Minimal Representations of Order Types by Geometric Graphs

Aichholzer[1], Balko[2], Hoffmann[3], Kynčl[2], Mulzer[4], Parada[1], Pilz[1,3], Scheucher[5], Valtr[2], Vogtenhuber[1], and Welzl[3]

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[2] Charles University, Prague
[3] ETH Zürich
[4] FU Berlin
[5] TU Berlin

Combinatorics of Point Sets

Infinite number of point sets $\Rightarrow$ Finite number classes

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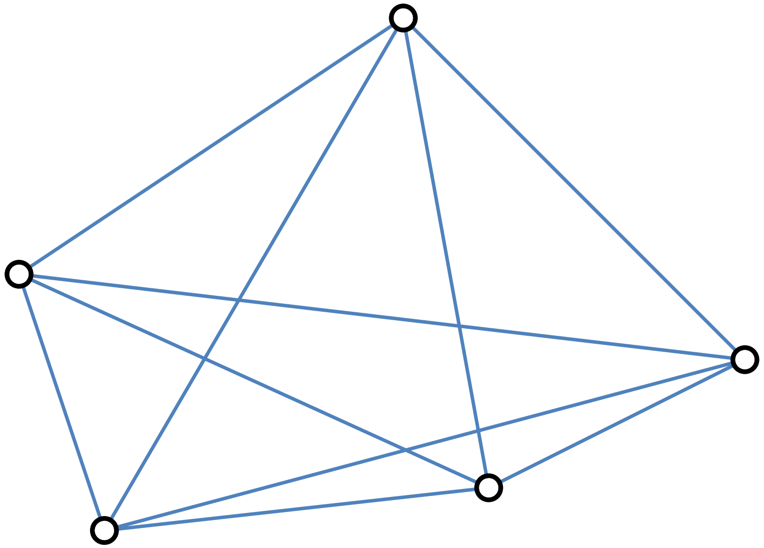
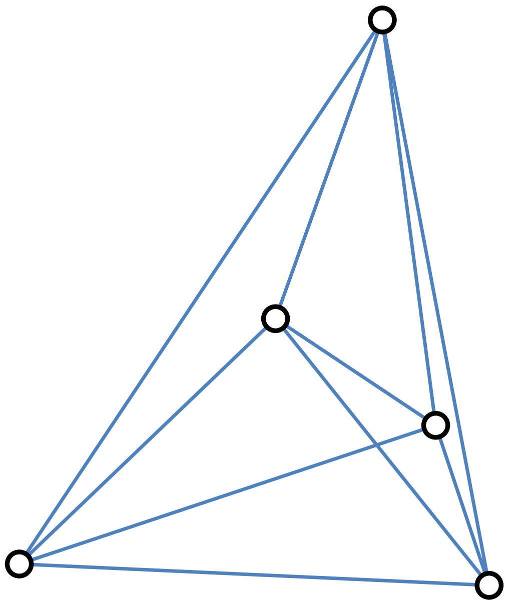
Infinite number of point sets ⇒ Finite number classes





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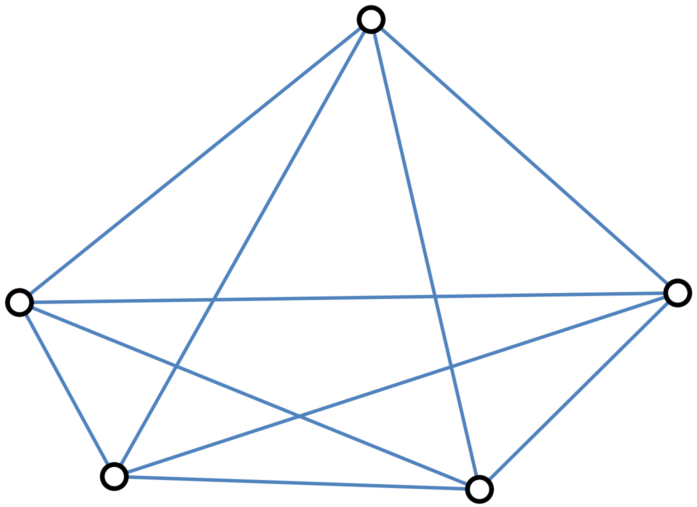

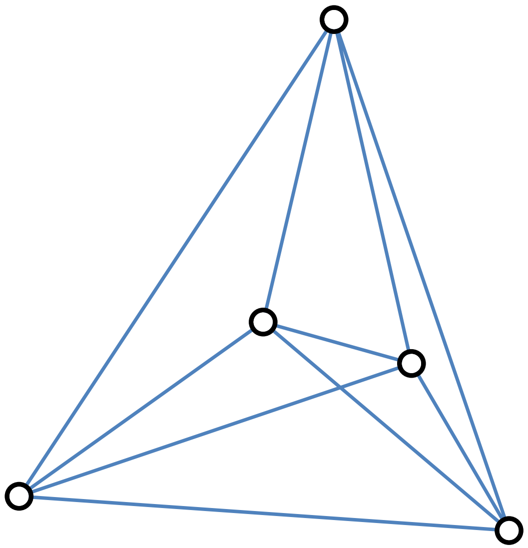

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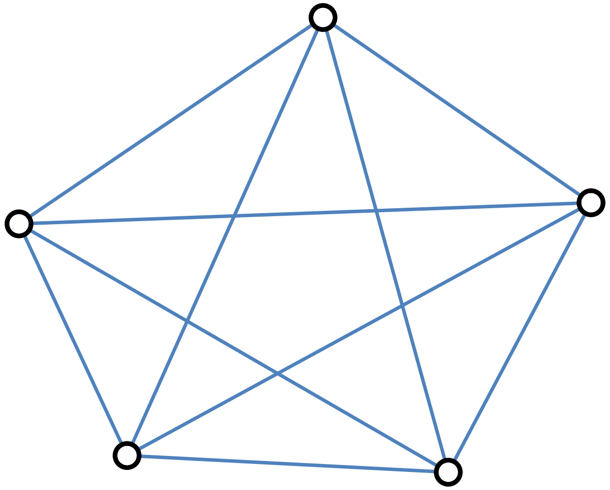

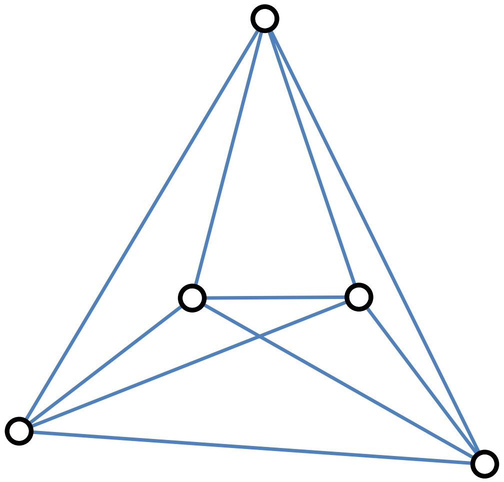

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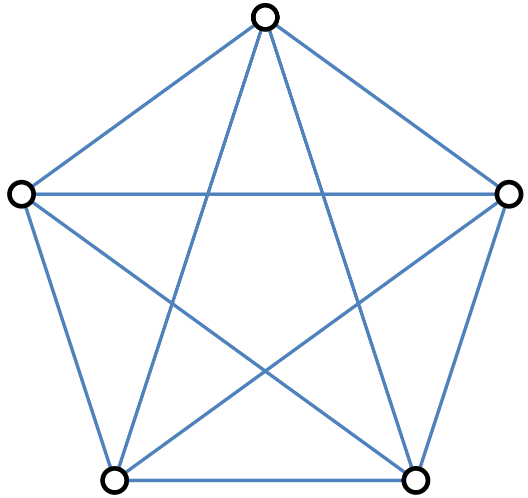
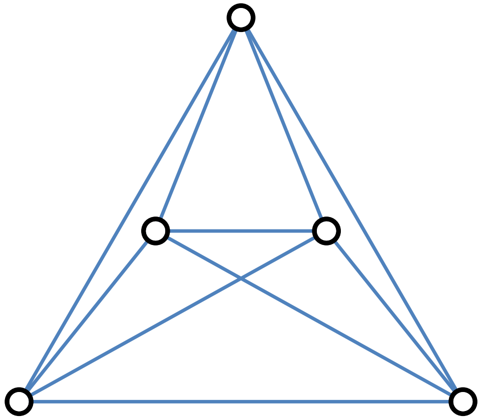
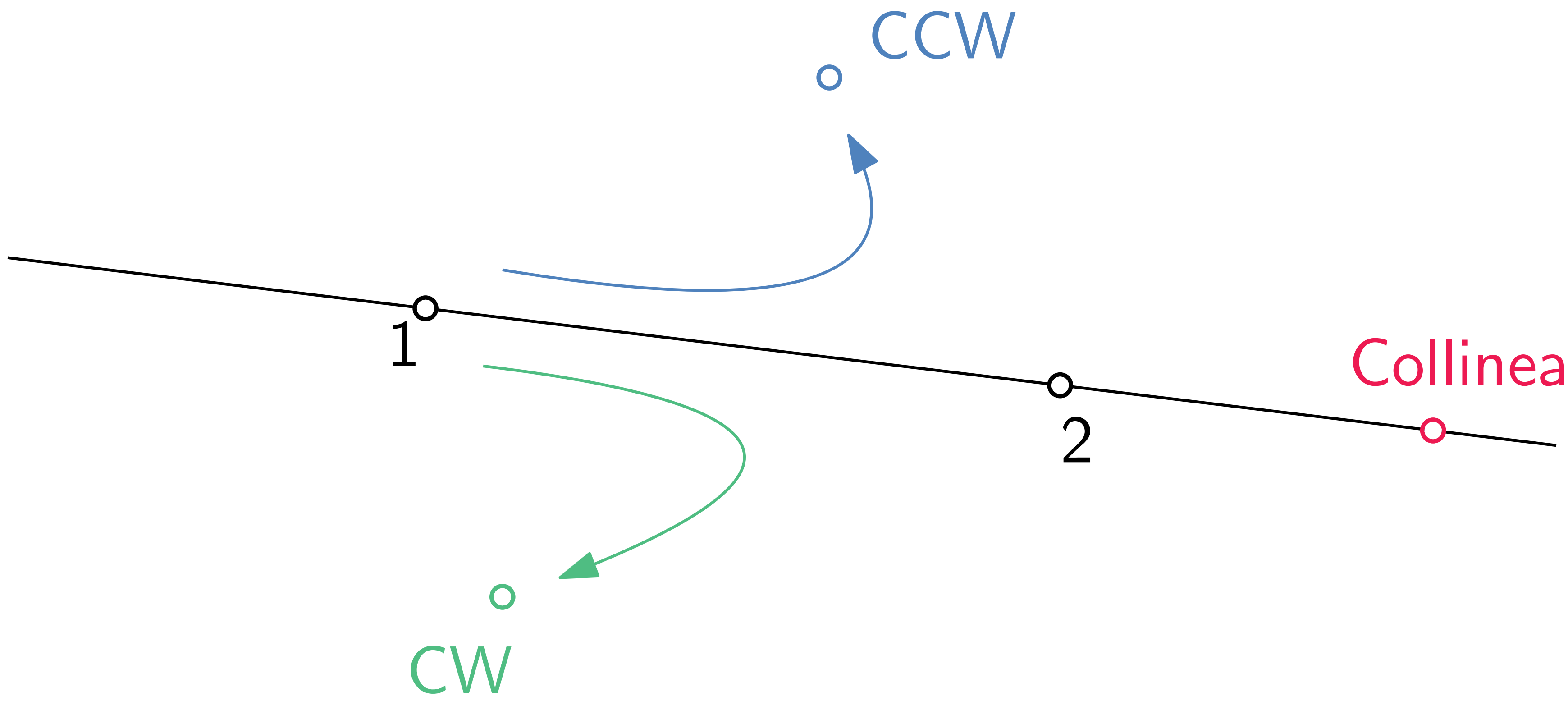




