Graph Drawing 2019

Průhonice, September 17-19

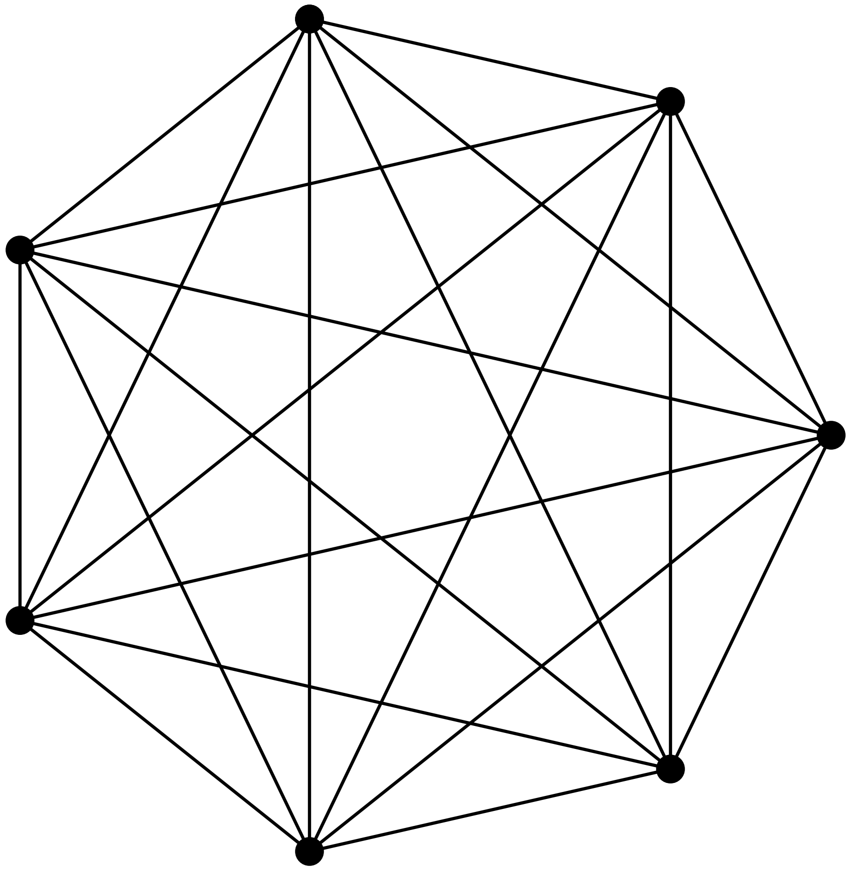
On the 2-Colored Crossing Number

Oswin Aichholzer[1], Ruy Fabila-Monroy[2],
Adrian Fuchs[1], Carlos Hidalgo-Toscano[2], Irene Parada[1],
Birgit Vogtenhuber[1], and Francisco Zaragoza[3]

[1] Graz University of Technology, Austria
[2] Cinvestav, Mexico
[3] Universidad Autónoma Metropolitana, Mexico

Rectilinear Crossing Number

Given: (straight-line drawing D of) graph $G = (V, E)$.



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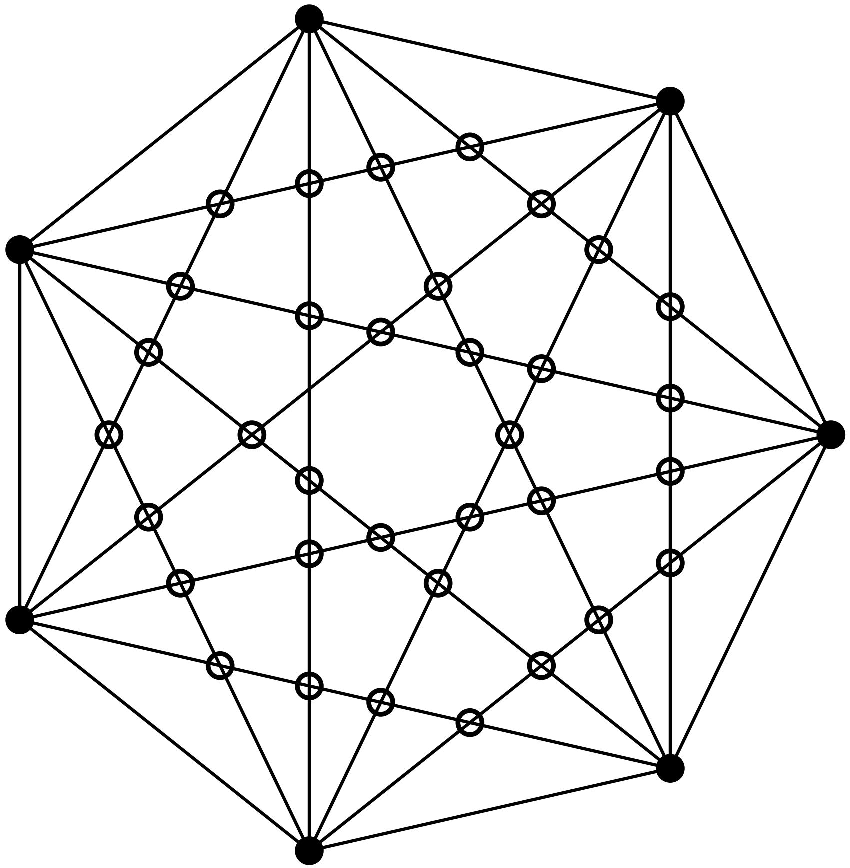

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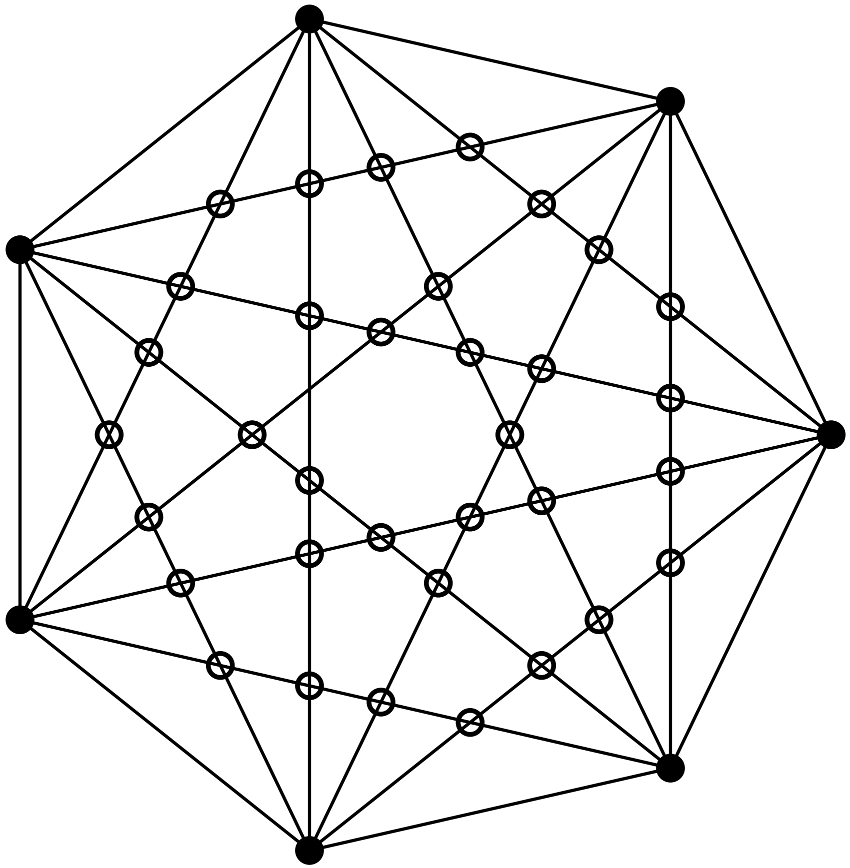

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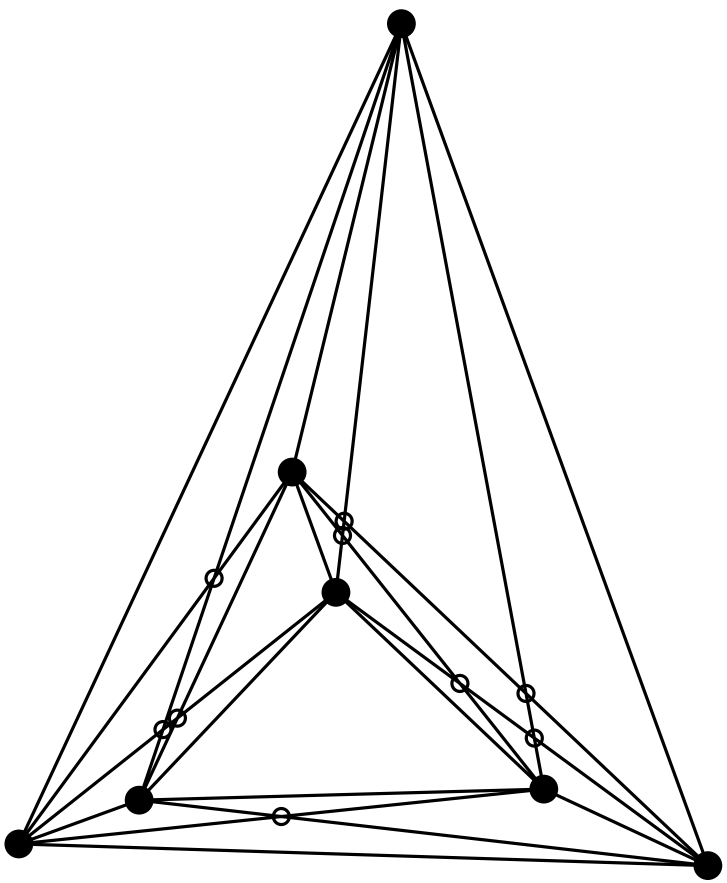

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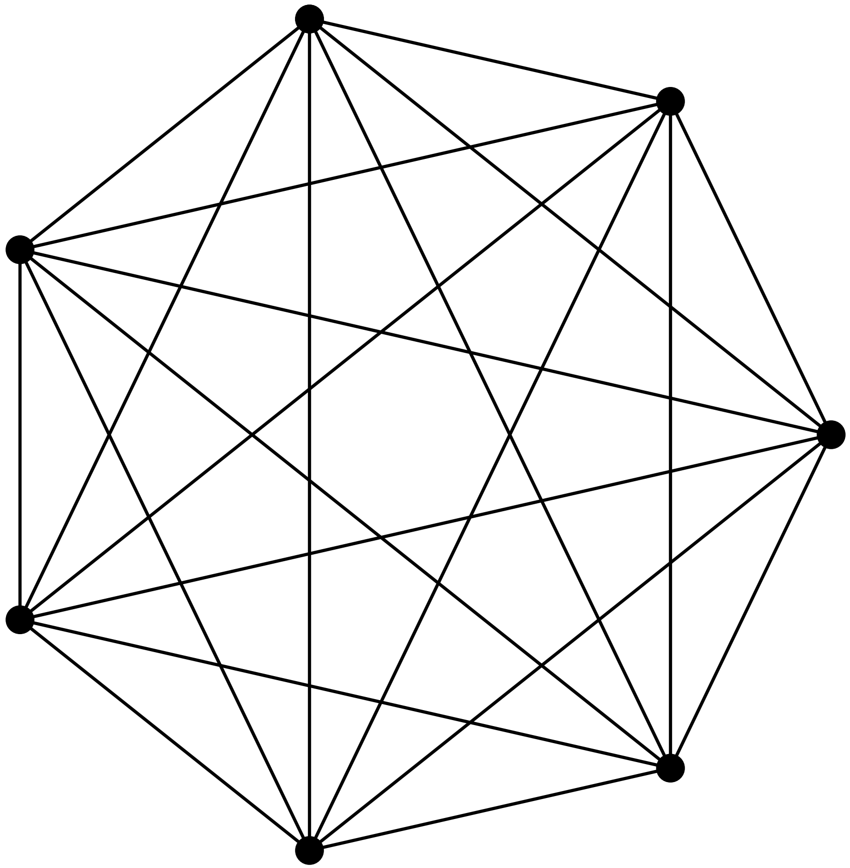

