Representing Graphs and Hypergraphs by Touching Polygons in 3D

Paweł Rzążewski, Noushin Saeedi

joint work with William Evans, Chan-Su Shin, and Alexander Wolff

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- non-crossing drawings

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- at most two polygons touch in one point

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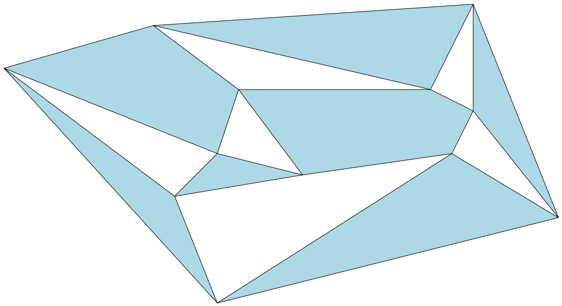

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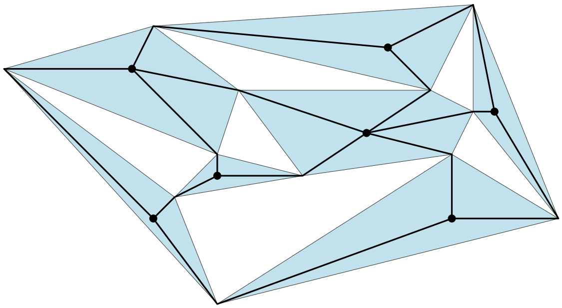

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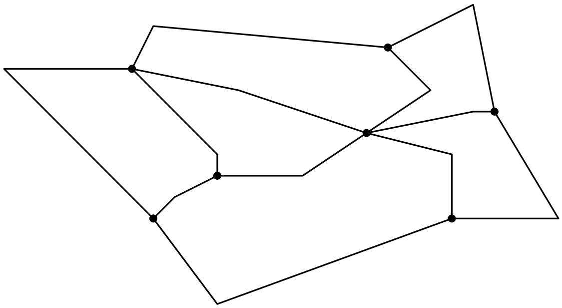

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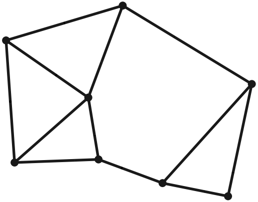

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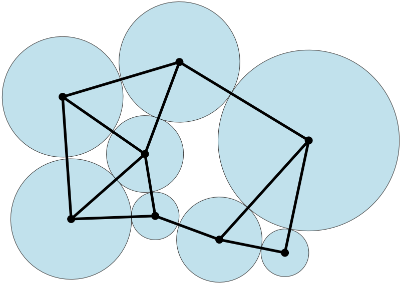

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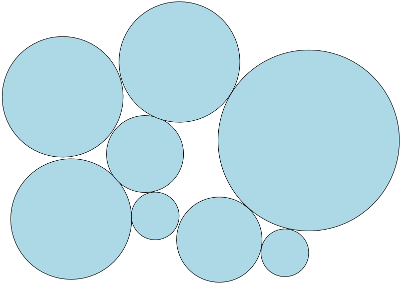

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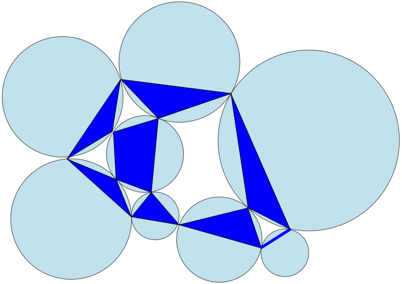

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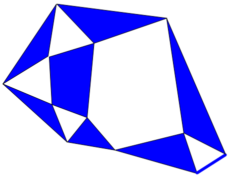

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Theorem. Recognizing segment intersection graphs in 3d is $\exists\mathbb{R}$-complete.

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Contact representations by touching polygons

Theorem. Every graph can be represented by touching convex polygons in 3d.

- in particular, this is an intersection representation by convex sets

Key lemma

Lemma. For every $n \geq 3$ there is an arrangement of lines $\ell_1, \ell_2, \ldots, \ell_n$, such that:

a) ℓ_i intersects $\ell_1, \ell_2, \ldots, \ell_n$ in this ordering ($p_{i,j} := \ell_i \cap \ell_j$),

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$$dist(p_{i,j-1}, p_{i,j}) \geq 2dist(p_{i,j}, p_{i,j+1}).$$

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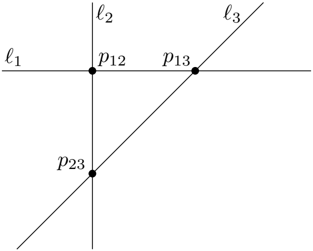