Order Types

Order types are the equivalence classes of point sets in the plane with respect to their triple-orientations.

Triple orientations: clockwise, counter clockwise, collinear



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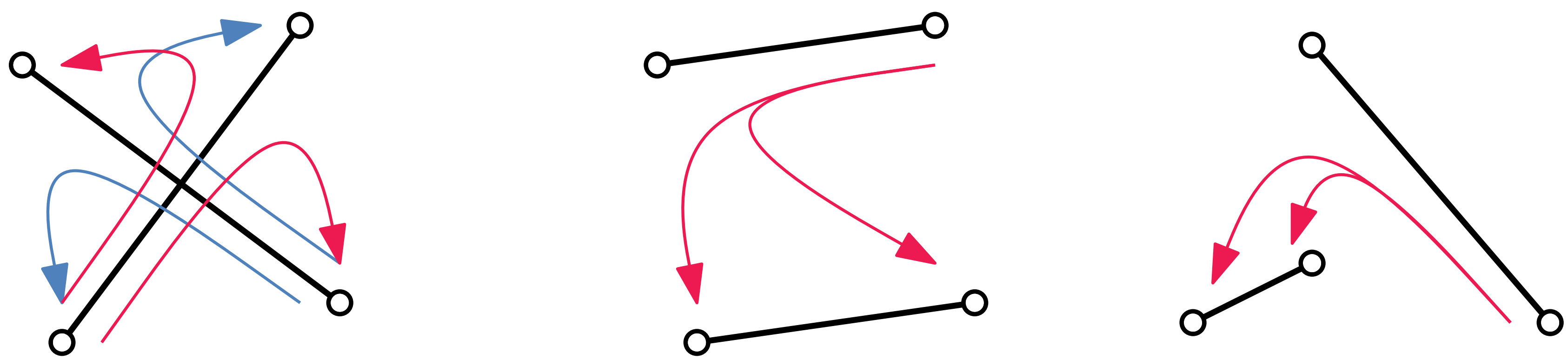
We can determine whether two edges cross from the triple orientations

No crossing:



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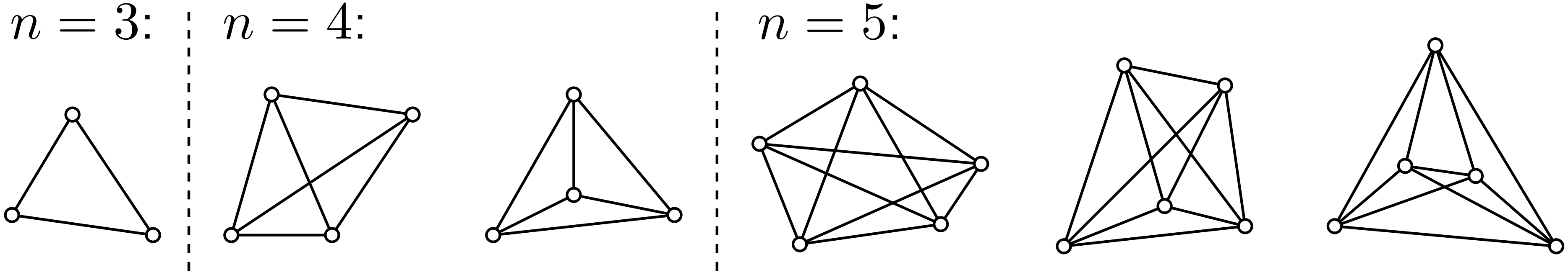
$n = 3$: | $n = 4$: | $n = 5$:



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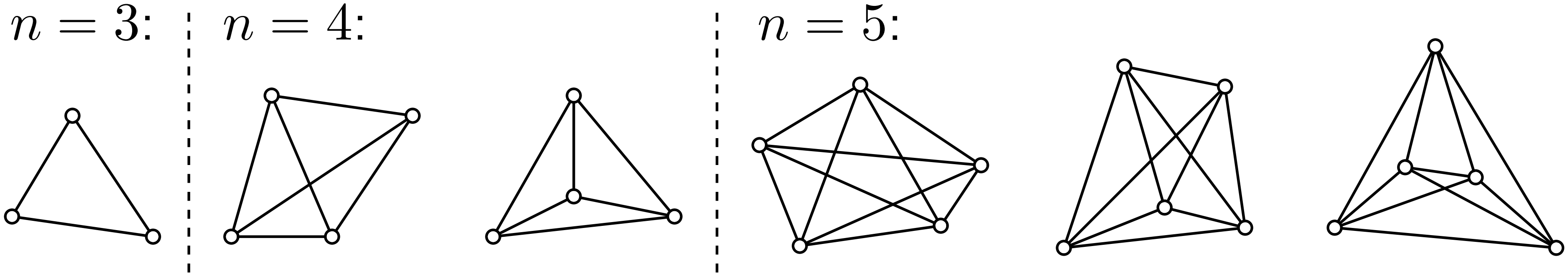

n	3	4	5	6	7	8	9	10	11
OT	1	2	3	16	135	3 315	158 817	14 309 547	2 334 512 907

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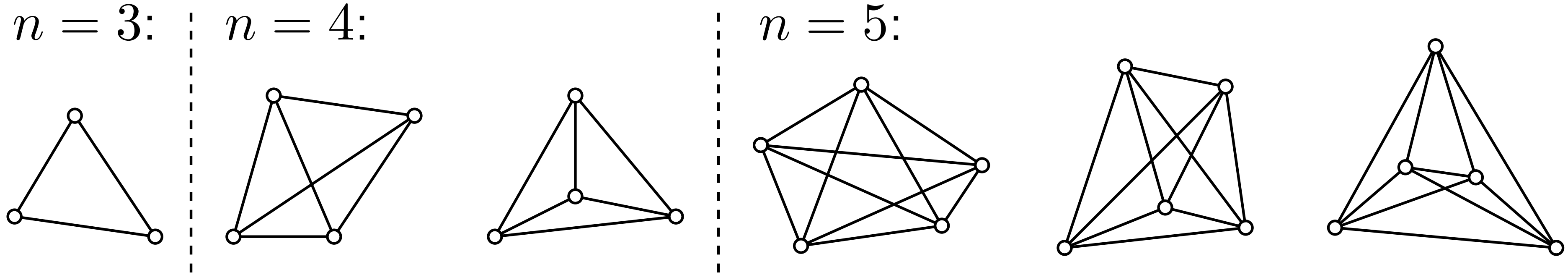

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Nr. of order types: $n^{4n+O(n/\log n)}$ [Goodman & Pollack '86]

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- Explicit coordinates

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- Triple orientations ☹️
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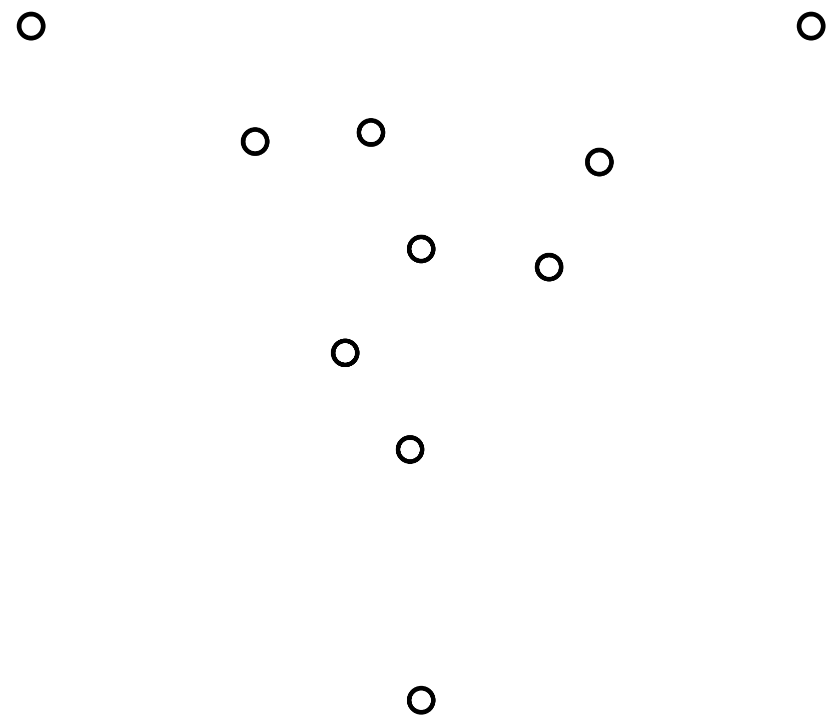
- Figure of the point set