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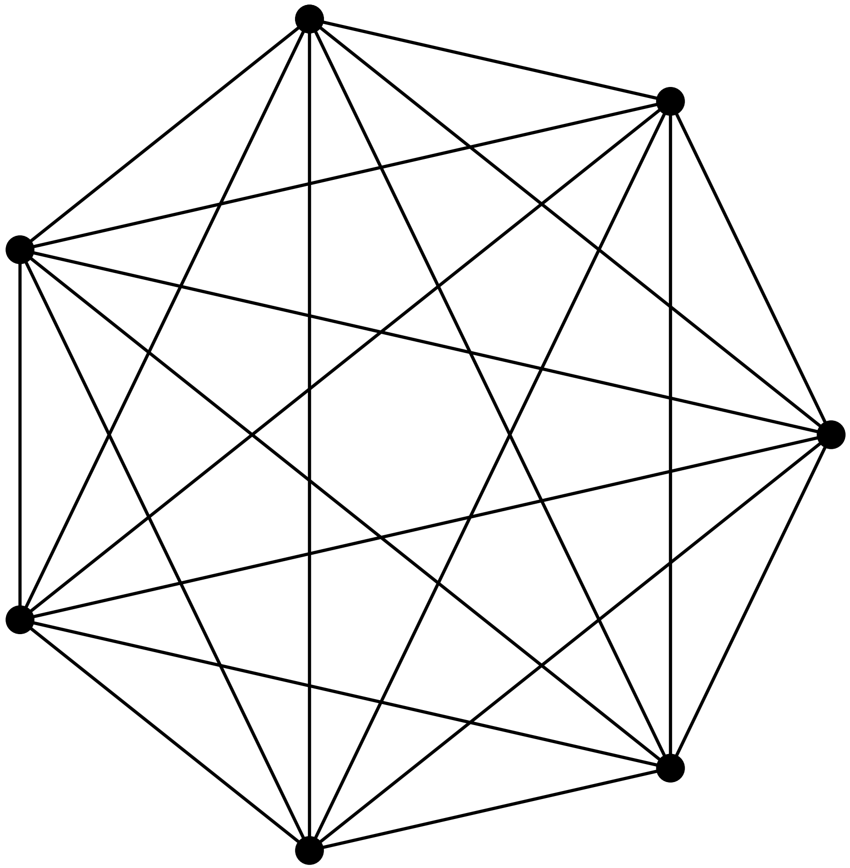

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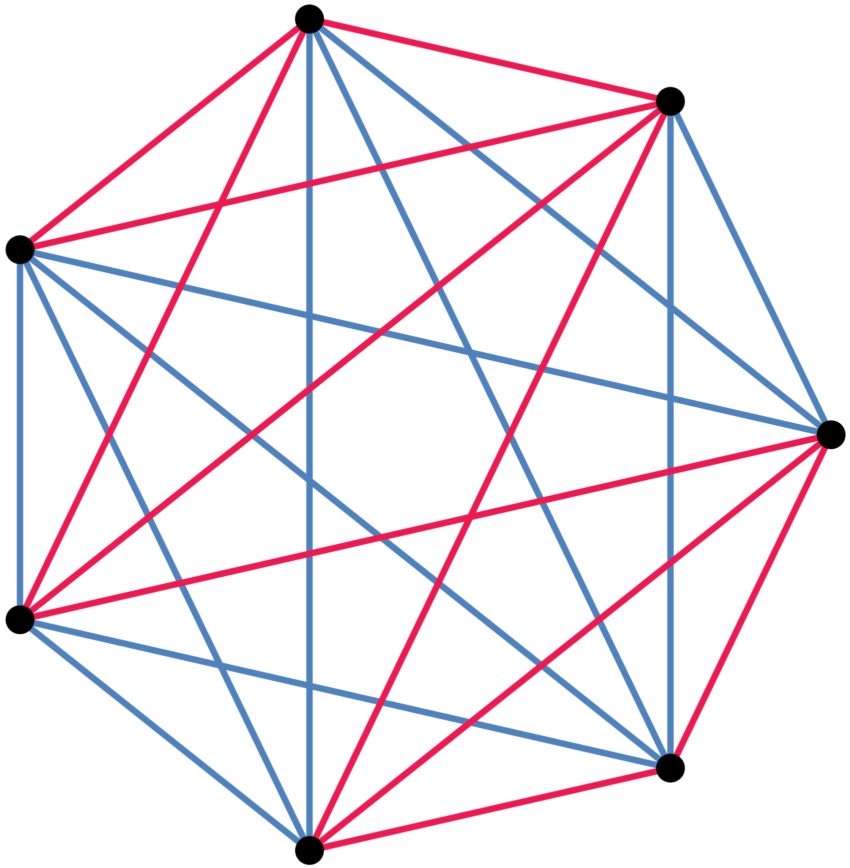

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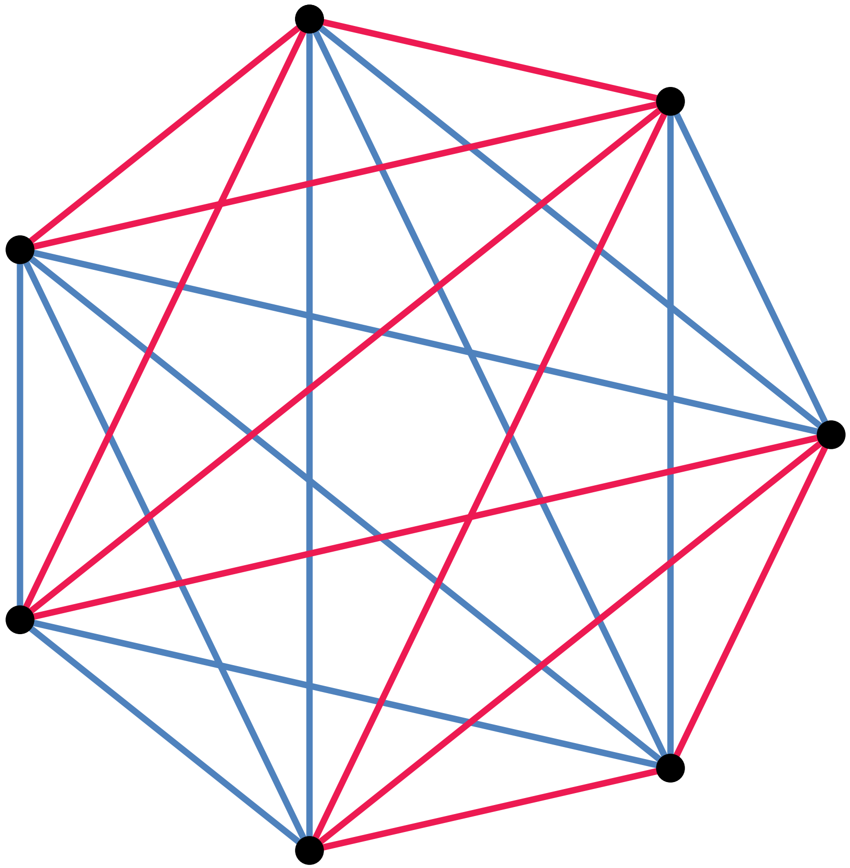

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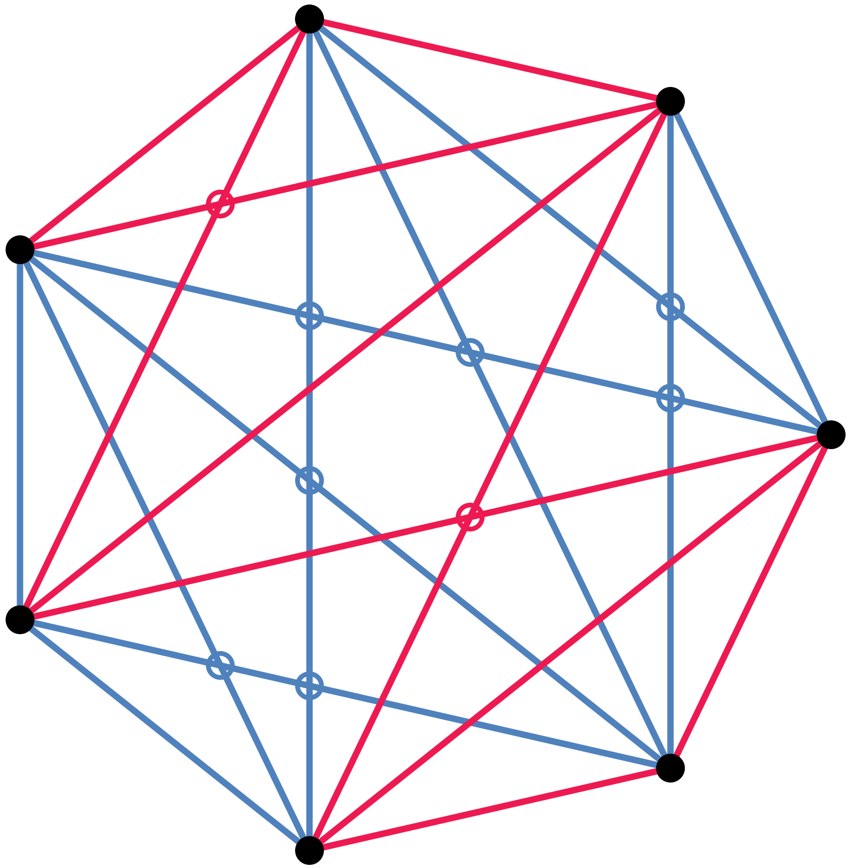