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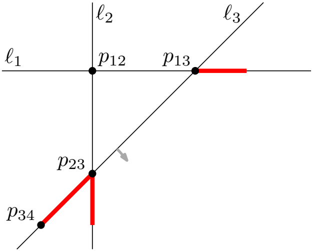

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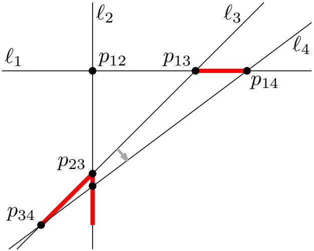

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- assume G is complete
- set height of $p_{i,j}$ to $\min(i, j)$
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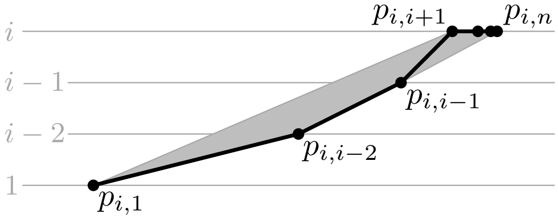

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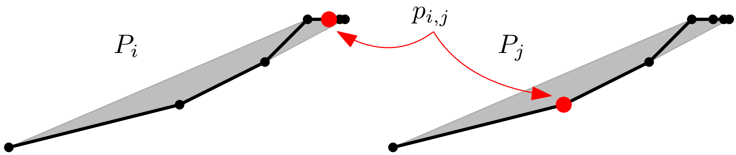

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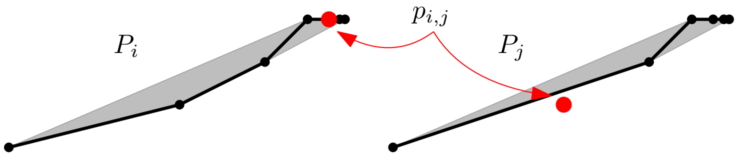
- for arbitrary graphs: if $v_i v_j$ is a non-edge, remove $p_{i,j}$ from P_i and P_j



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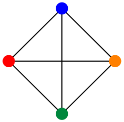
Grid size

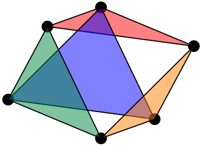
- our representation requires exponential-sized grid
- we consider also special classes of graphs

Graph class	general	bipartite	1-plane cubic	subcubic
Grid volume	super-poly	$O(n^4)$	$O(n^2)$	$O(n^3)$
Running time	$O(n^2)$	linear	linear	$O(n \log^2 n)$

Drawing Hypergraphs

Graph $G = (V, E)$





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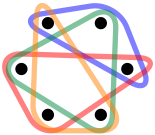

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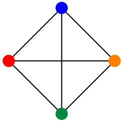

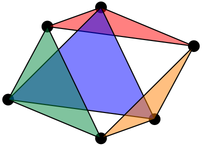

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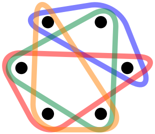

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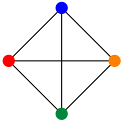

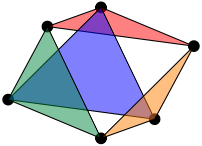

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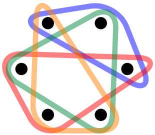

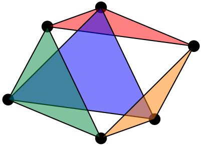

Complete 3-uniform Hypergraphs

A hypergraph is 3-uniform if all its hyperedges are of cardinality 3.

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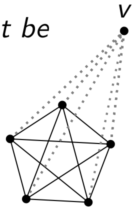

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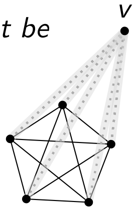

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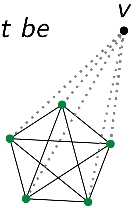

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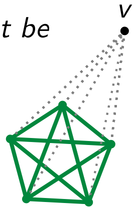

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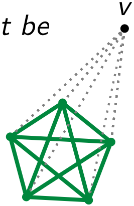

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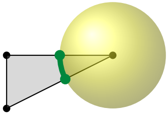
- If there is a non-crossing drawing, the link graph at any vertex must be planar.



Steiner Systems

A Steiner system $S(t, k, n)$ is an n-element set S together with a set of k-element subsets of S, called blocks, such that each t-element subset of S is contained in exactly one block.