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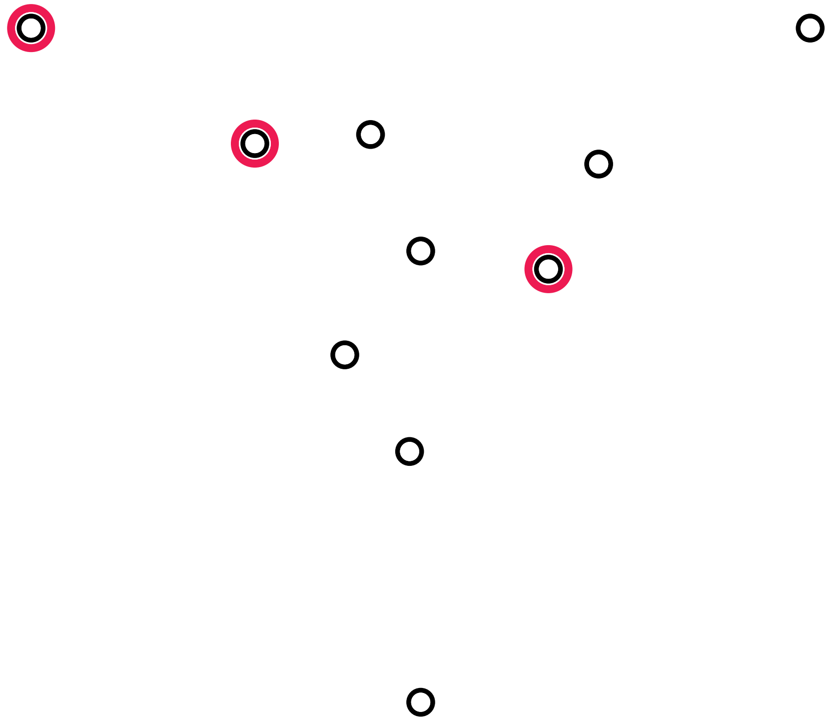

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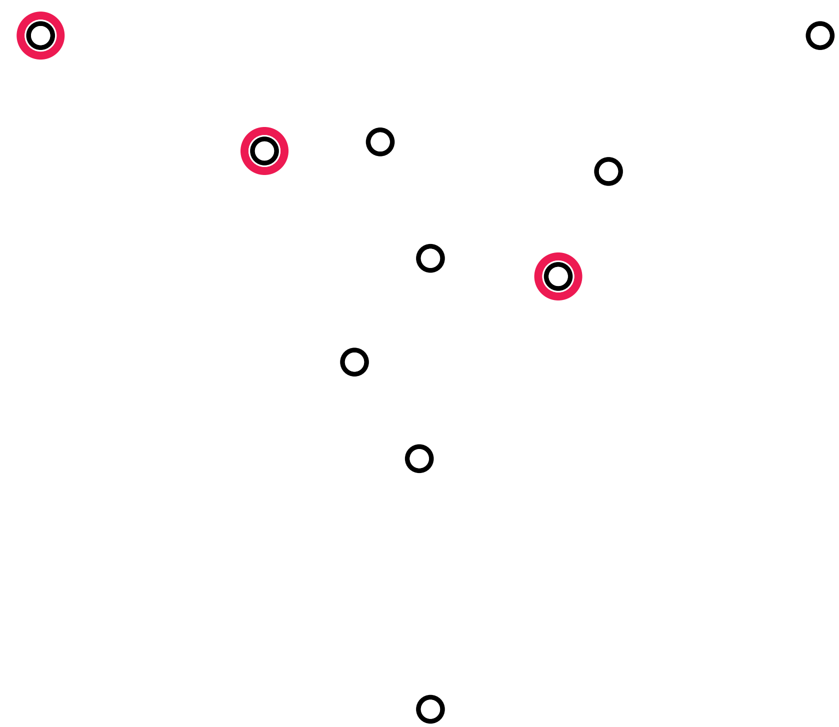

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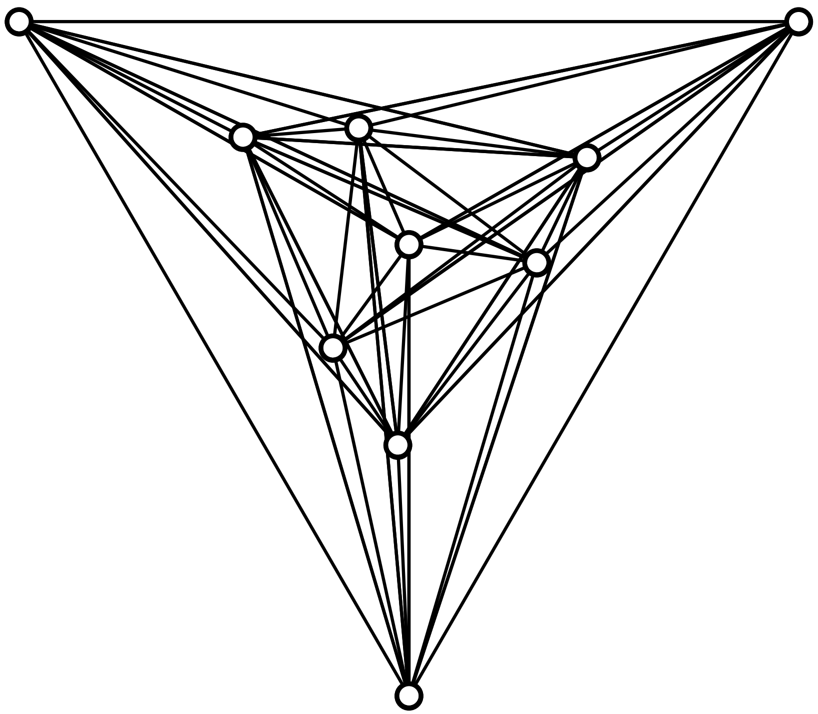
- + spanned lines/segments

Complete *geometric graph*: vertices mapped points, edges drawn as straight-line segments.



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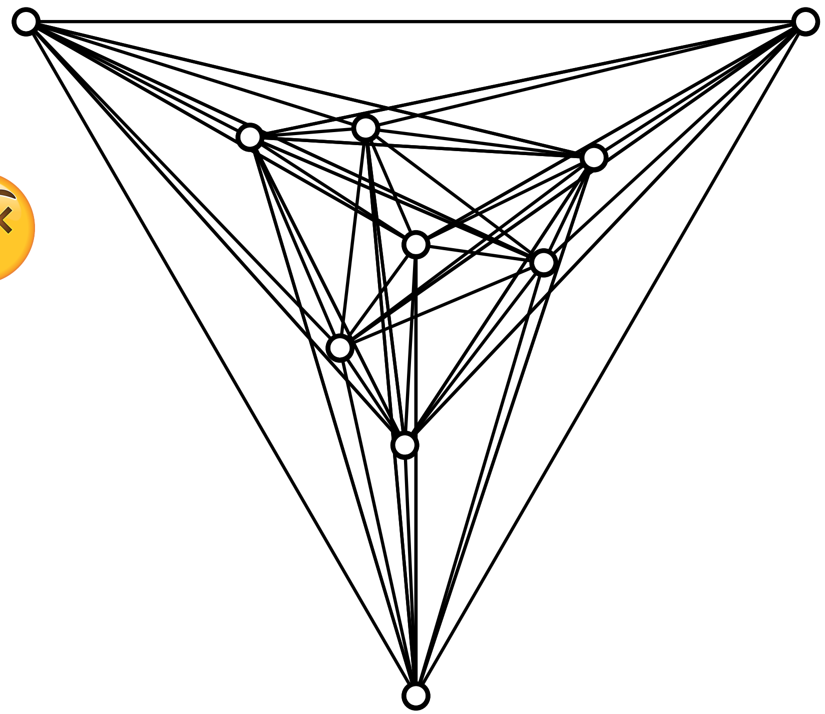

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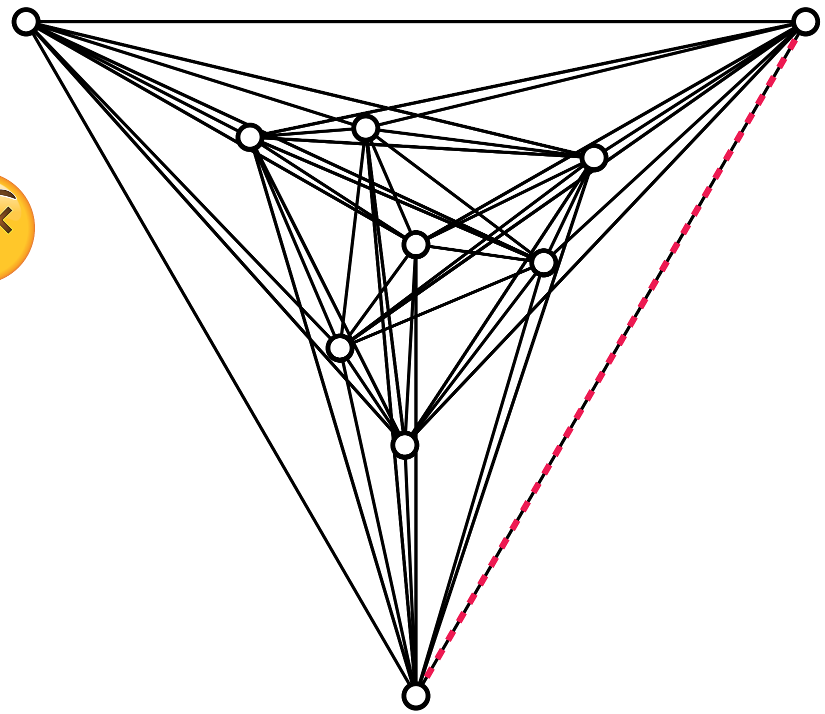
- Points + non-redundant edges



