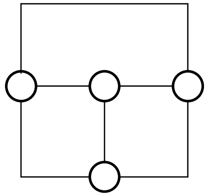
Orthogonal Drawings of Graphs and Their Relatives

Part 2 – Orthogonal drawings in the variable embedding setting

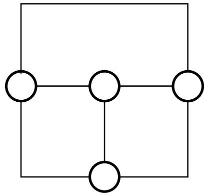
Walter Didimo
University of Perugia
walter.didimo@unipg.it

Summary

- The SQPR-tree data structure
- Bend-minimization of planar 3-graphs
  - Efficient algorithms
- Bend-minimization of planar 4-graphs
  - Exponential-time approaches



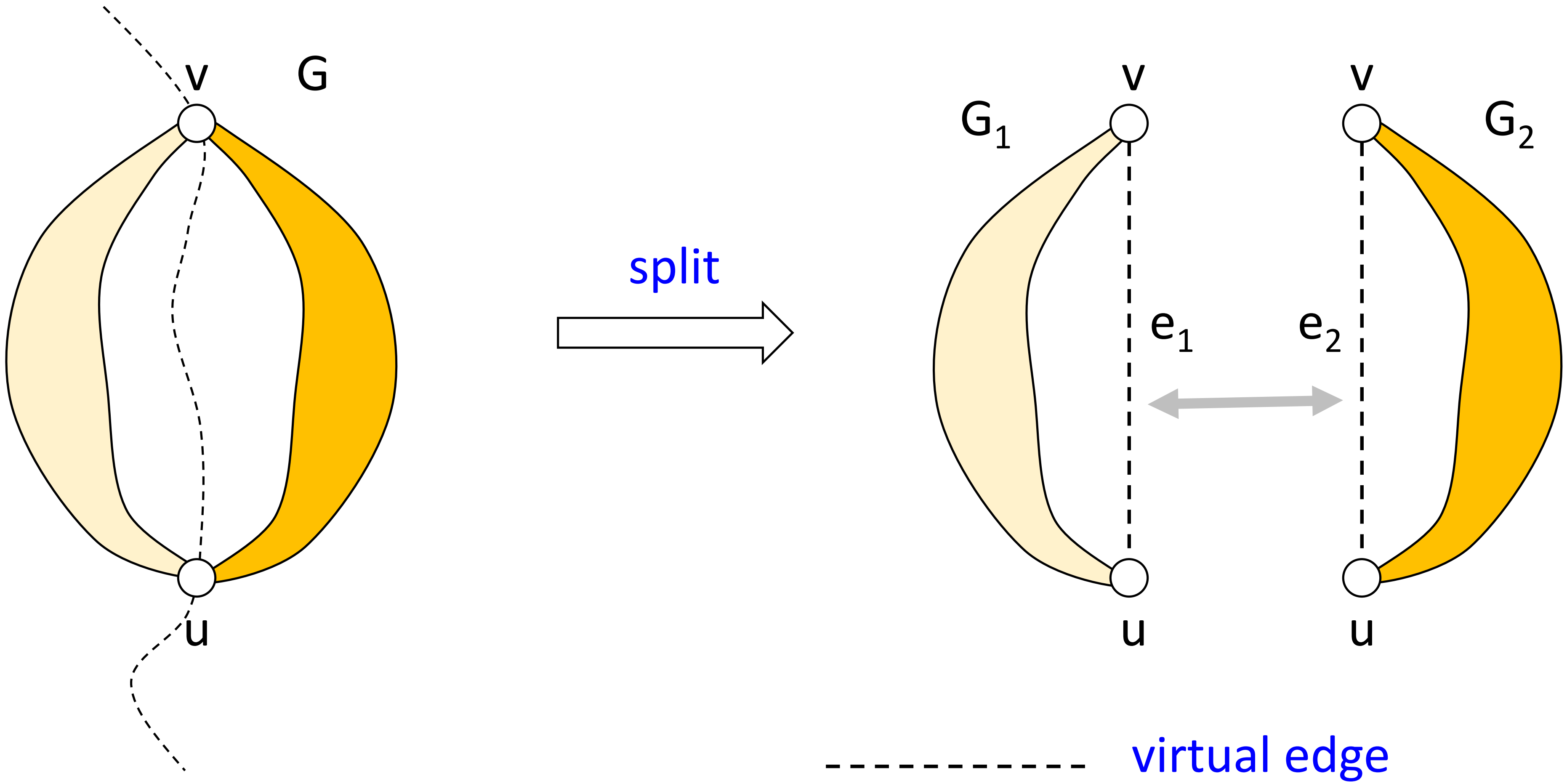
SPQR-trees



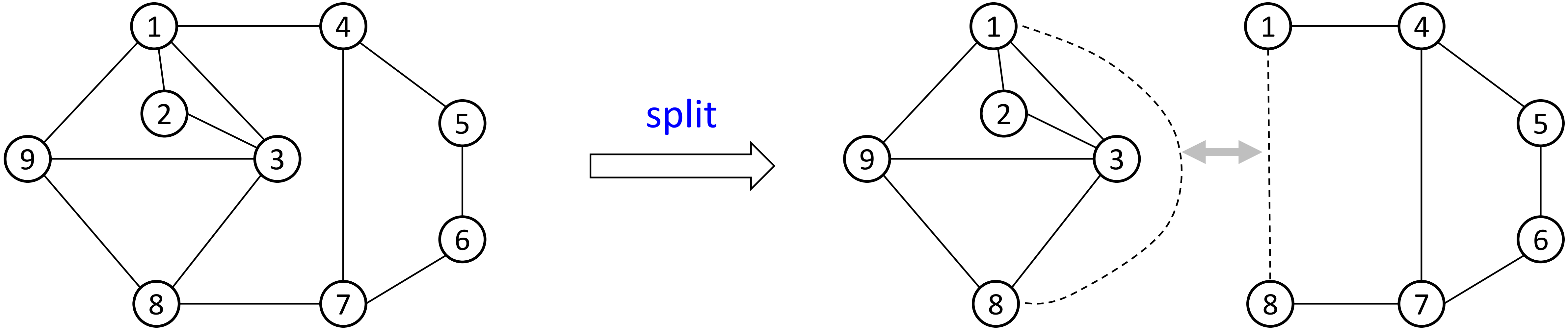
Triconnected components and SPQR-trees