Steiner Triple Systems[1]

$S(2,3,7)$
1 2 3
1 4 7
1 5 6
2 4 6
2 5 7
3 4 5
3 6 7

$S(2,3,9)$	
1 2 3	1 5 9
4 5 6	2 6 7
7 8 9	3 4 8
1 4 7	1 6 8
2 5 8	2 4 9
3 6 9	3 5 7

Steiner Quadruple System

$S(3,4,8)$	
1 2 4 8	3 5 6 7
2 3 5 8	1 4 6 7
3 4 6 8	1 2 5 7
4 5 7 8	1 2 3 6
1 5 6 8	2 3 4 7
2 6 7 8	1 3 4 5
1 3 7 8	2 4 5 6

[1] Ossona de Mendez [JGAA'02] shows that any 3-uniform hypergraph with incidence poset dimension 4 has a non-crossing drawing with triangles. This implies the existence of 3d representations (with exponential coordinates) for the two smallest Steiner triple systems.

Steiner Triple Systems

Theorem

The Fano plane $S(2,3,7)$ has a non-crossing drawing.

$S(2,3,7)$
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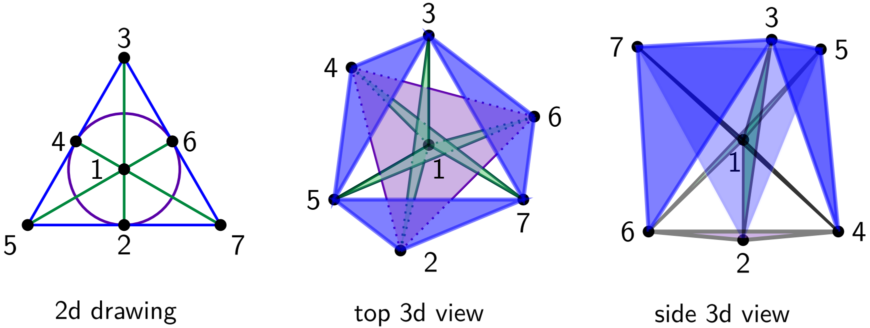

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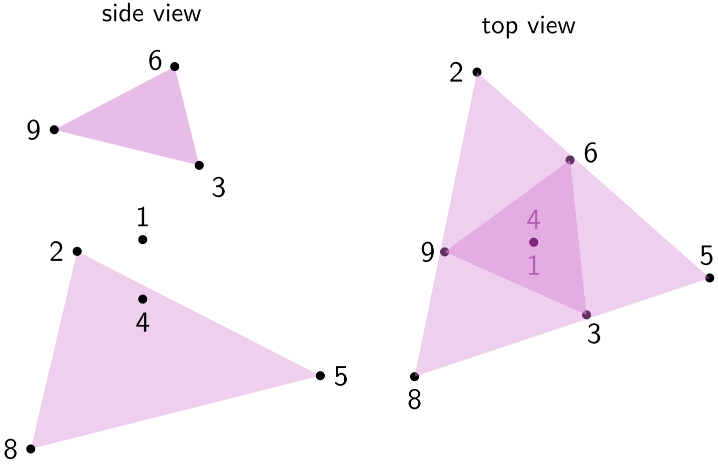

$S(2,3,9)$
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4 5 6
7 8 9
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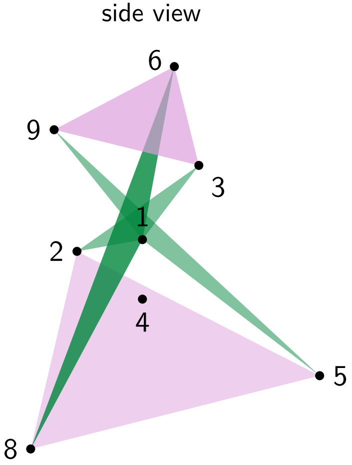
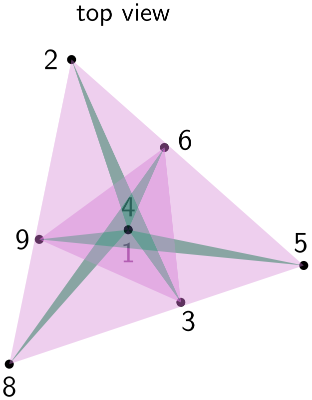
Theorem

The Steiner triple system $S(2,3,9)$ has a non-crossing drawing.

Steiner Triple Systems (cont.)





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Steiner Triple Systems (cont.)