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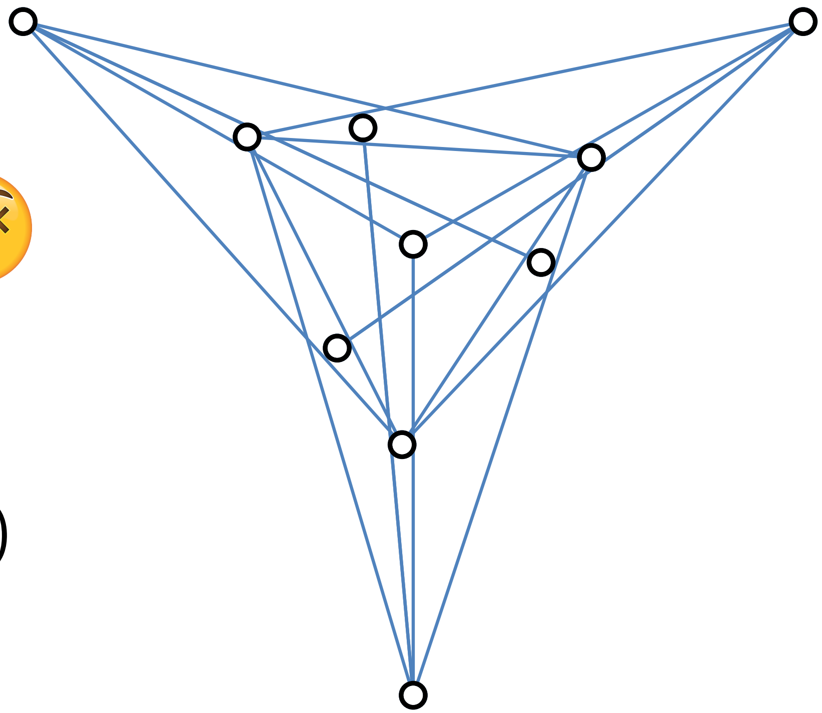
  15 edges drawn (total: $45 = \binom{10}{2}$)



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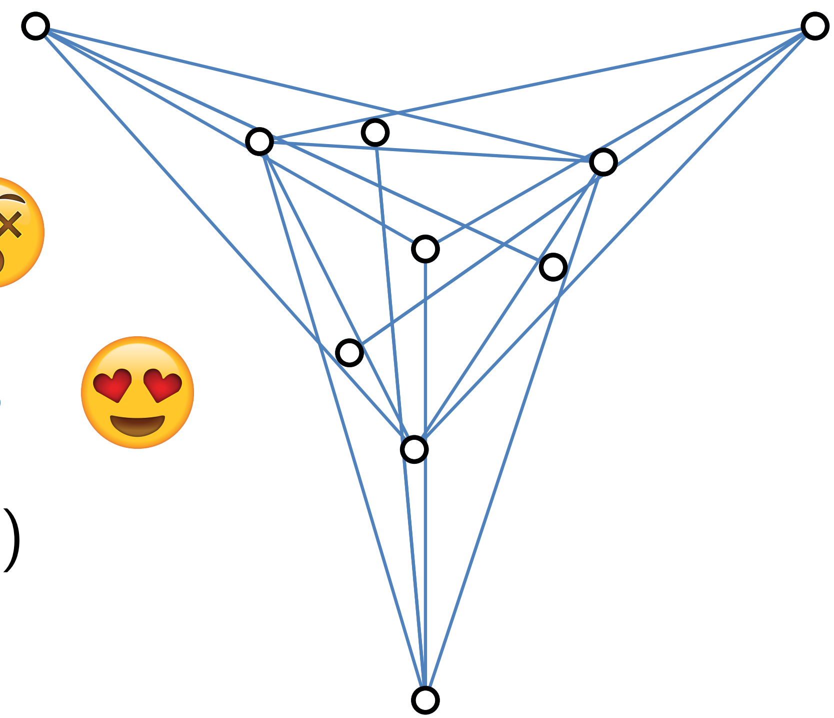


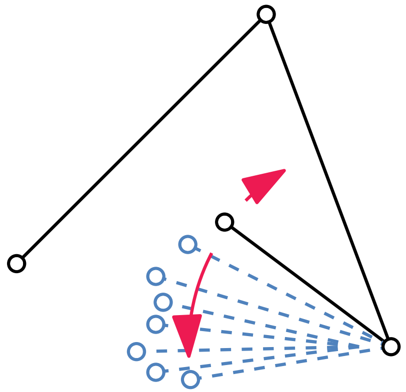
Geometric Graphs Supporting Point Sets

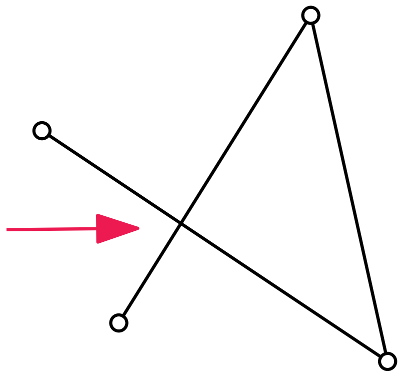
We consider "topology-preserving deformations".

A geometric graph G *supports* a set S of points if every "continuous deformation" that

- keeps edges straight and
- allows at most 3 points to be collinear at the same time

also preserves the order type of the vertex set.





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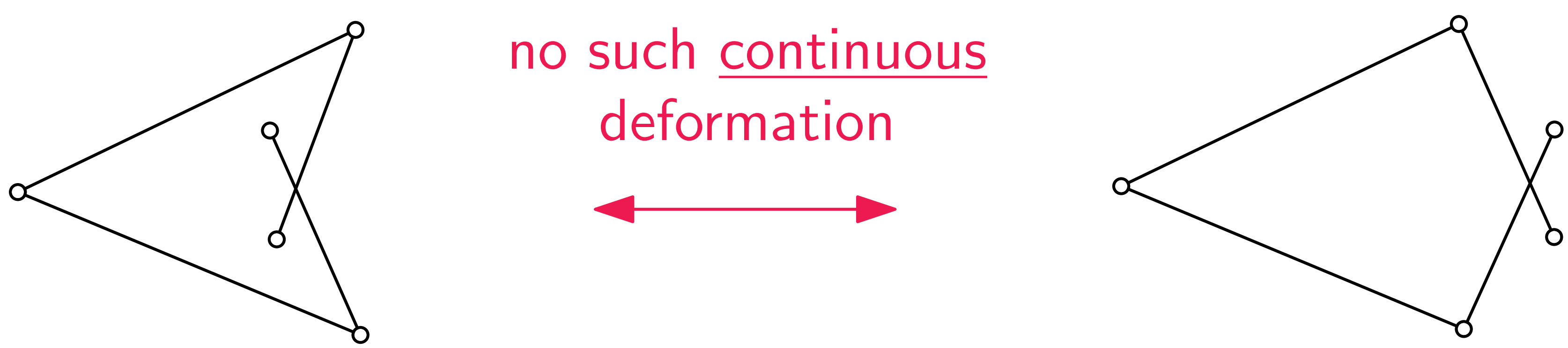

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An *ambient isotopy* of the real plane is a continuous map $f\colon \mathbb{R}^2 \times [0,1] \to \mathbb{R}^2$ such that $f(\cdot,t)$ is homeomorphism for all $t \in [0,1]$ and $f(\cdot,0) = Id$.

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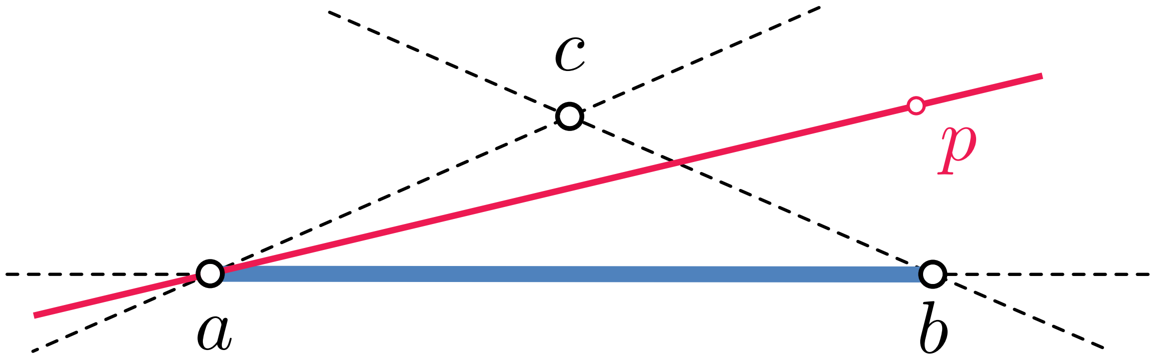
Every complete geometric graph is supporting.

Exit Edges: Definition

S set of $n \geq 4$ points in general position (no 3 collinear).

The edge ab is an *exit edge with witness* c if there is no point $p \in S$ such that the line $\overline{ap}$ separates b from c or the line $\overline{bp}$ separates a from c.



