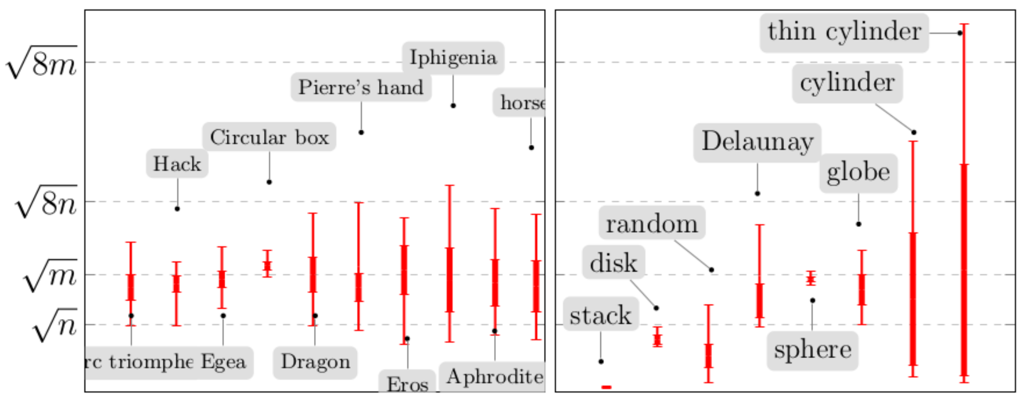
Def (**small balanced cycle separators**)

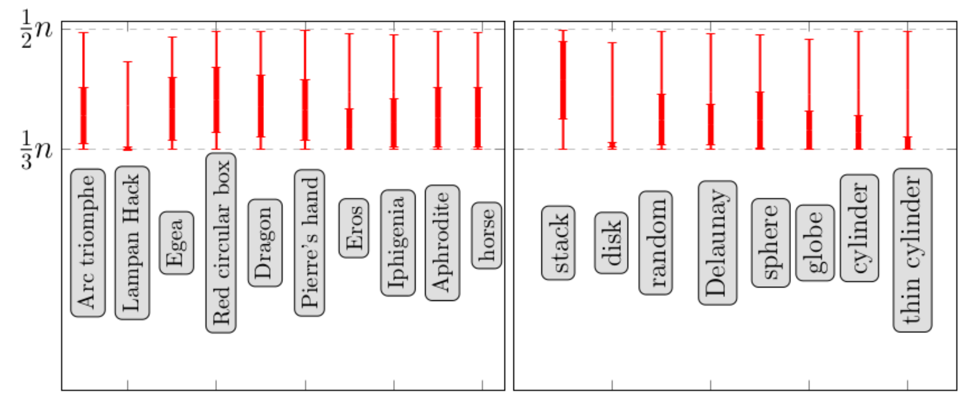
A partition (A, B, S) of $V(G)$ such that:

- S defines a simple cycle
- A and B are balanced: $|A| \leq \frac{2}{3}n$, $|B| \leq \frac{2}{3}n$
- the separator is small: $|S| \leq \sqrt{8m}$

choose the best index i and vertex v s.t. $\begin{cases} S = P_i(v) \cup P_{i+2}(v) \cup \{v\} \text{ is minimized} \\ A = Int(R_i(v) \cup R_{i+2}(v)) \quad |A| \leq \frac{2}{3}n \\ B = Int(R_{i+1}(v)) \quad |B| \leq \frac{2}{3}n \end{cases}$