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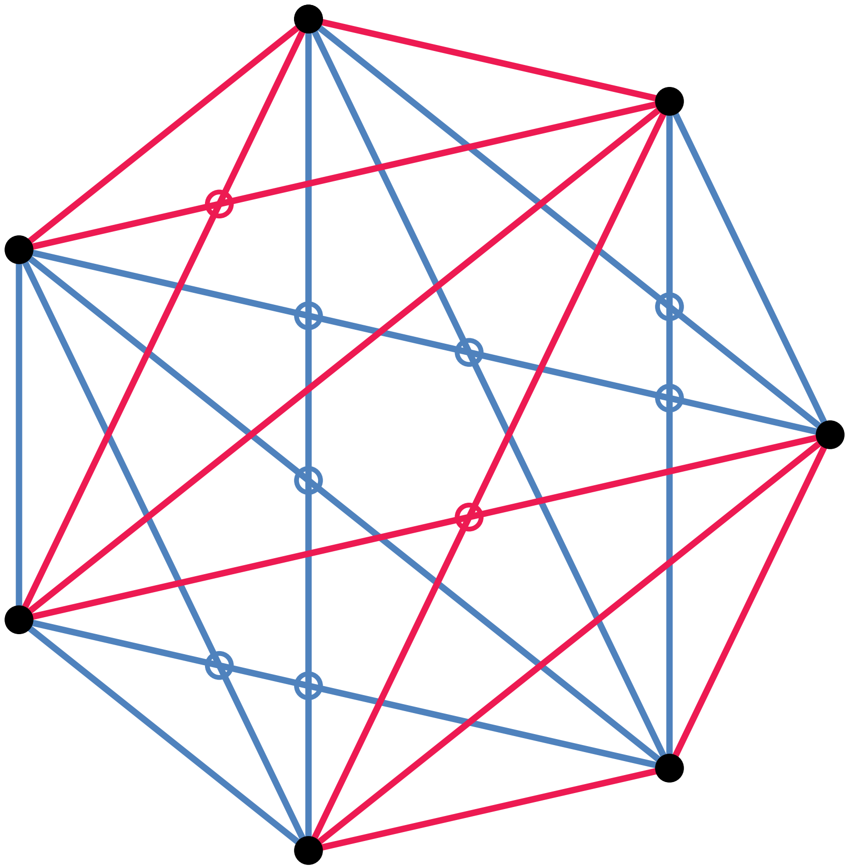

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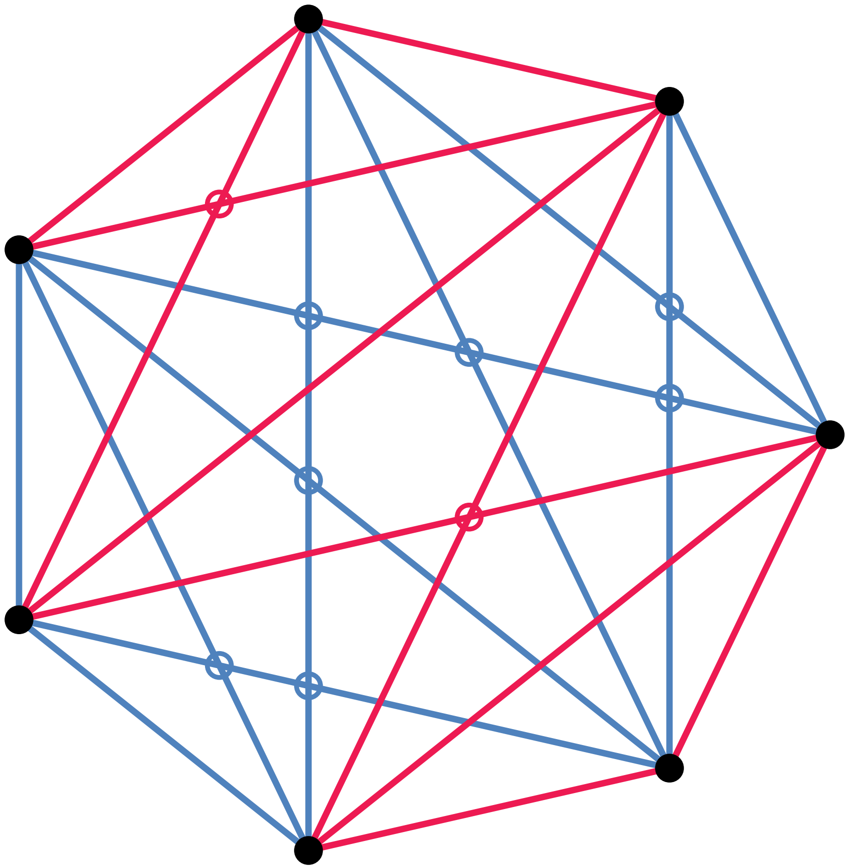

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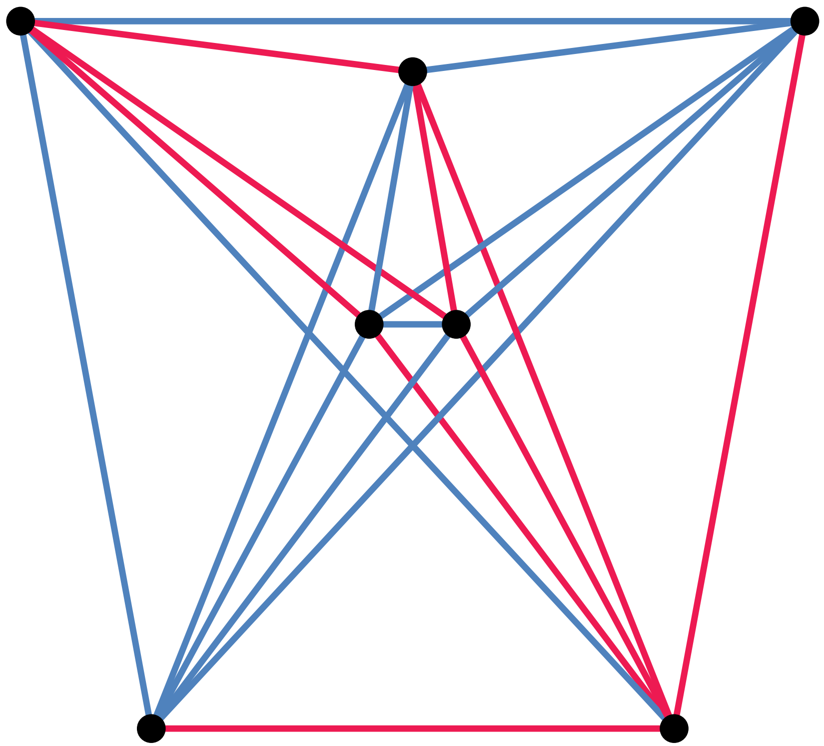

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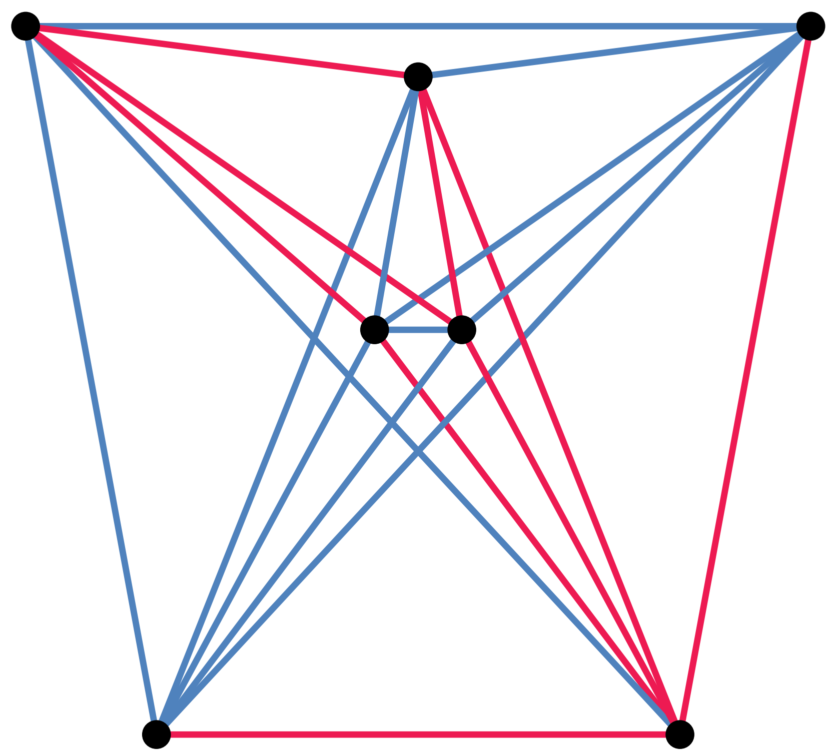
- Determining $\overline{\mathrm{cr}}_2(G)$ and even $\overline{\mathrm{cr}}_2(D)$ is NP-hard



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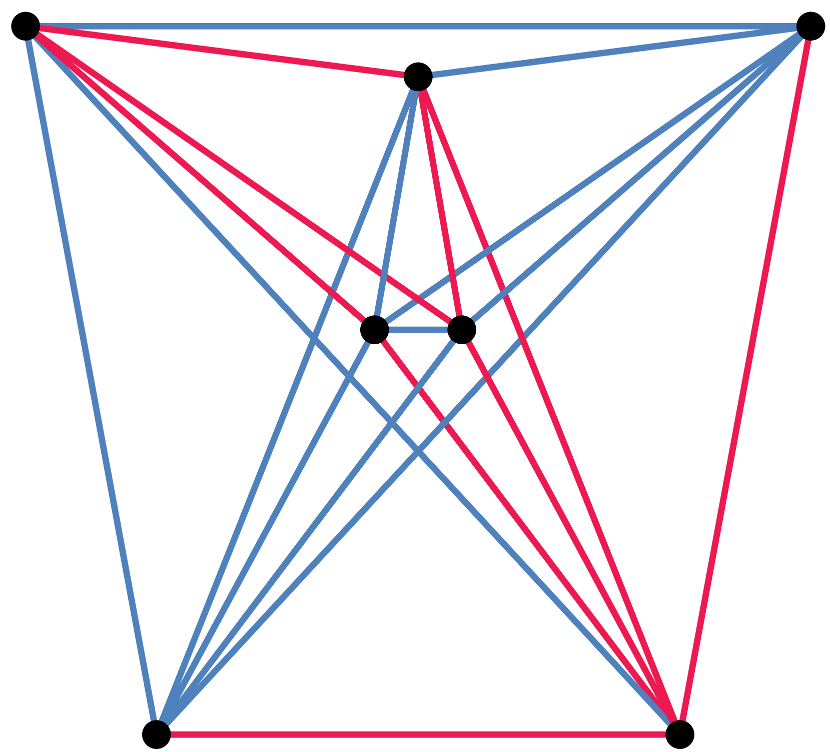

- Goal: find bounds on $\overline{\text{cr}}_2(G)$ and $\overline{\text{cr}}_2(D)$ for $G = K_n$.