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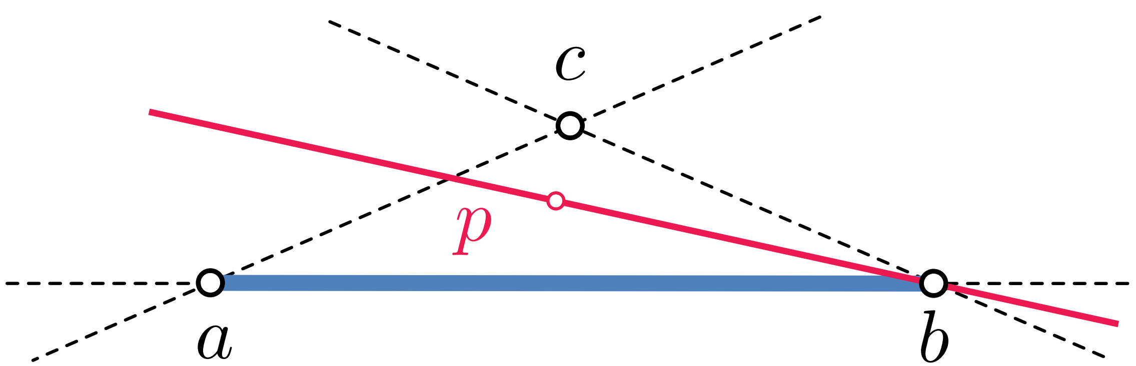

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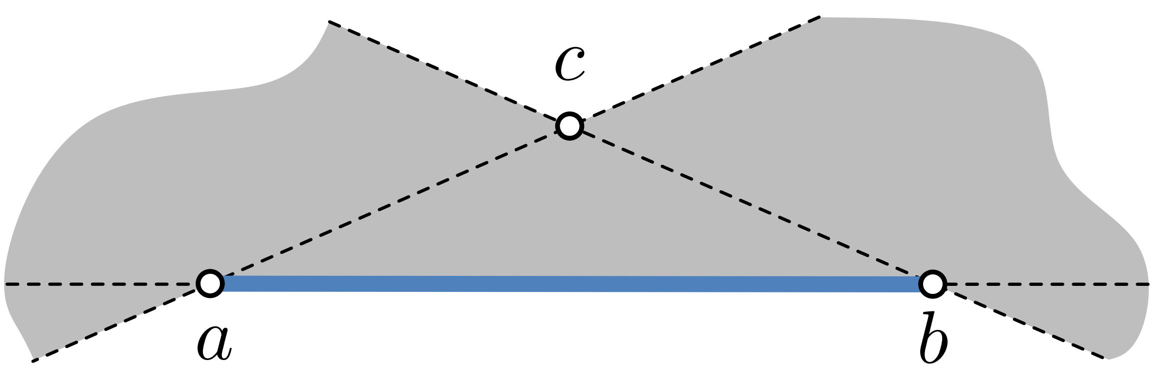
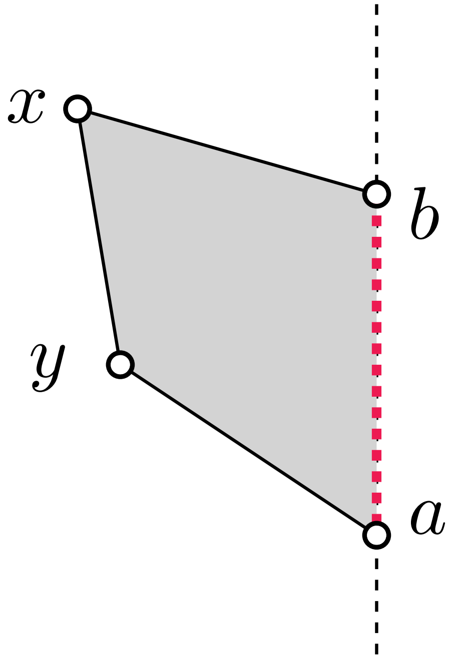
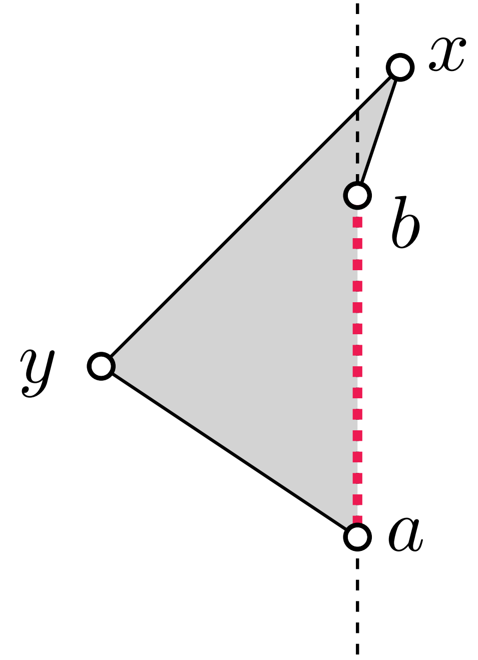


Exit Edges: Alternative Characterization

The edge ab is <u>not</u> an exit edge if and only if:

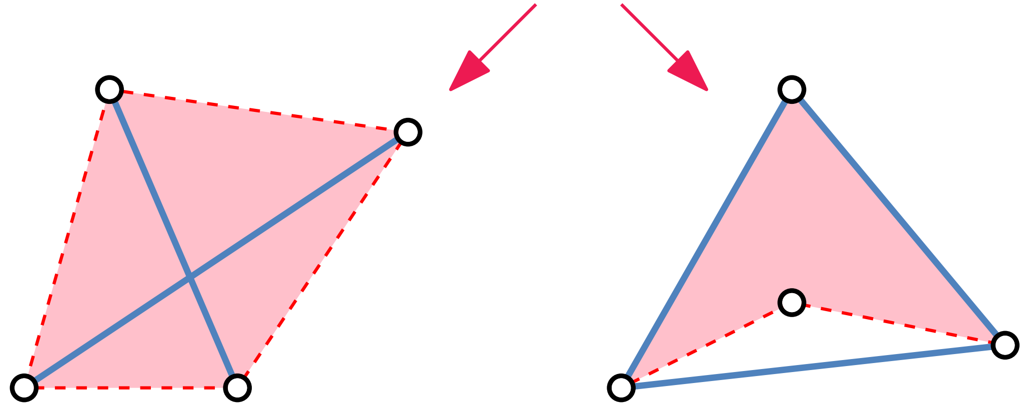
- ab external & incident to convex 4-hole or
- ab internal & incident to general 4-hole on each side, with the reflex angle (if any) incient to ab.





Exit Edges: Small Point Sets

$n = 4$: classification via 4-holes



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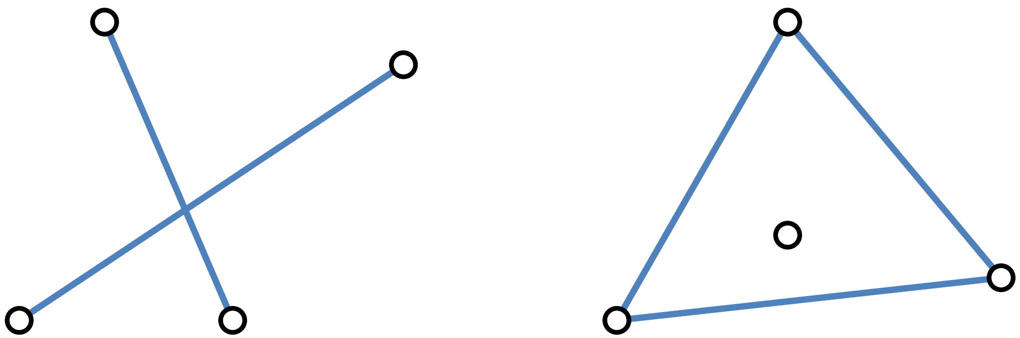
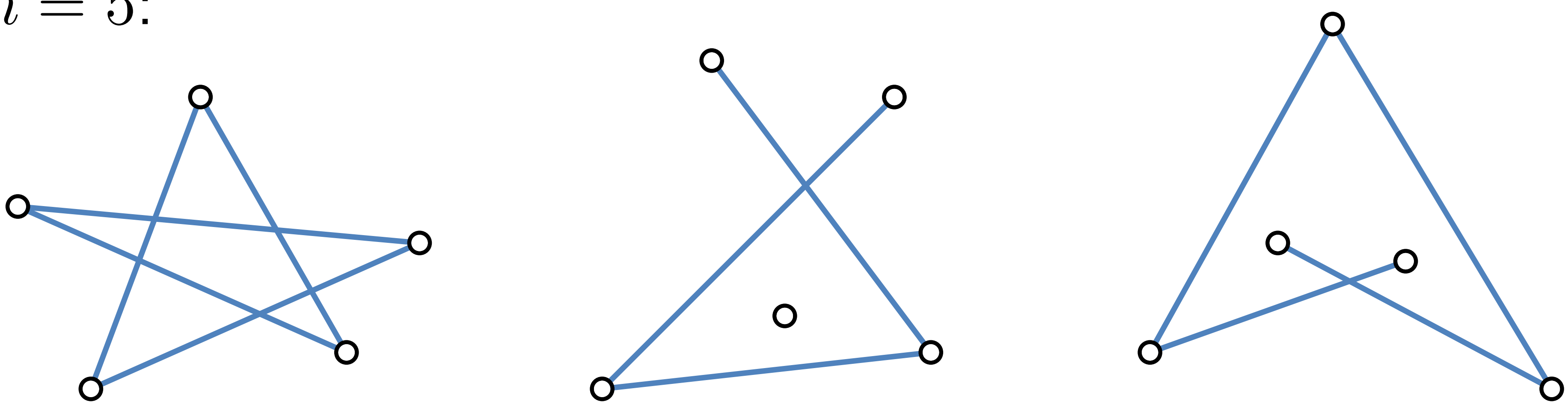
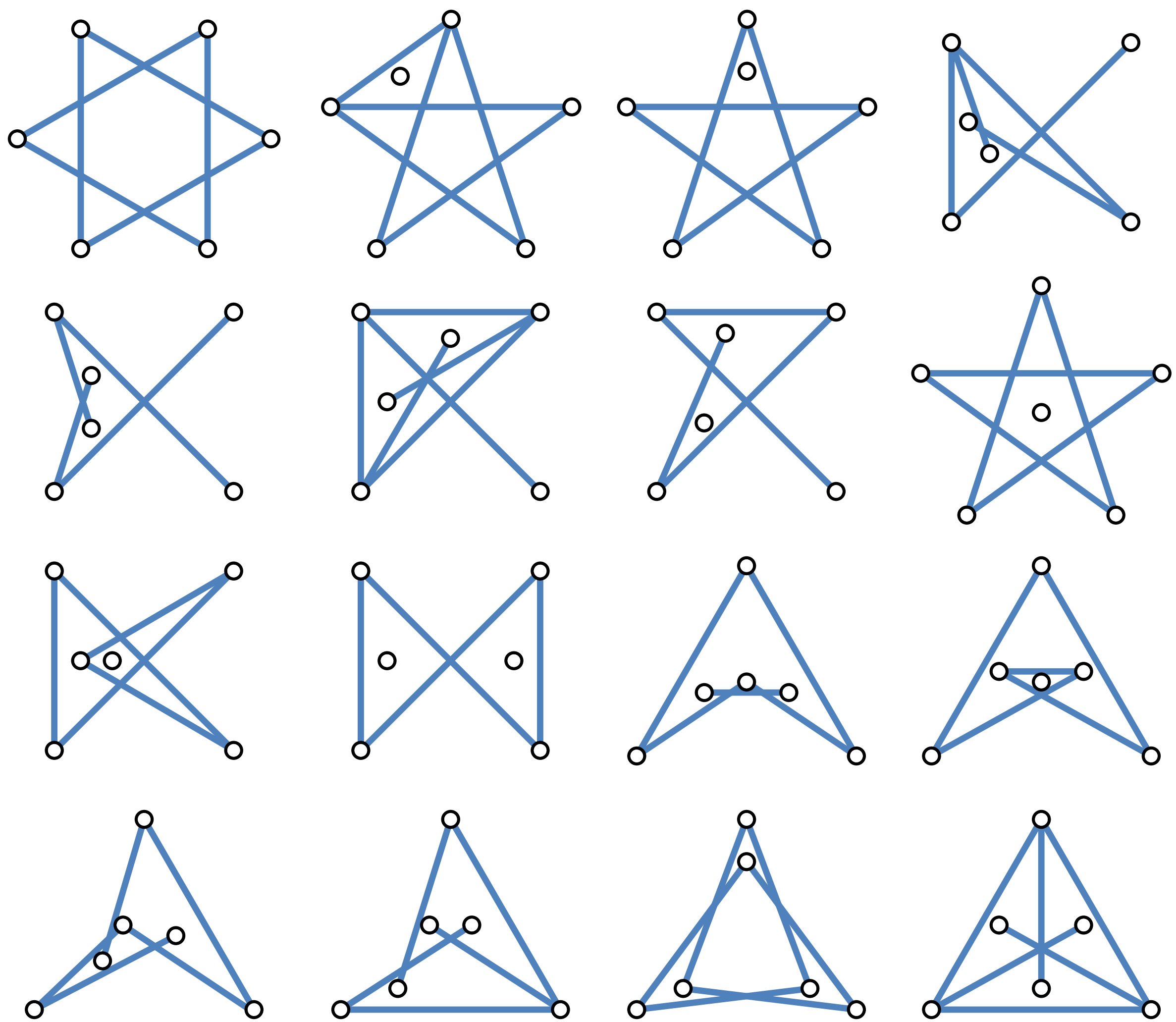