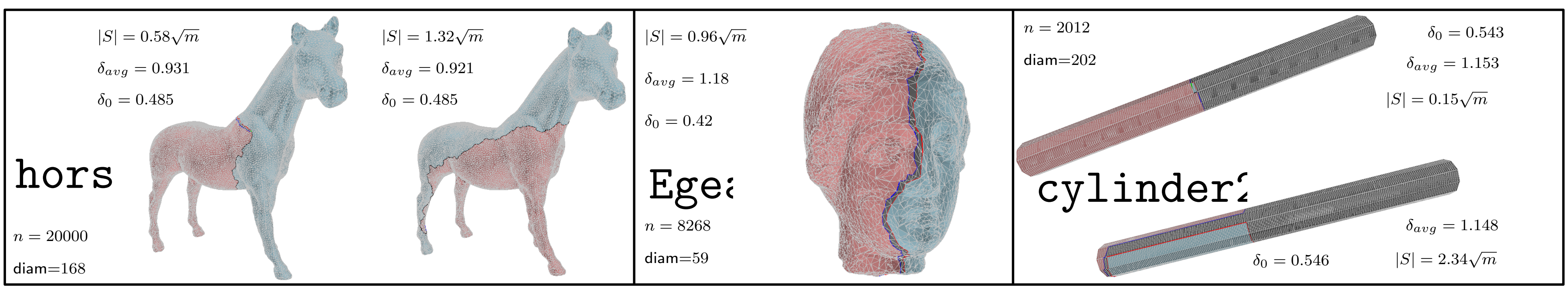
n = number of vertices

m = number of edges

Boundary size

Separator balance

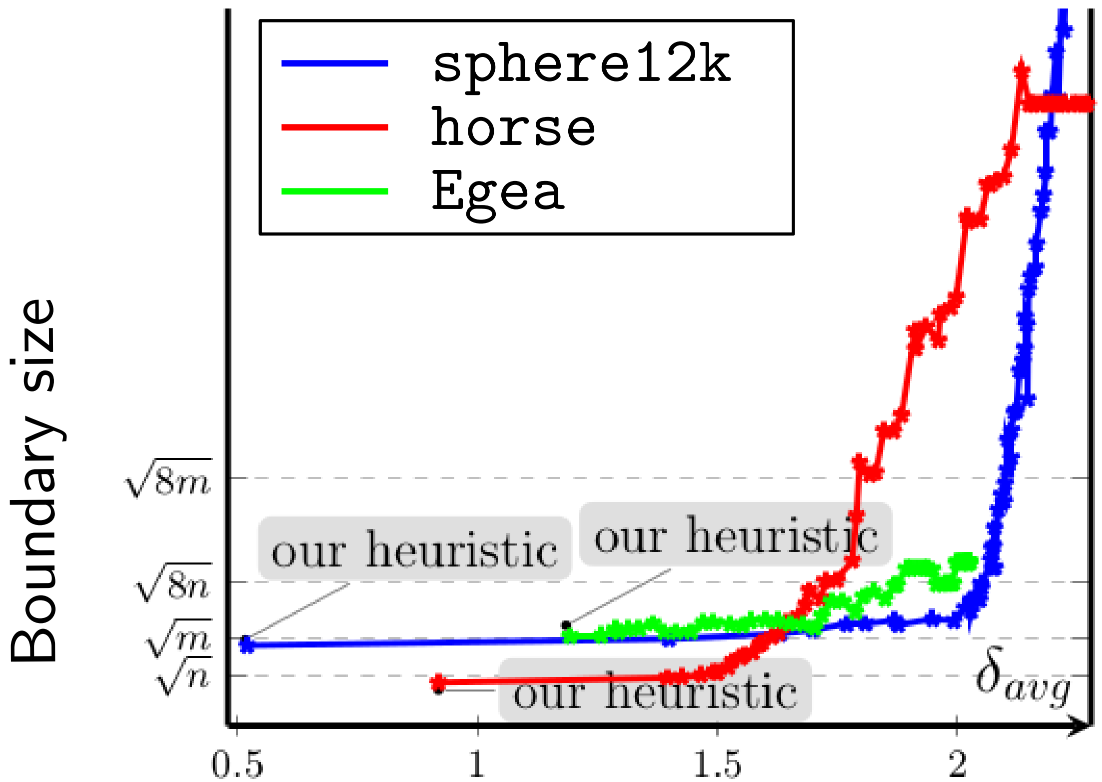
(tests are repated with 200 random choices of the initial **seed**, the root face)



From Schnyder woods to cycle separators

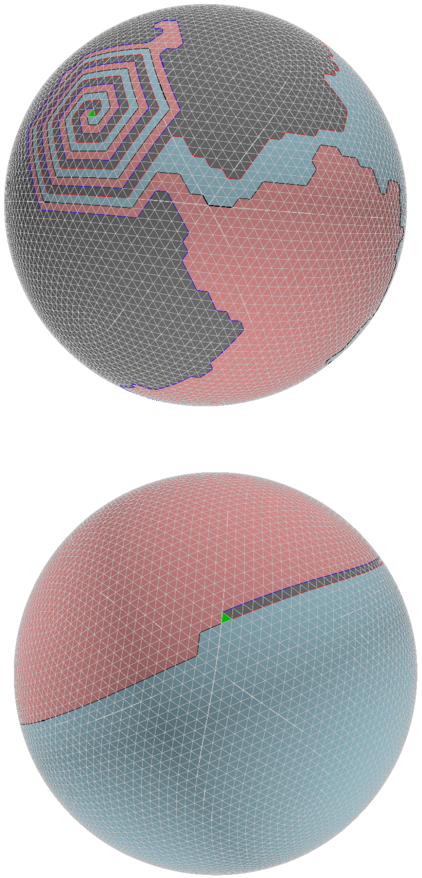
How the separator quality depends on the balance