Main Results

- Lower and upper bounds on $\overline{\mathrm{cr}}_2(K_n)$:

$$\frac{1}{33}\binom{n}{4} + \Theta(n^3) < \overline{\mathrm{cr}}_2(K_n) < 0.11798016\binom{n}{4} + \Theta(n^3)$$

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- Ratio between $\overline{\mathrm{cr}}_2(K_n)$ and $\overline{\mathrm{cr}}(K_n)$:

$$\lim_{n\to\infty} \frac{\overline{\mathrm{cr}}_2(K_n)}{\overline{\mathrm{cr}}(K_n)} < 0.31049652$$

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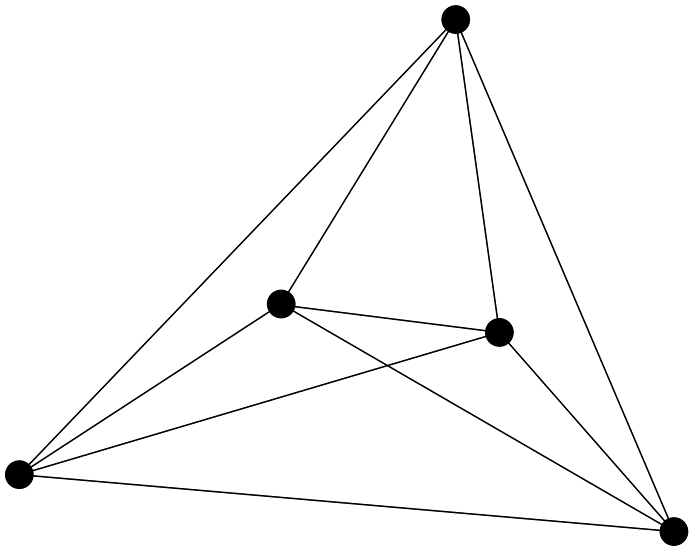
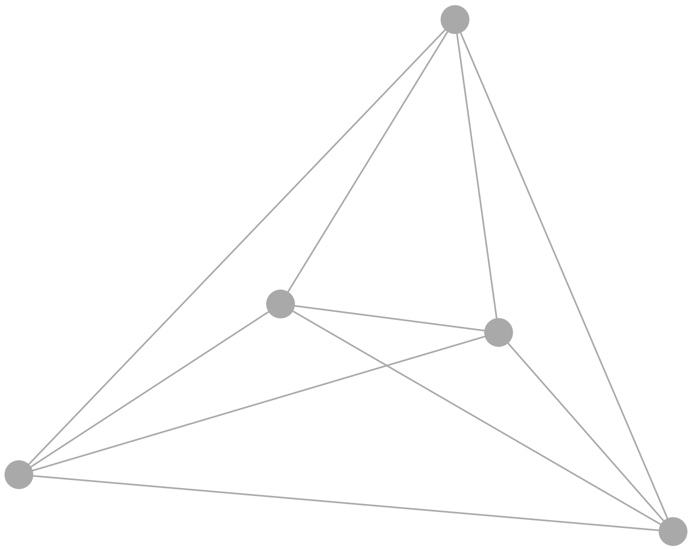
- Ratio for any fixed straight-line drawing D of K_n with sufficiently large n:

$$\frac{\overline{\text{cr}}_2(D)}{\overline{\text{cr}}(D)} < \frac{1}{2} - c \qquad \text{for some const. } c > 0$$

Duplication Process

- Duplication: drawing D of $K_m \longrightarrow$ drawing D' of K_{2m}





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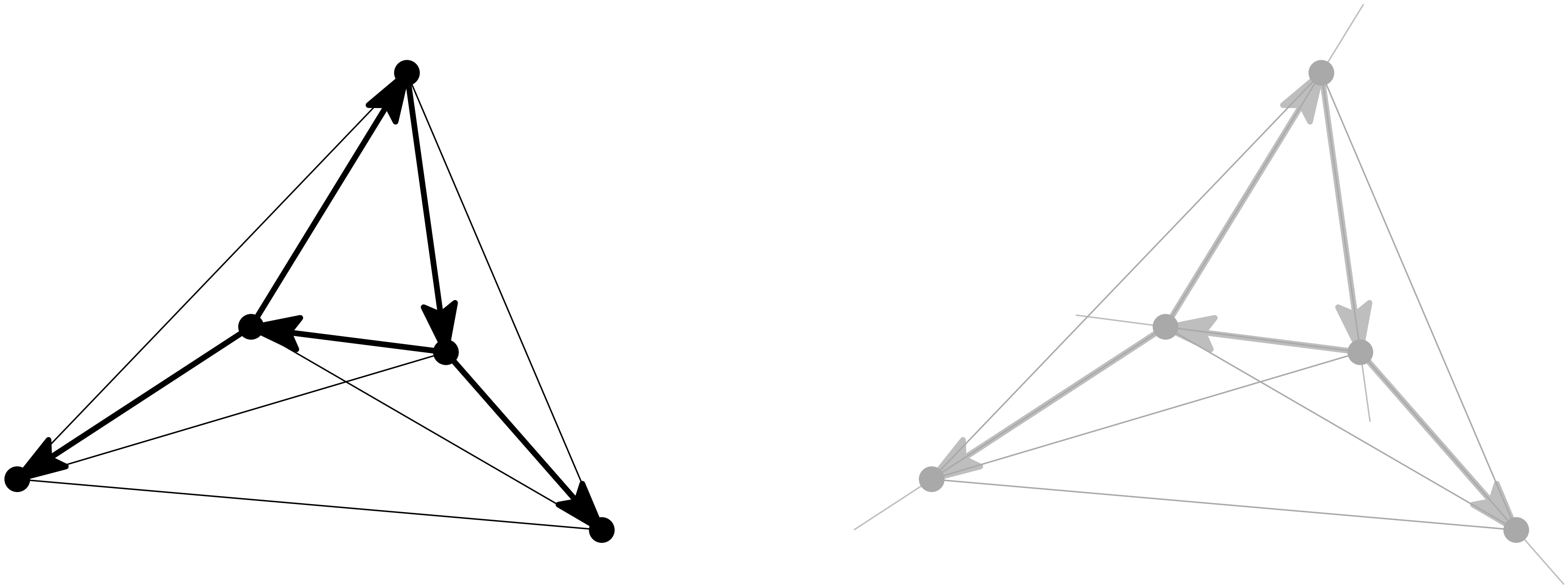

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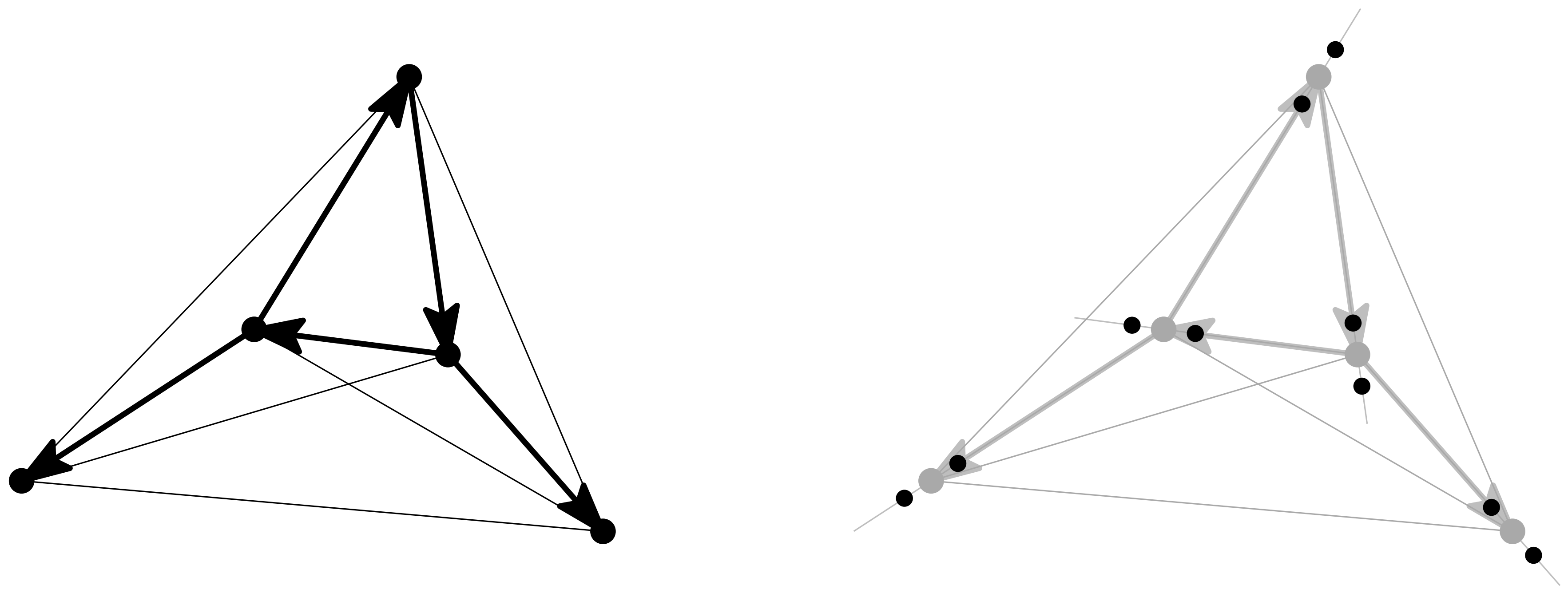

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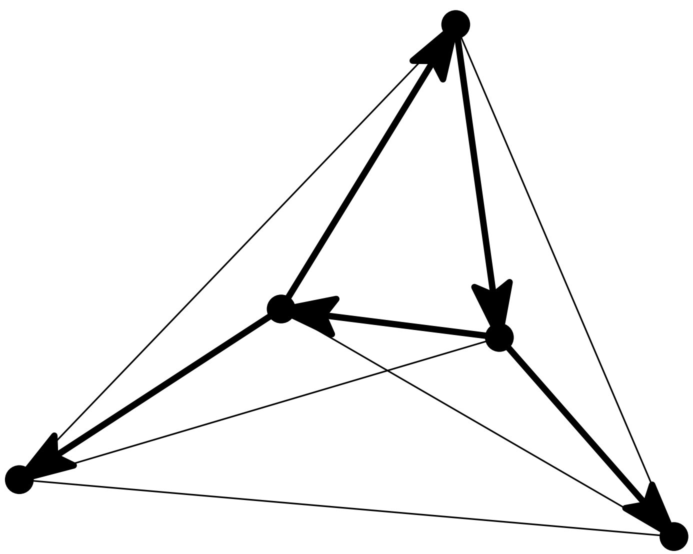

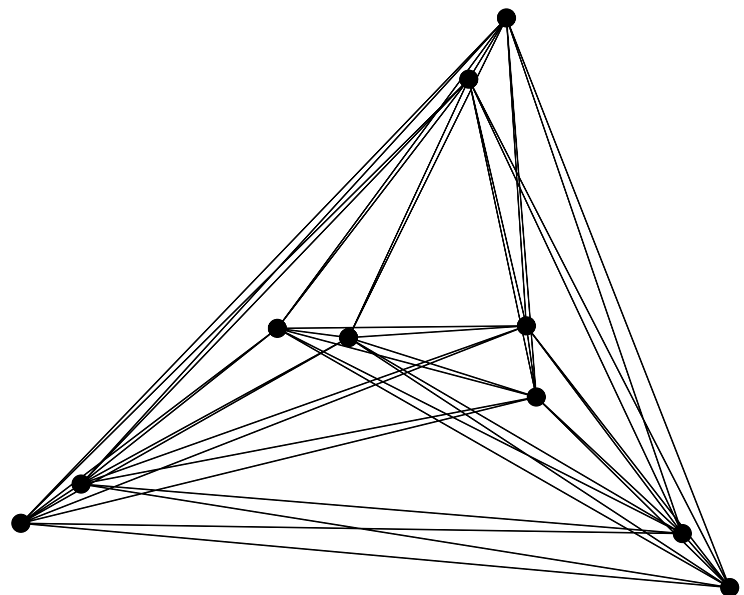

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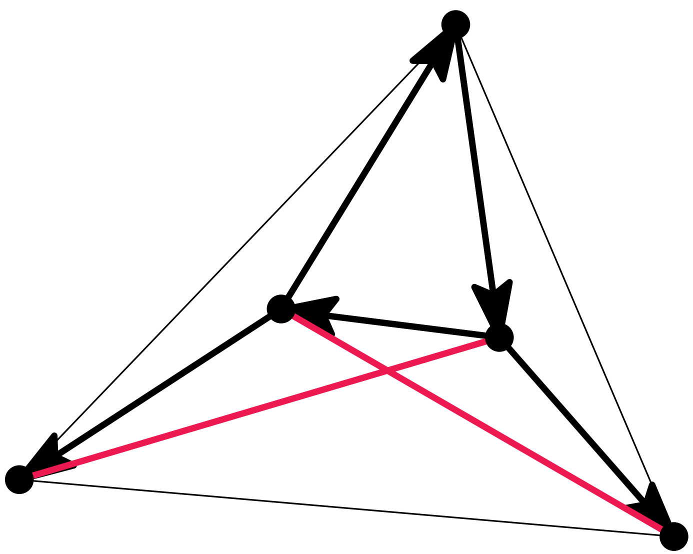