$n = 5$:



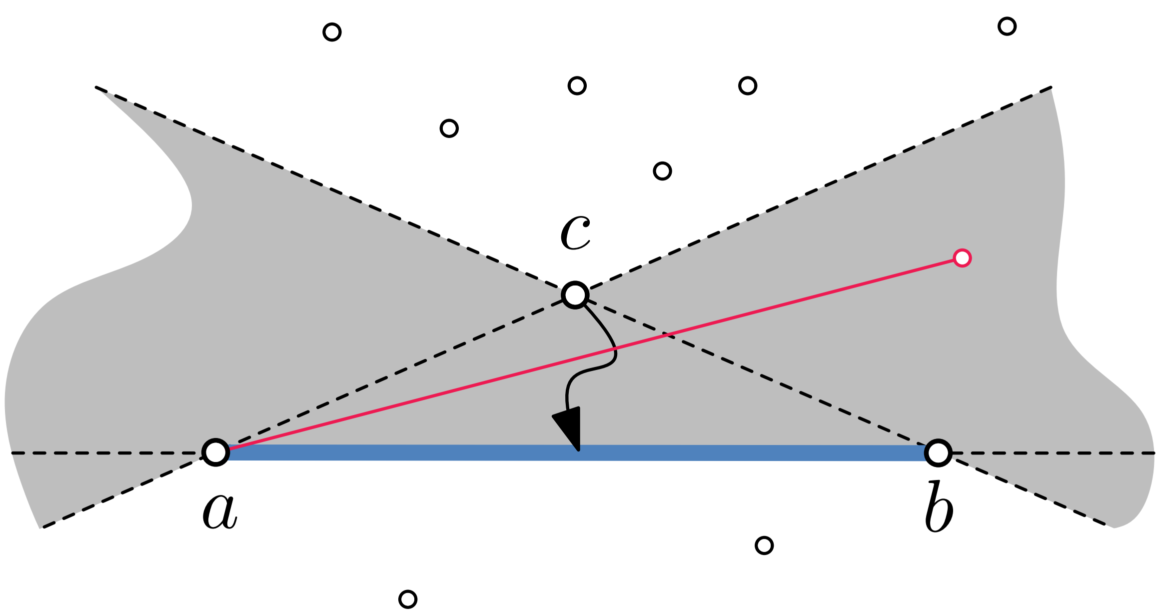
Exit Edges: Small Point Sets

$n = 6$:



Exit Graph Is Supporting

Let $S(t)$ be a continuous deformation of S at time t.
Let (a, b, c) be the first triple to become collinear, at $t_0 > 0$.
If c lies on the segment ab in $S(t_0)$,
then ab is an exit edge in $S(0)$ with witness c.



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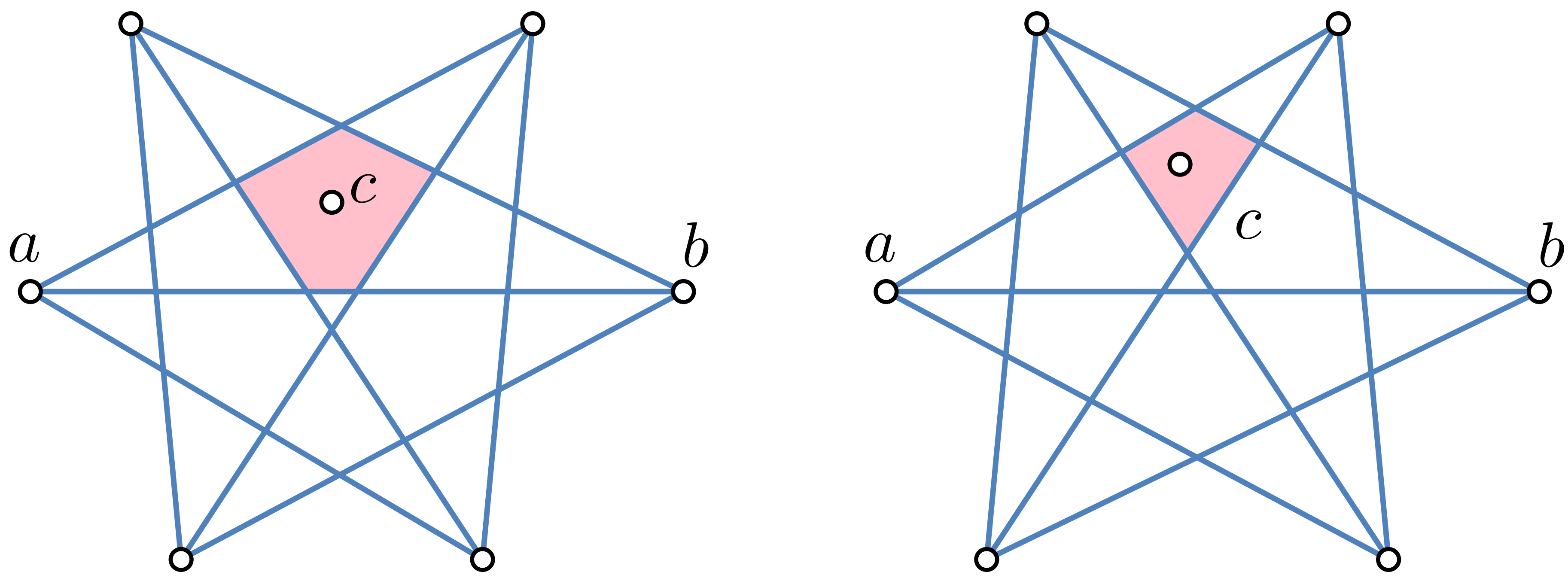
The graph of exit graph (whose edges are the exit edges) of every point set is supporting.