(lower values are better)



$\delta_{avg} := \frac{1}{n}\sum_v \delta(v)$ (average vertex defect)

unbalanced

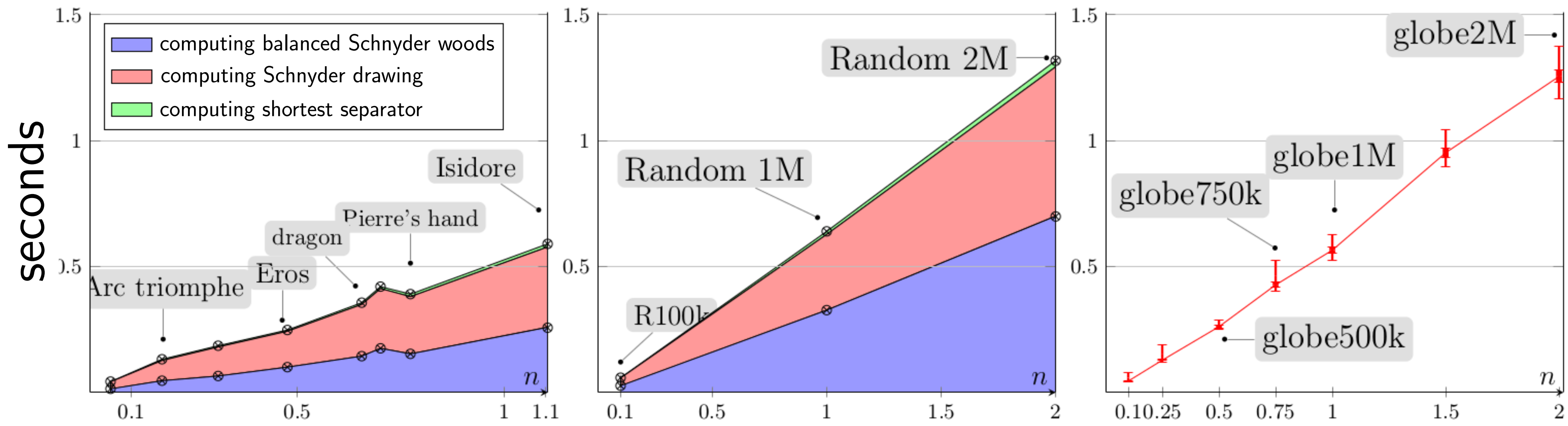


well balanced (our heuristic)

Evaluation of timing costs

average timings (over 100 executions)

total timing costs
(100 choice of random seeds)

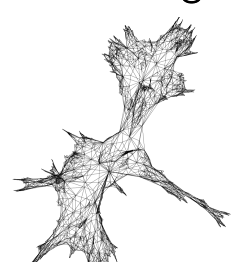


- **Our performances** (pure **Java**, on a core i7-5600 U, 2.60GHz, 1GB Ram):
  We can process $\approx 1.43M - 1.92M$ vertices/seconds
- Metis can process $\approx 0.7M$ vertices/seconds (**C**, on a Intel core i7-5600 2.60GHz)
- Previous works can process $\approx 0.54M - 0.62M$ vertices/seconds
  (Fox-Epstein et al. 2016, Holzer et al. 2009) (**C/C++**, on a Xeon X5650 2.67GHz)

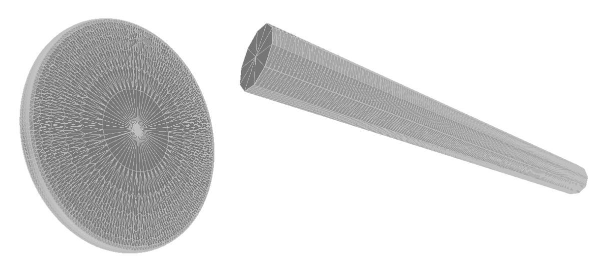
Our datasets
(several tens of real-world, random and synthetic graphs)

3d meshes from **aim@shape** and **Thingi 10k**

Random triangulations



Synthetic graphs



Thanks

Improving the balance (returning oriented cycles)