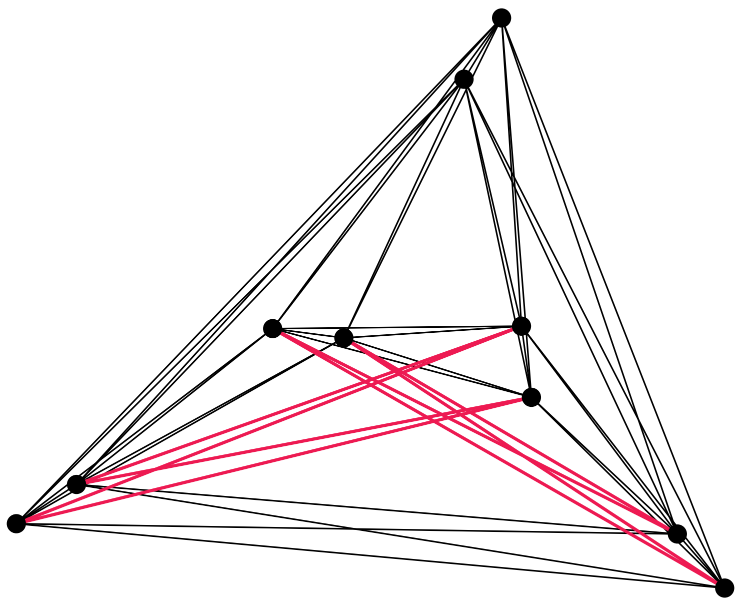

per original crossing: 16 crossings

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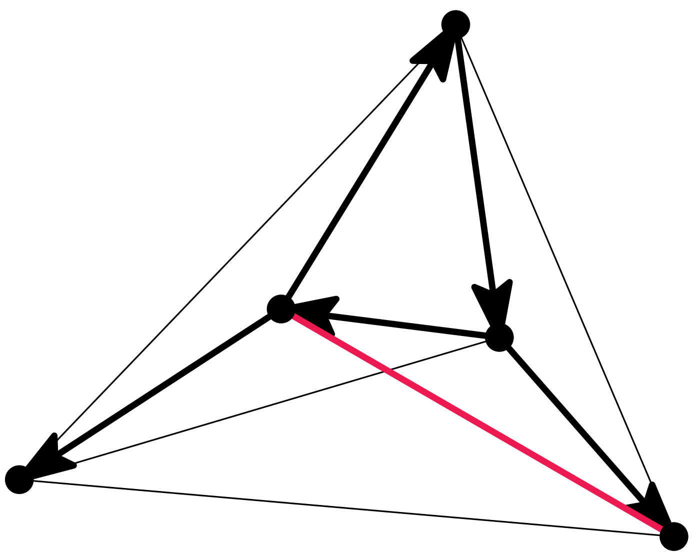

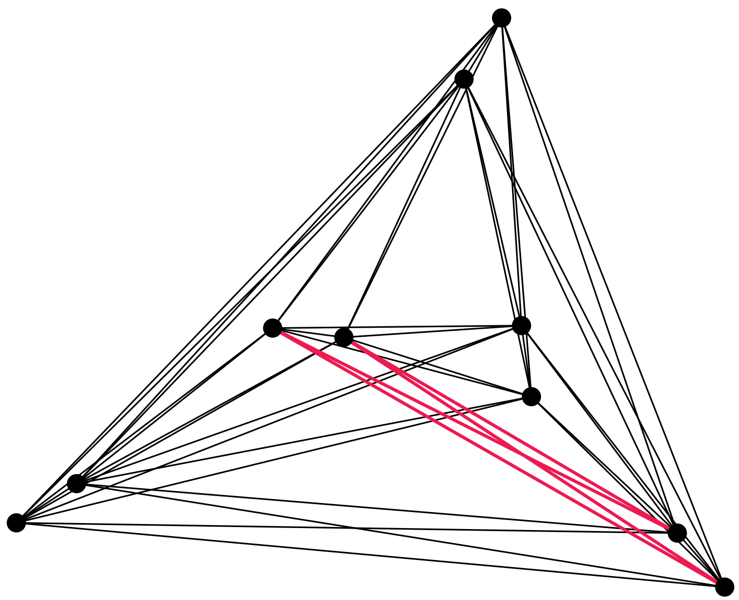

per original edge: 1 crossing

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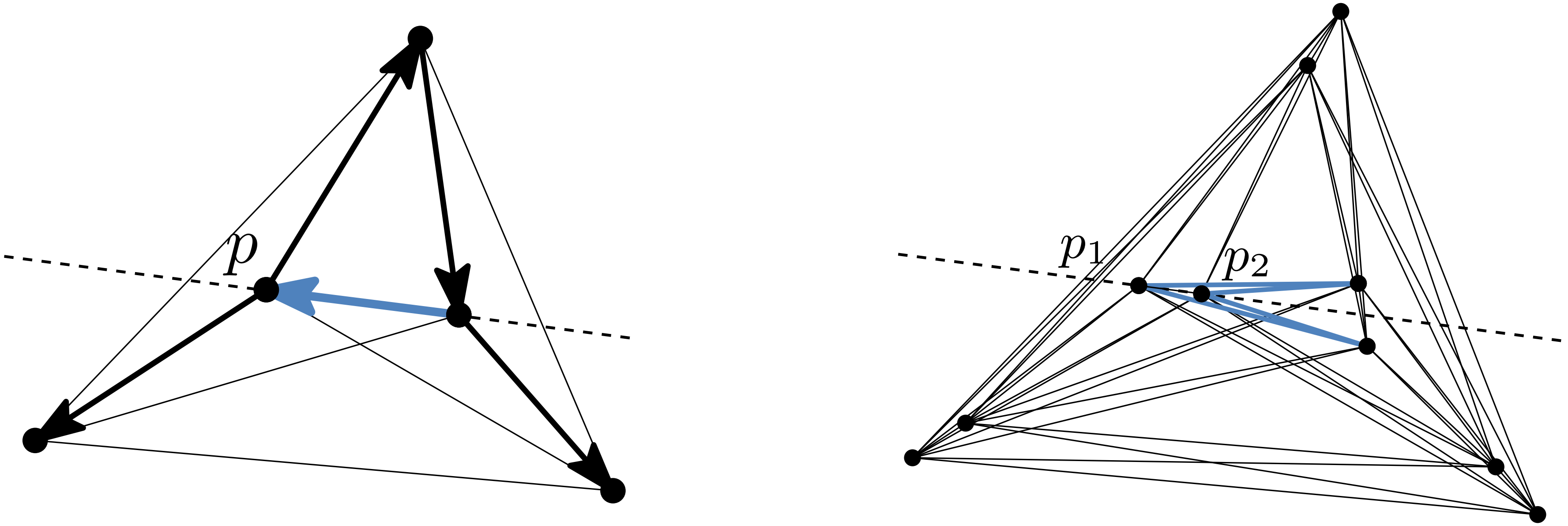

except for matching edges

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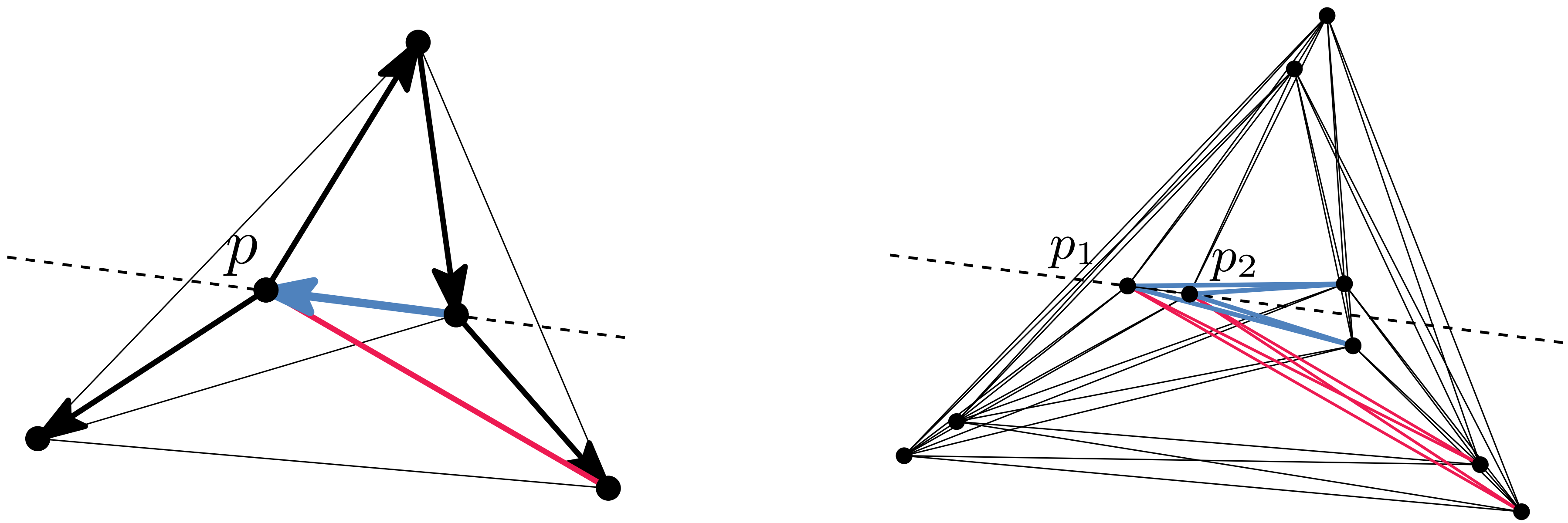