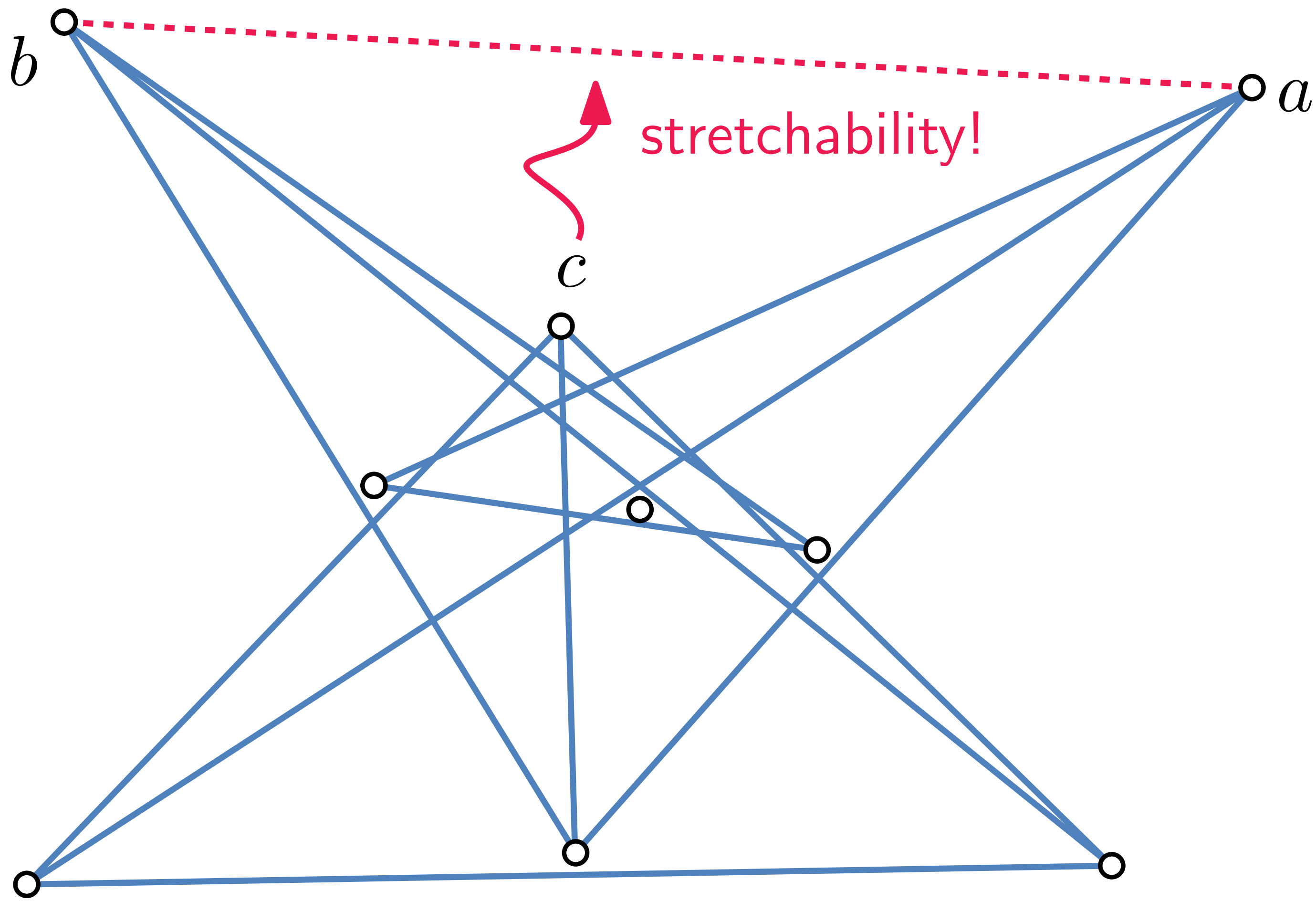
Exit Edges: Observations

It is not always possible to make the witness c reach the corresponding exit edge ab.



Exit Edges: Observations

Exit edges are not necessary in a supporting graph.



Duality

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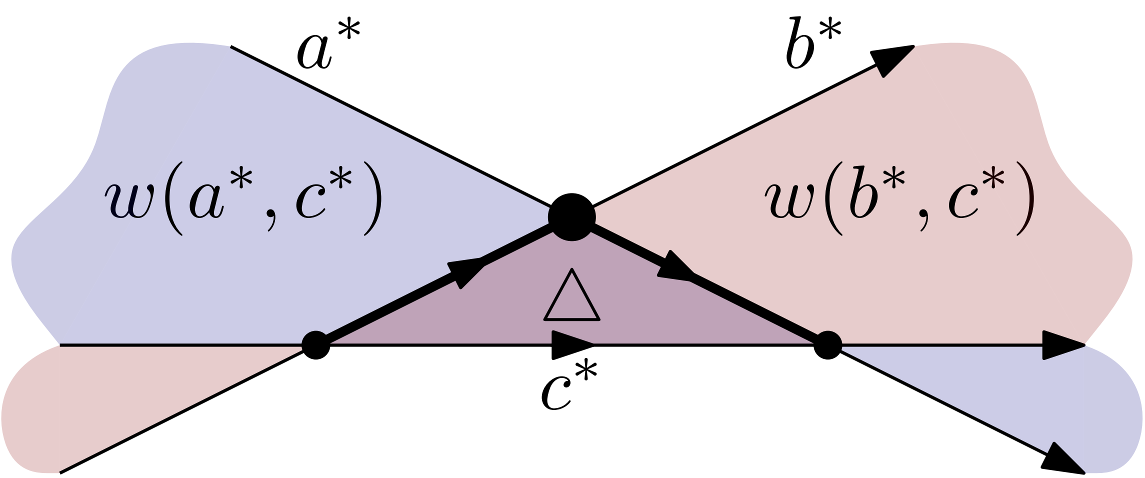
The edge ab is an exit edge with witness point c in S if and only if the lines a^*, b^*, and c^* bound an unmarked triangular cell in S^* with c^* being the *witness line* and $\overline{ab}^*$ $(= a^* \cap b^*)$ being the *exit vertex*.



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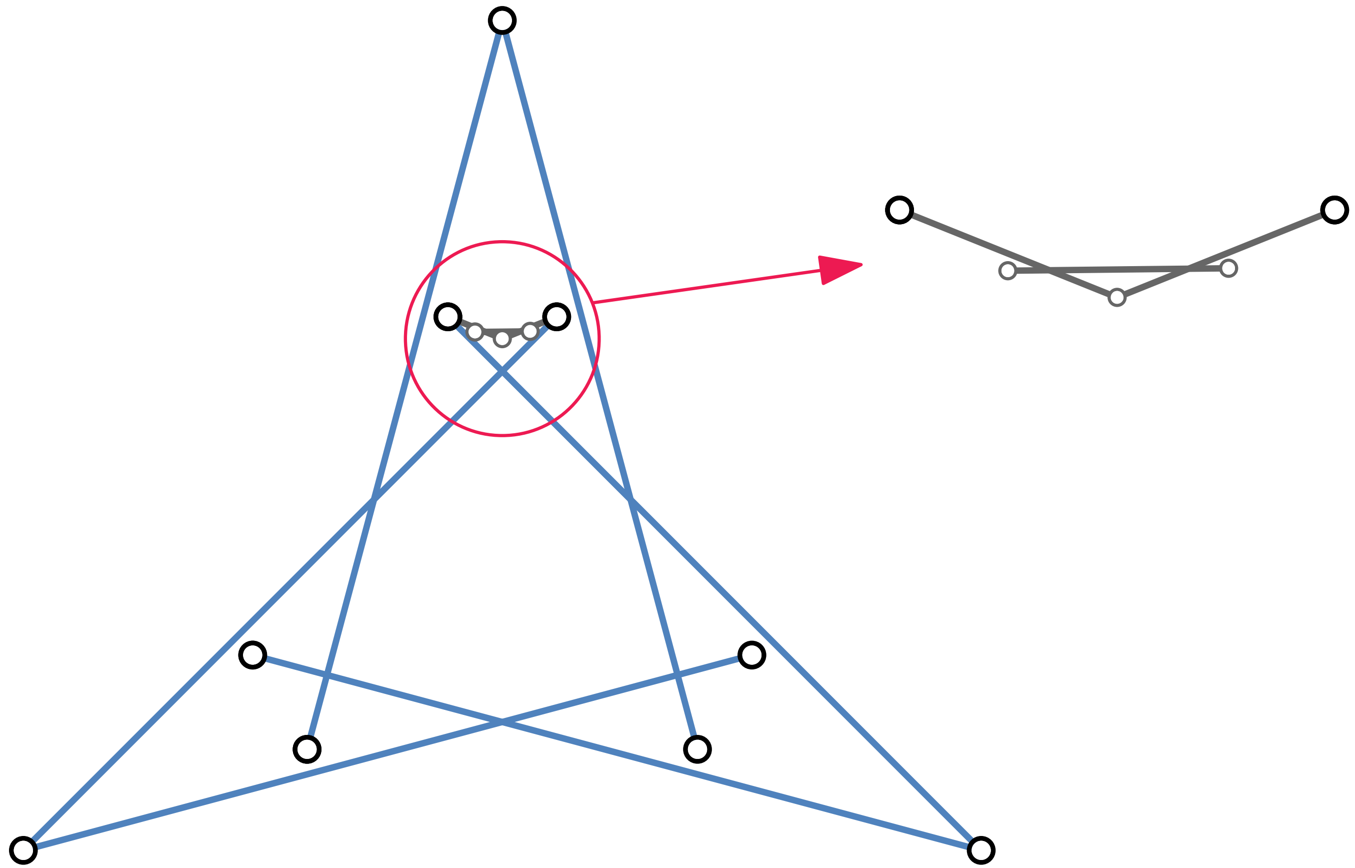
Construction with $n-3$ exit edges.



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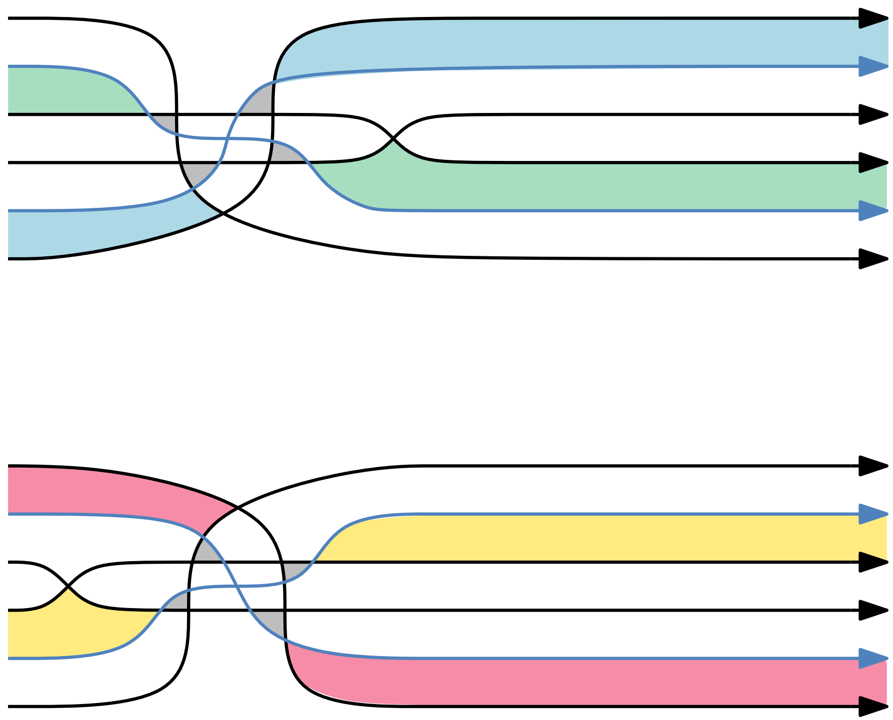
$\Rightarrow$ Construction with $\Theta(n^2)$ exit edges.