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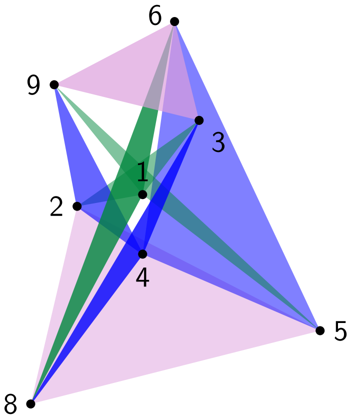
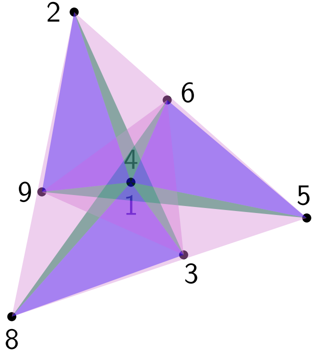


top view



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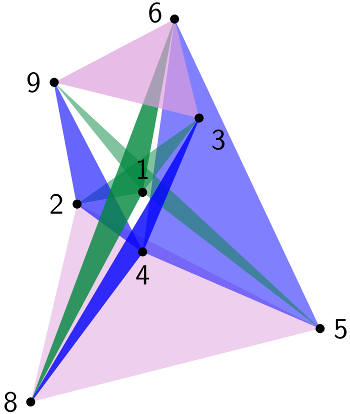

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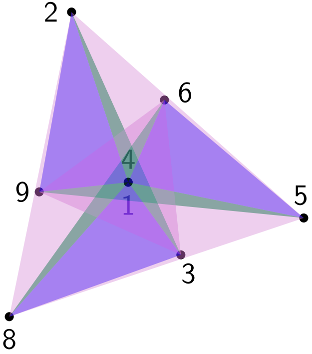

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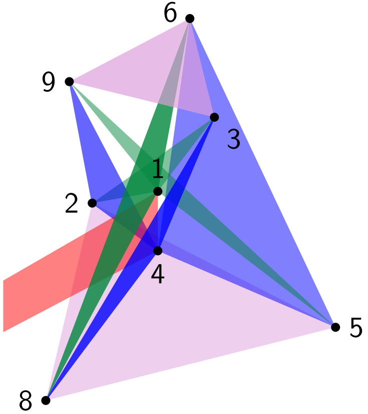

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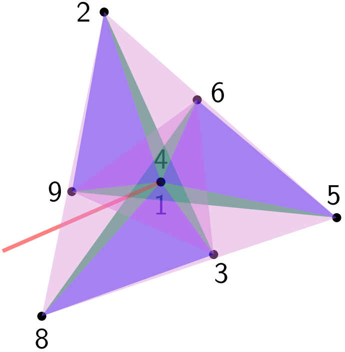

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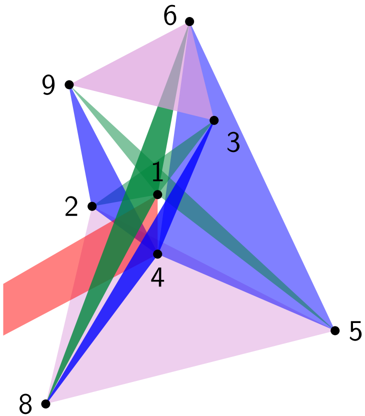
3 5 7

Theorem

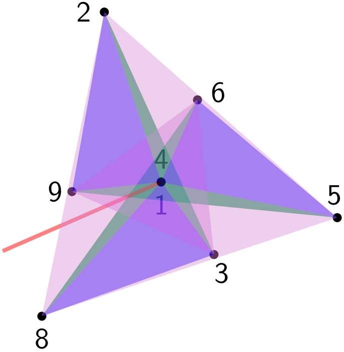
The Steiner triple system $S(2,3,9)$ has a non-crossing drawing.

Steiner Triple Systems (cont.)

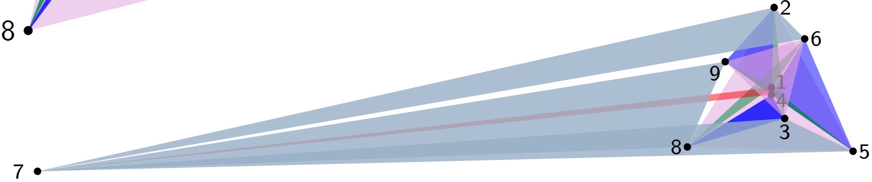
side view



top view



$S(2,3,9)$
1 2 3
4 5 6
7 8 9
1 4 7
2 5 8
3 6 9
1 5 9
2 6 7
3 4 8
1 6 8
2 4 9
3 5 7



Steiner Quadruple Systems

Theorem

The Steiner quadruple system $S(3,4,8)$ does not have a non-crossing drawing.