incident edges: 2 additional crossings with matching edge

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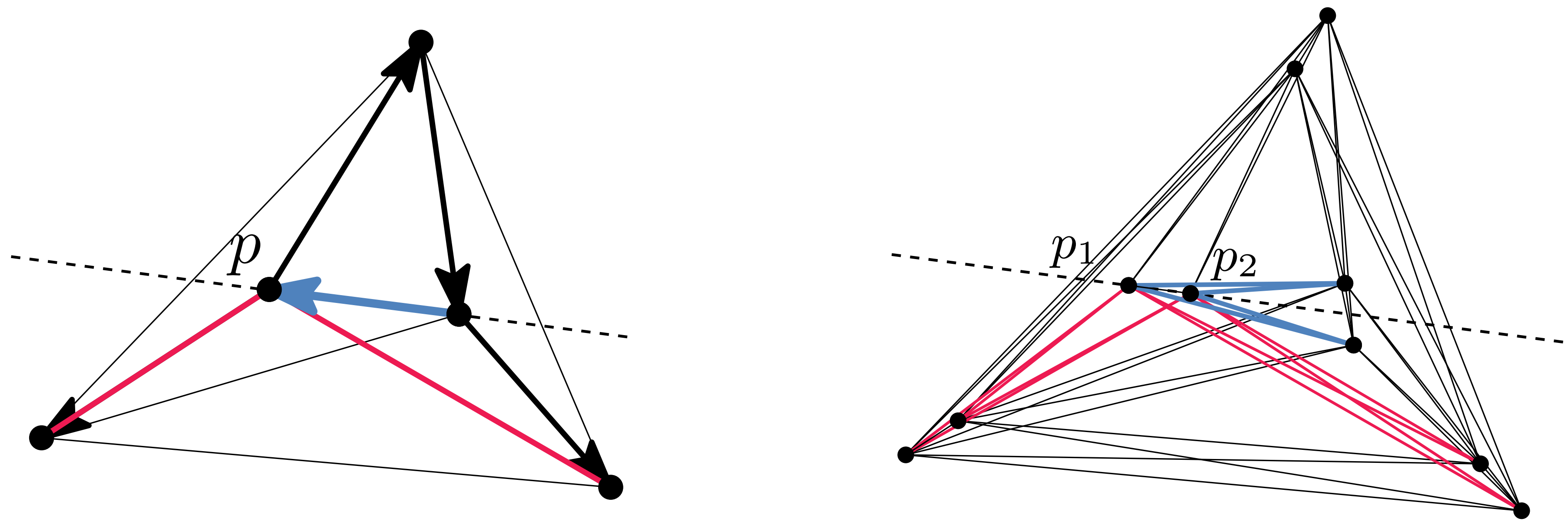

incident edge pairs: 4 additional crossings

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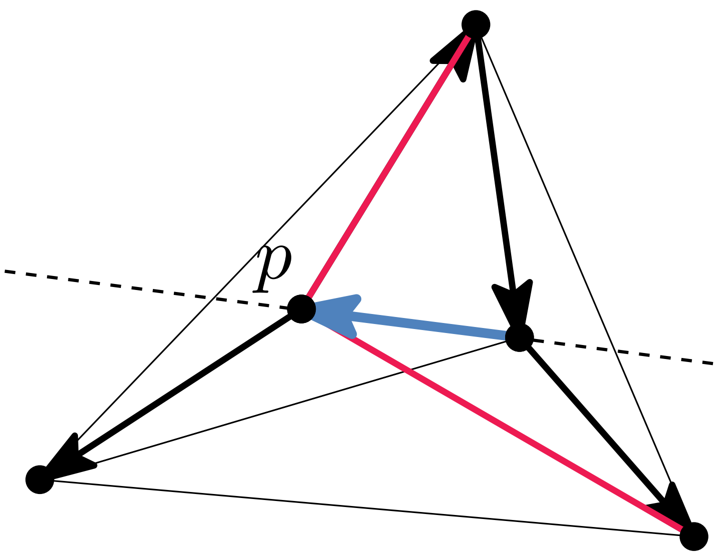

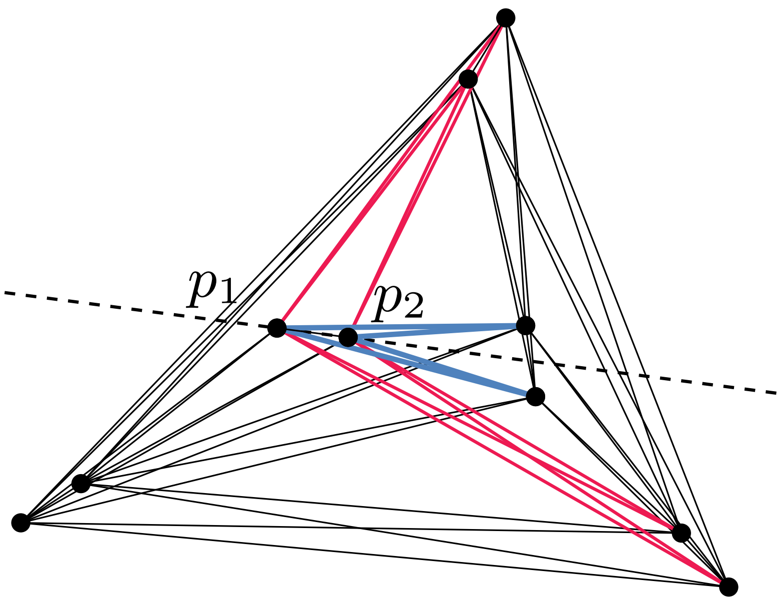

opposite incident edge pairs: no additional crossings

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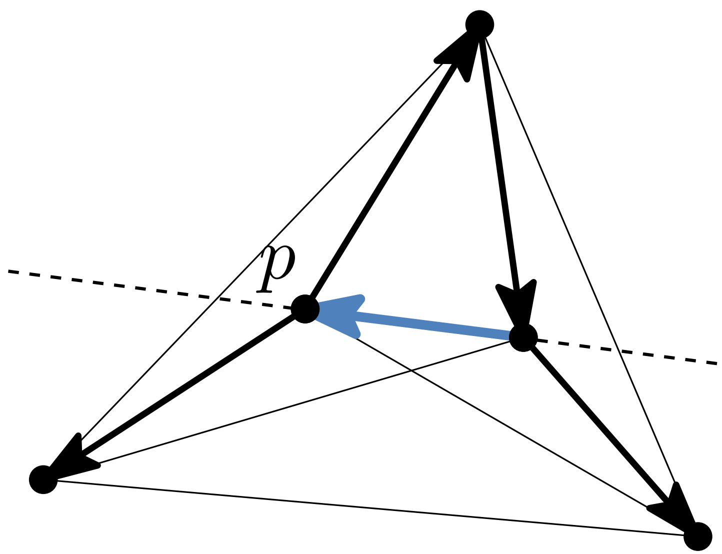

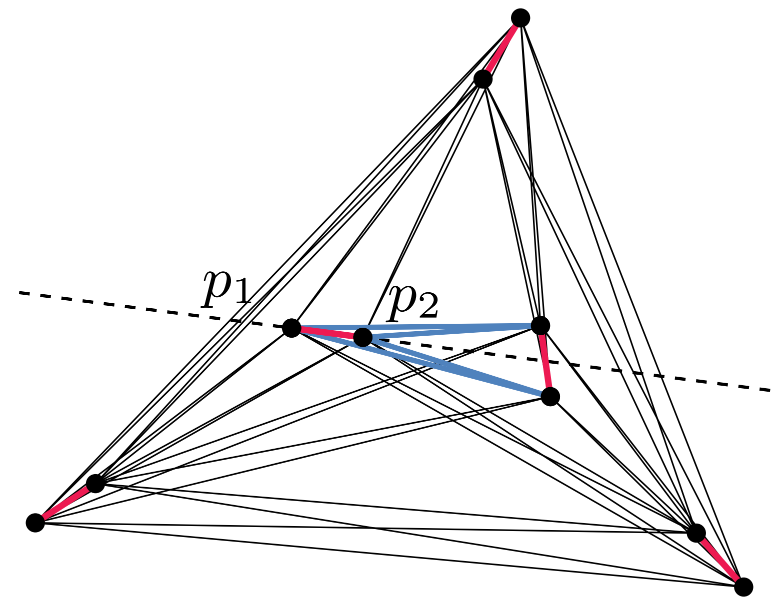

small edges: no crossings

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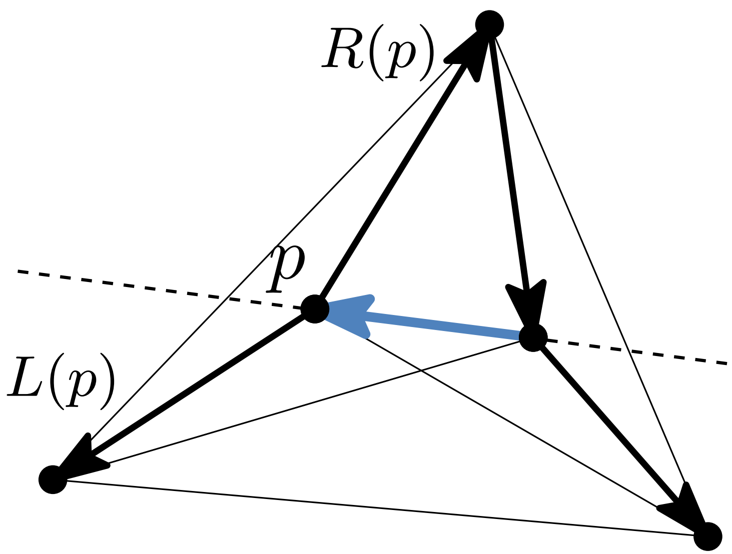

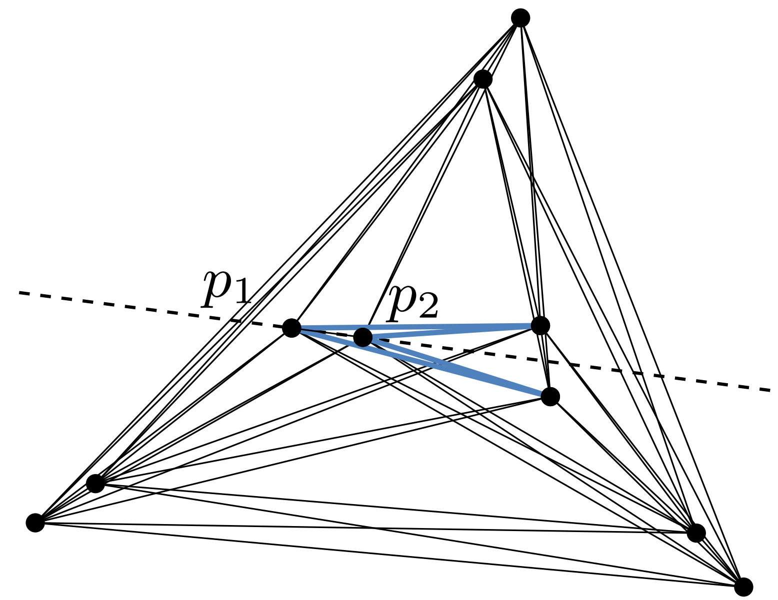

$$\overline{\text{cr}}(D') = \underbrace{16\overline{\text{cr}}(D)}_{\text{crossings in } D} + \underbrace{\binom{n}{2}}_{\text{edges in } D} - n + \sum_{p \in D} \underbrace{\left(4\binom{L(p)}{2} + 4\binom{R(p)}{2} + 2\,(L(p)+R(p))\right)}_{\text{edgepairs incident to } p \in D}$$