Different Order Type, Same Exit Triples

Even if we are given all the exit edges and their witnesses (in the dual, having all triangles and their orientations), we cannot always infer the order type.



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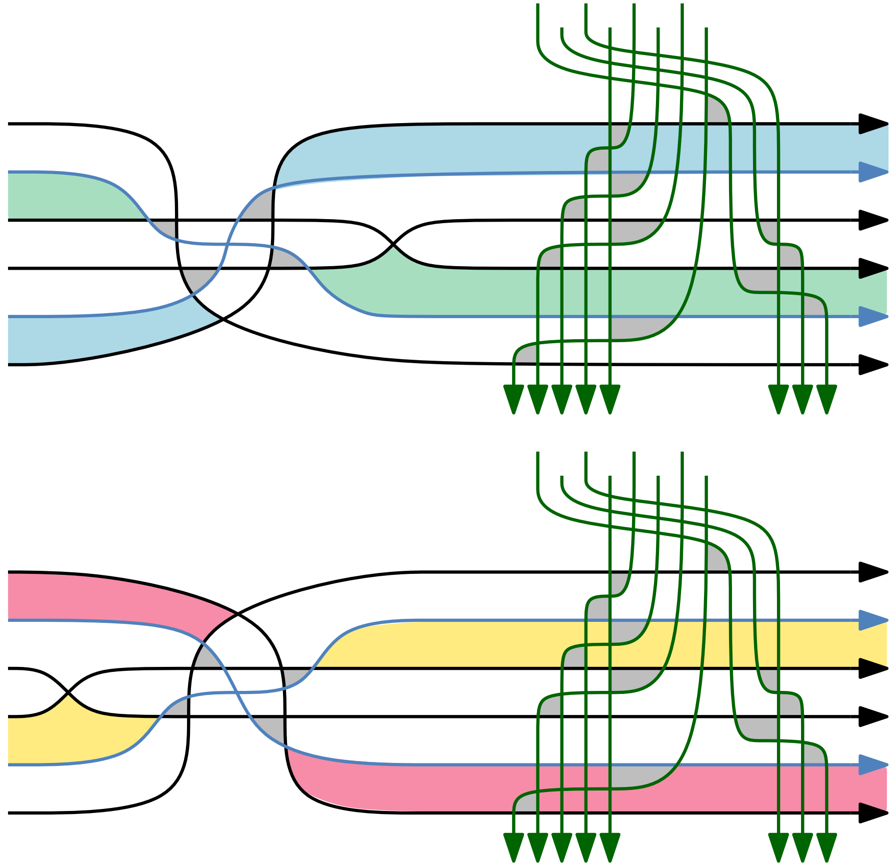

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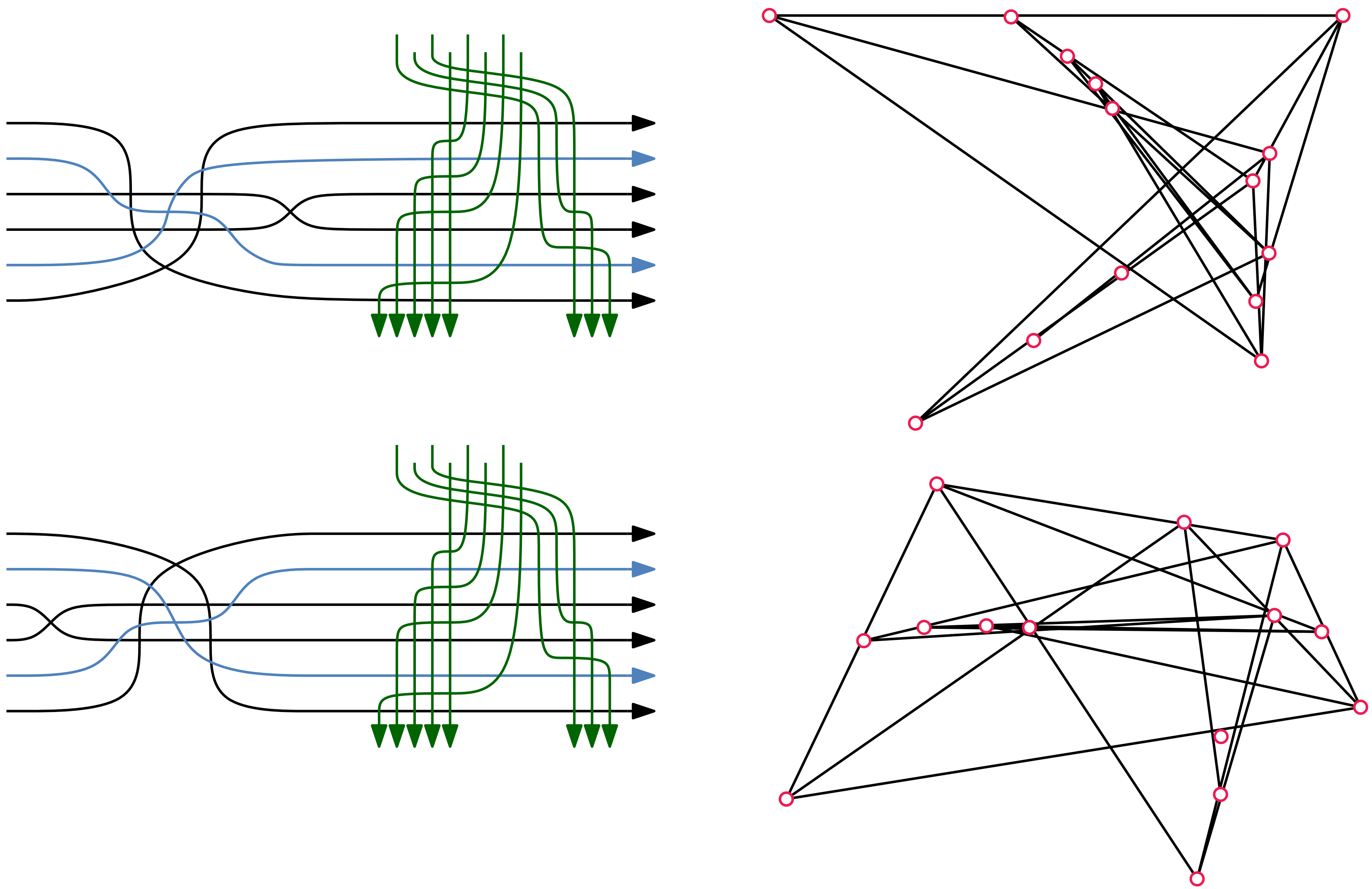

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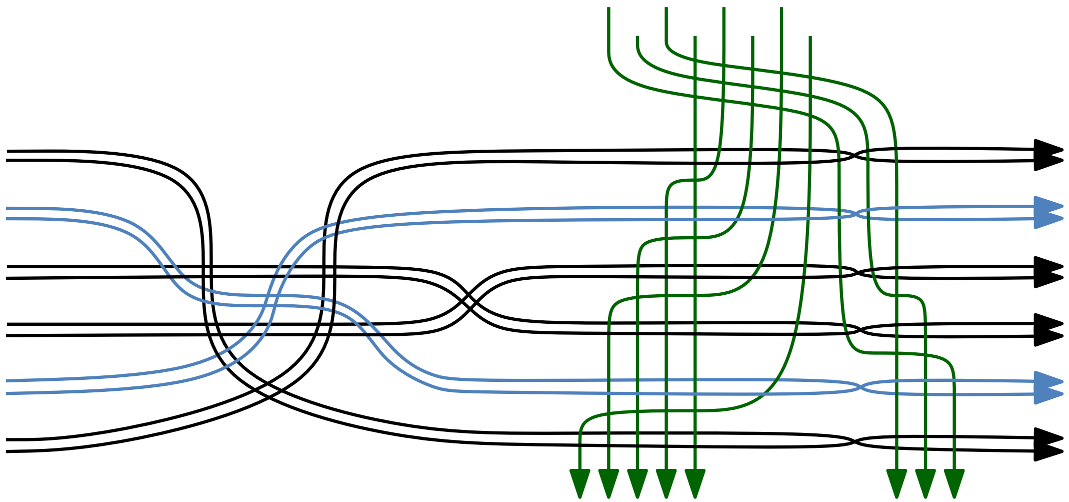

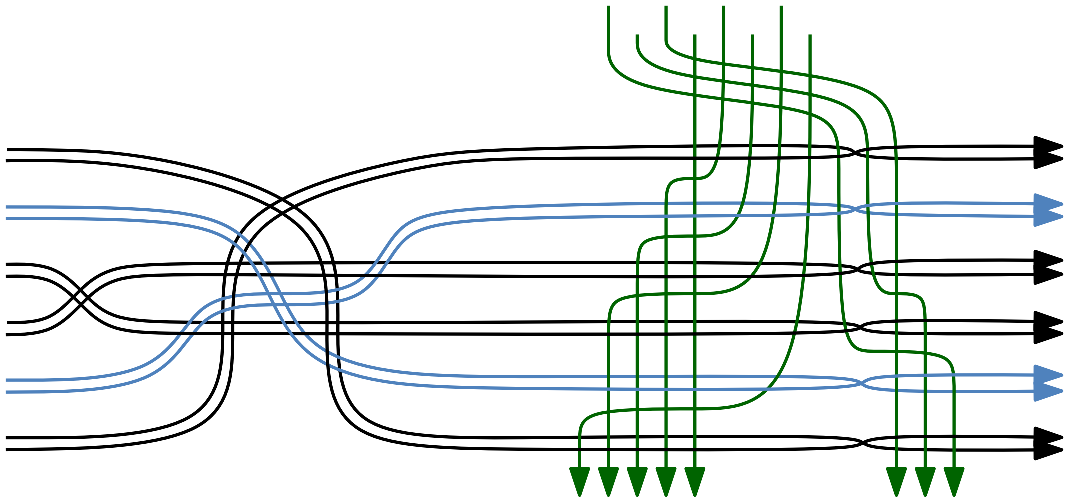

Conclusions

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