$S(3,4,8)$	
1 2 4 8	3 5 6 7
2 3 5 8	1 4 6 7
3 4 6 8	1 2 5 7
4 5 7 8	1 2 3 6
1 5 6 8	2 3 4 7
2 6 7 8	1 3 4 5
1 3 7 8	2 4 5 6

Steiner Quadruple Systems

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$S(3,4,8)$	
1 2 4 8	3 5 6 7
2 3 5 8	1 4 6 7
3 4 6 8	1 2 5 7
4 5 7 8	1 2 3 6
1 5 6 8	2 3 4 7
2 6 7 8	1 3 4 5
1 3 7 8	2 4 5 6

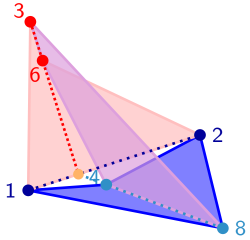
Steiner Quadruple Systems

Theorem

The Steiner quadruple system $S(3,4,8)$ does not have a non-crossing drawing.

1248	36

$P_{1236} \cap P_{3468} = l_{36}$ and $l_{12} \cap l_{48} \in l_{36}$



$S(3,4,8)$	
1 2 4 8	3 5 6 7
2 3 5 8	1 4 6 7
3 4 6 8	1 2 5 7
4 5 7 8	1 2 3 6
1 5 6 8	2 3 4 7
2 6 7 8	1 3 4 5
1 3 7 8	2 4 5 6

Steiner Quadruple Systems

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1248	36	$P_{1236} \cap P_{3468} = l_{36}$ and $l_{12} \cap l_{48} \in l_{36}$
1248	37	$P_{1378} \cap P_{2347} = l_{37}$ and $l_{18} \cap l_{24} \in l_{37}$
1248	67	$P_{1467} \cap P_{2678} = l_{67}$ and $l_{14} \cap l_{28} \in l_{67}$

$S(3,4,8)$	
1 2 4 8	3 5 6 7
2 3 5 8	1 4 6 7
3 4 6 8	1 2 5 7
4 5 7 8	1 2 3 6
1 5 6 8	2 3 4 7
2 6 7 8	1 3 4 5
1 3 7 8	2 4 5 6

If there is a drawing,

- 3,6, and 7 are all placed at the same point.
- 3567 is degenerate; a contradiction.
  (In fact, we can show that 3567 is just a point.) □

Steiner Quadruple Systems

Theorem

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2 3 5 8	1 4 6 7
3 4 6 8	1 2 5 7
4 5 7 8	1 2 3 6
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If there is a drawing,

- 3,6, and 7 are all placed at the same point.
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  (In fact, we can show that 3567 is just a point.) □

Theorem

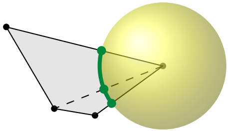
The Steiner quadruple system $S(3,4,10)$ cannot be drawn using all convex or all non-convex non-crossing quadrilaterals.

Steiner Quadruple Systems (cont.)

Theorem

No Steiner quadruple system can be drawn using convex quadrilaterals[2].

- Any vertex v is incident to $\frac{(n-1)(n-2)}{6}$ quadrilaterals.
- Add the diagonals incident to v to get a simplicial 2-complex.
- The link graph at v has $\frac{(n-1)(n-2)}{3}$ edges and $n-1$ vertices.
- For $n > 8$, the link graph is not planar.



[2]We thank Arnaud de Mesmay and Eric Sedgwick for pointing us to a lemma of Dey and Edelsbrunner [DCG'94], which uses the same proof idea.

Steiner Quadruple Systems (cont.)

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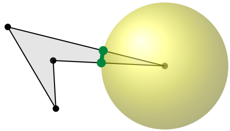
Theorem

No Steiner quadruple system with 20 *or more vertices can be drawn using quadrilaterals.*



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Steiner Quadruple Systems (cont.)

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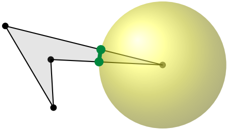

Conjecture

No Steiner quadruple system can be drawn using non-crossing quadrilaterals.

[2] We thank Arnaud de Mesmay and Eric Sedgwick for pointing us to a lemma of Dey and Edelsbrunner [DCG'94], which uses the same proof idea.

Open problems

Other hypergraphs	Larger Steiner triple systems/projective planes.
Hardness	Is deciding whether a 3-uniform hypergraph has a non-crossing drawing with triangles NP-hard?
Grid size	Can any graph be represented with convex polygons on a polynomial sized grid?
Nicer drawings	Small aspect ratio, large angle resolution, etc.