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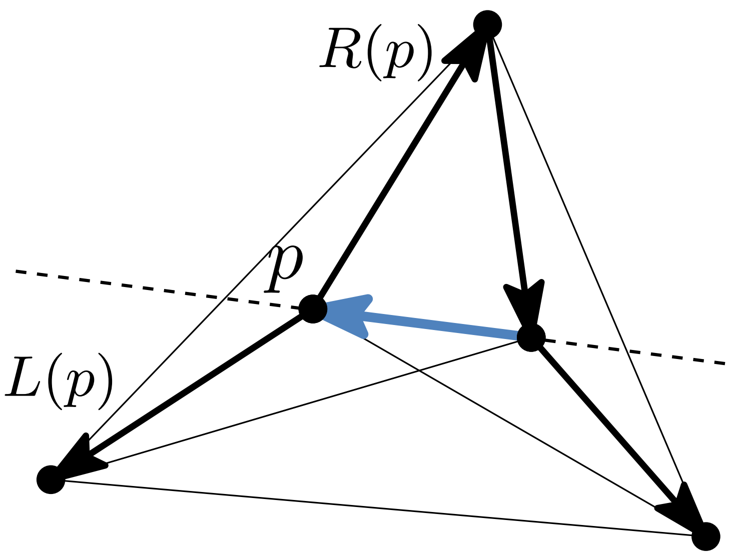

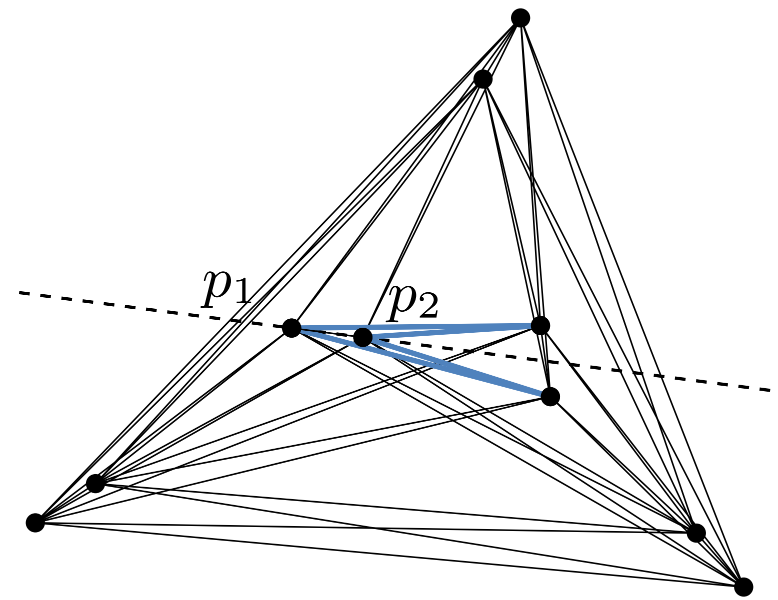

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- Best matching edges: half of the edges on each side

Duplication Process

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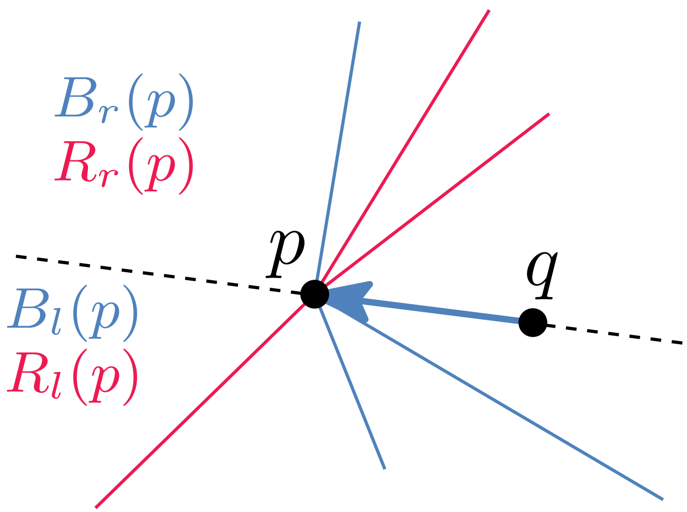

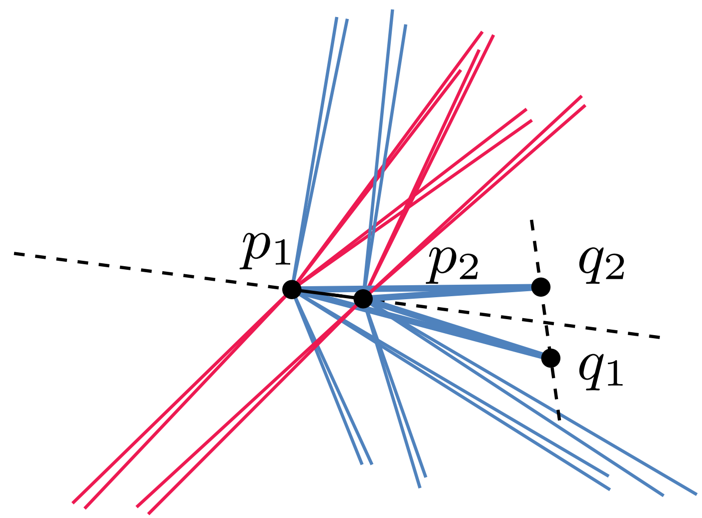

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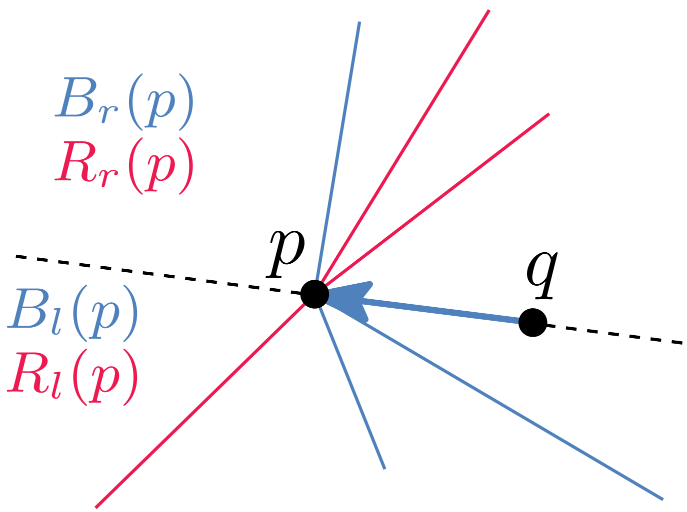

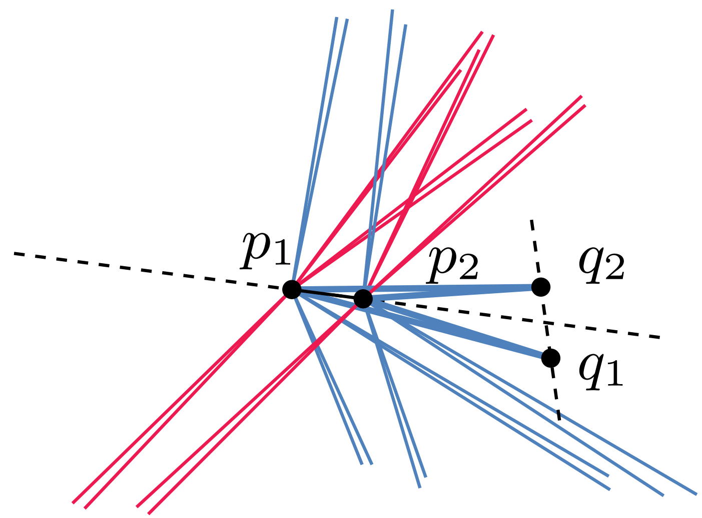

$$\begin{aligned}\overline{\text{cr}}_2(D',\chi') \;=\;& 16\overline{\text{cr}}_2(D,\chi) + \binom{n}{2} - n \\ &+ \sum_{p\in D} 4\left(\binom{B_l(p)}{2} + \binom{R_l(p)}{2} + \binom{B_r(p)}{2} + \binom{R_r(p)}{2}\right) \\ &+ \sum_{p\in D} 2\,(H_l(p)+H_r(p)) \qquad \Leftarrow \qquad H_i \in \{B_i, R_i\}\end{aligned}$$

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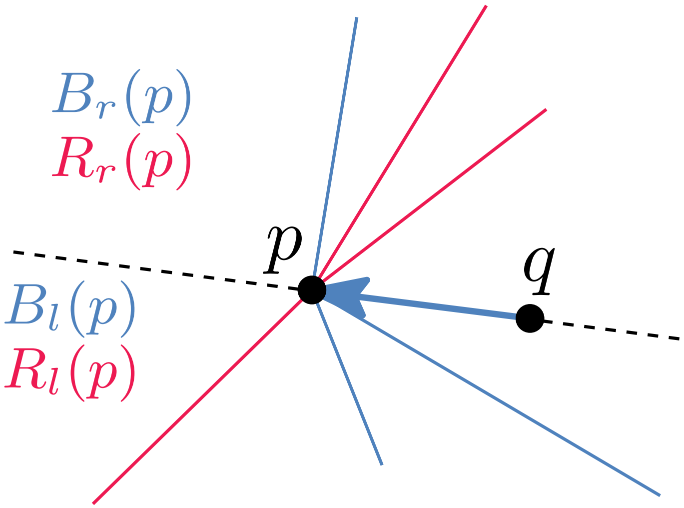
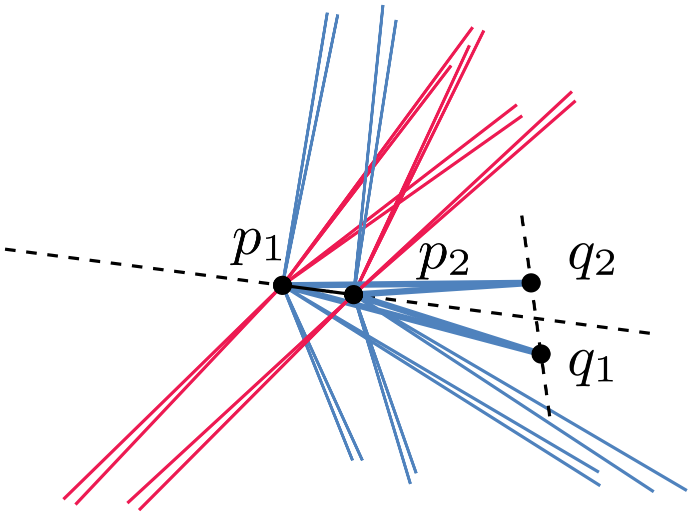
  $\overline{\text{cr}}_2$ of D': independent of colors for small edges!





$$\overline{\text{cr}}_2(D', \chi') = 16\overline{\text{cr}}_2(D, \chi) + \binom{n}{2} - n$$
$$+ \sum_{p \in D} 4\left(\binom{B_l(p)}{2} + \binom{R_l(p)}{2} + \binom{B_r(p)}{2} + \binom{R_r(p)}{2}\right)$$
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  $\overline{\mathrm{cr}}_2$ of D': independent of colors for small edges!
- Best matching edges: half of the edges of each color on each side

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- "Nice" matching edges:

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  $\overline{\text{cr}}_2$ of D': independent of colors for small edges!
- Best matching edges: half of the edges of each color on each side $\Leftarrow$ in general not possible!
- "Nice" matching edges:
  - halve the larger color class at the point
  - split the smaller color class as good as possible

Upper Bound for $\overline{\mathrm{cr}}_2(K_n)$

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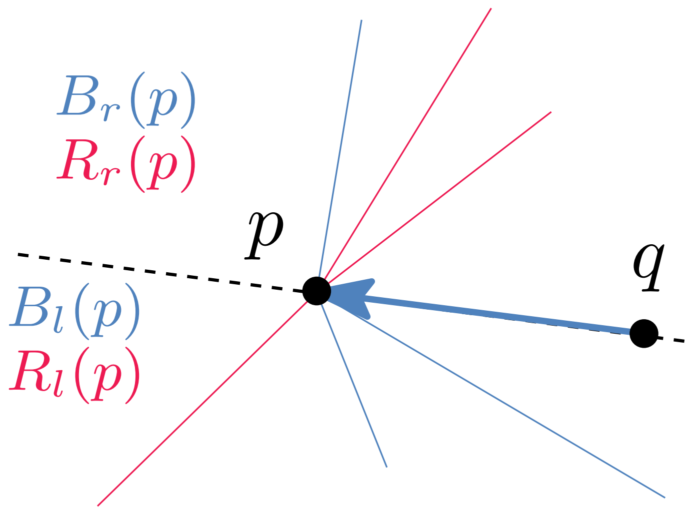

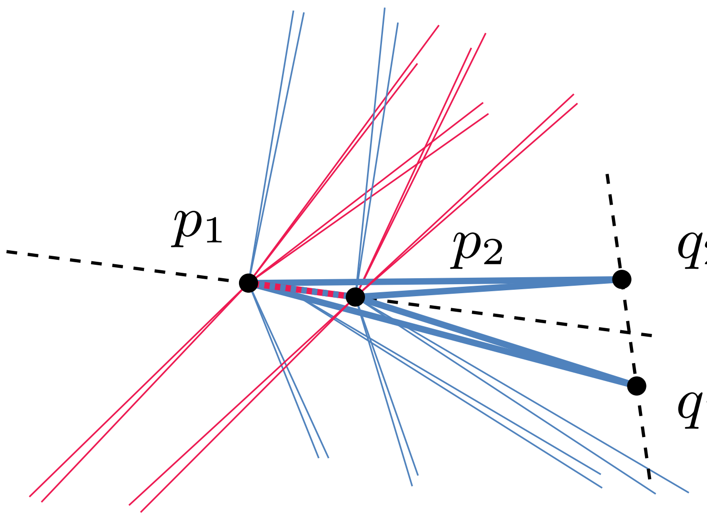

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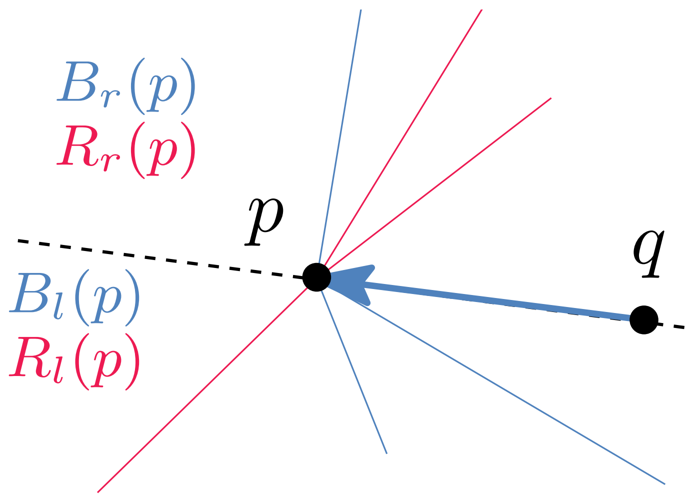

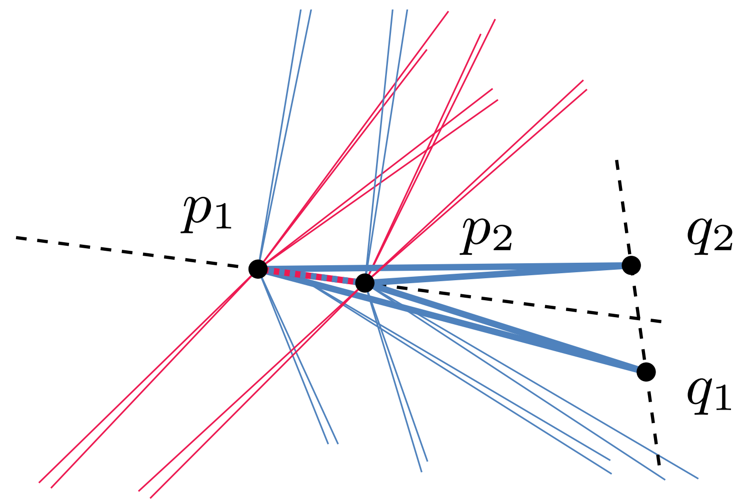

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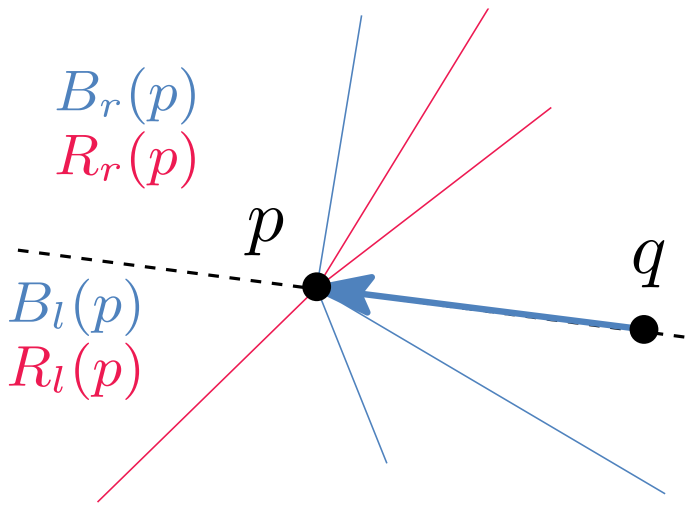

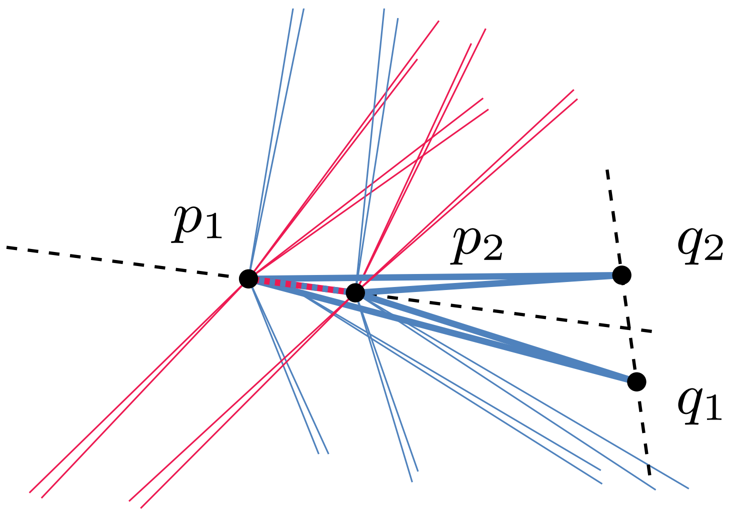

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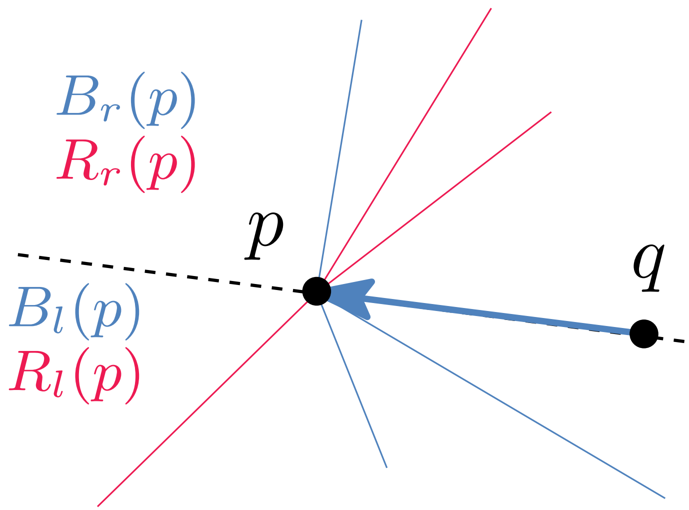
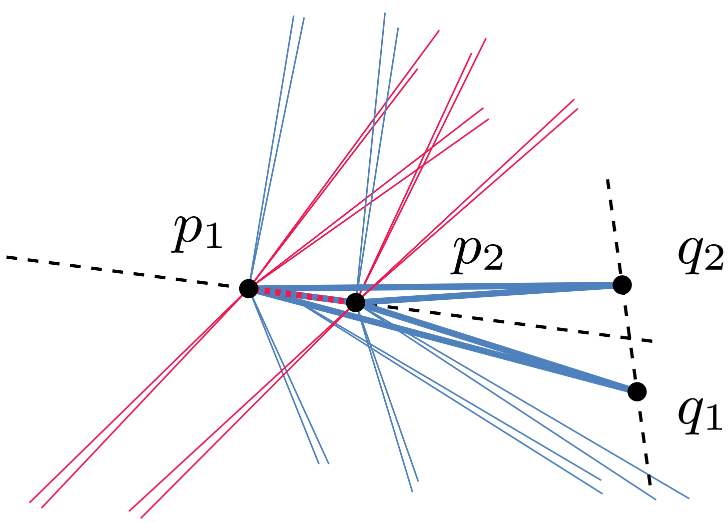
  - several cases, choices with good recursive behavior





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□

Main Results

- Lower and upper bounds on $\overline{\text{cr}}_2(K_n)$:

$$\frac{1}{33}\binom{n}{4} + \Theta(n^3) < \overline{\text{cr}}_2(K_n) < 0.11798016\binom{n}{4} + \Theta(n^3)$$

- Ratio between $\overline{\text{cr}}_2(K_n)$ and $\overline{\text{cr}}(K_n)$:

$$\lim_{n\to\infty} \frac{\overline{\text{cr}}_2(K_n)}{\overline{\text{cr}}(K_n)} < 0.31049652$$

- Ratio for any fixed straight-line drawing D of K_n with sufficiently large n:

$$\frac{\overline{\text{cr}}_2(D)}{\overline{\text{cr}}(D)} < \frac{1}{2} - c \qquad \text{for some } c > 0$$

Open Problems

- What is the computational complexity of determinging $\overline{\text{cr}}_2(D)$ for a given straight-line drawing D of K_n?
  $\Leftrightarrow$ How fast can we solve max-cut on the segment intersection graph induced by D?
- What can we say about the structure of point sets that minimize $\overline{\text{cr}}_2(K_n)$?
- Is it true that the maximum for $\overline{\text{cr}}_2(D)/\overline{\text{cr}}(D)$ is uniquely obtained for point sets in convex position?

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Thank you for your attention!