- A biconnected graph can be decomposed into triconnected components
  - *J. E. Hopcroft, R. E. Tarjan*: Dividing a Graph into Triconnected Components. SIAM J. Comput. 2(3): 135-158 (1973)
- If G is a planar graph, the planar embeddings of G depend on the planar embeddings of its triconnected components
  - the SPQR-tree data structure provides an implicit representation of the triconnected components of G and of all planar embeddings of G [*G. Di Battista, R. Tamassia*: On-Line Planarity Testing. SIAM J. Comput. 25(5): 956-997 (1996)]

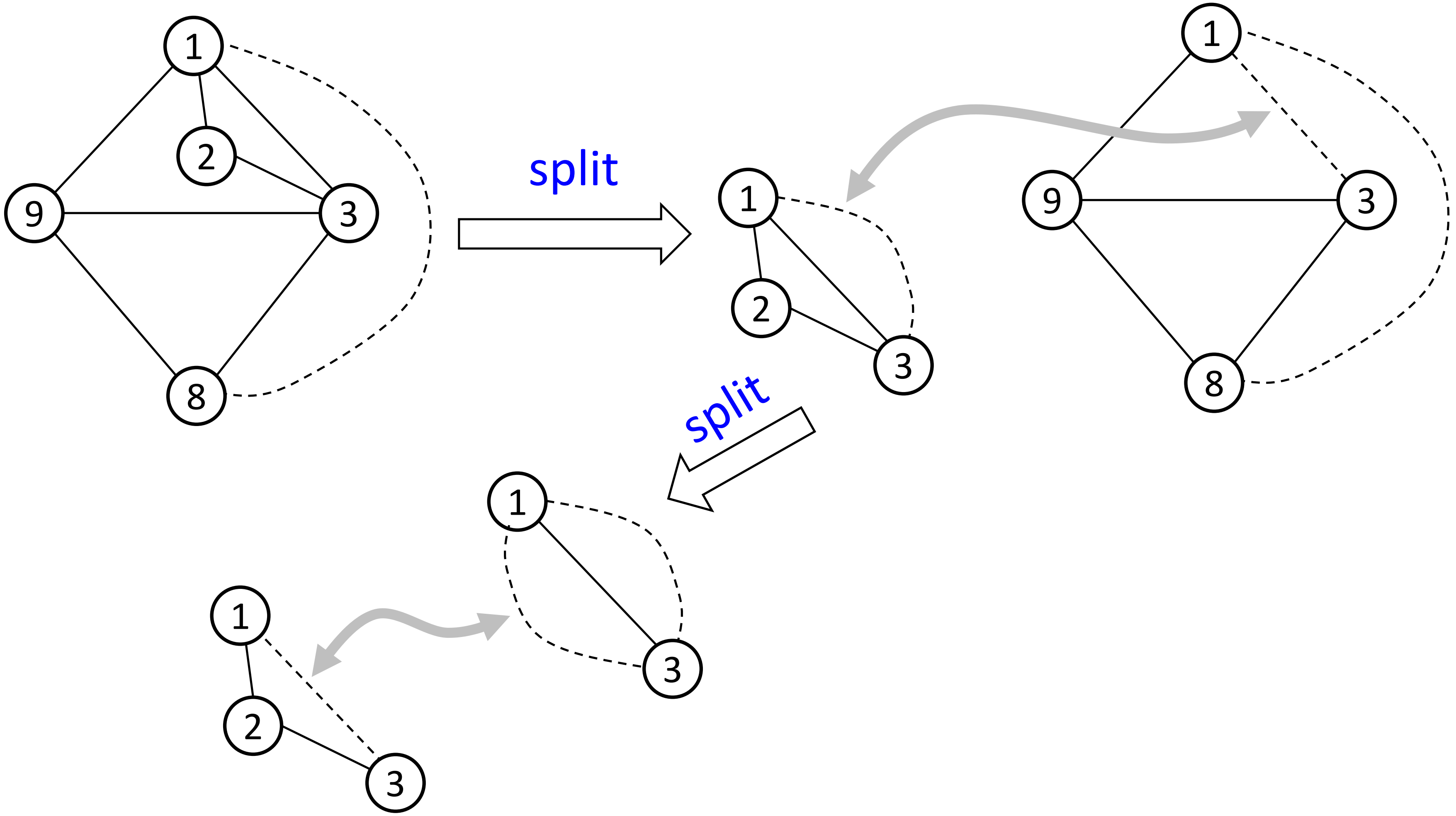
Separation pair and split operation



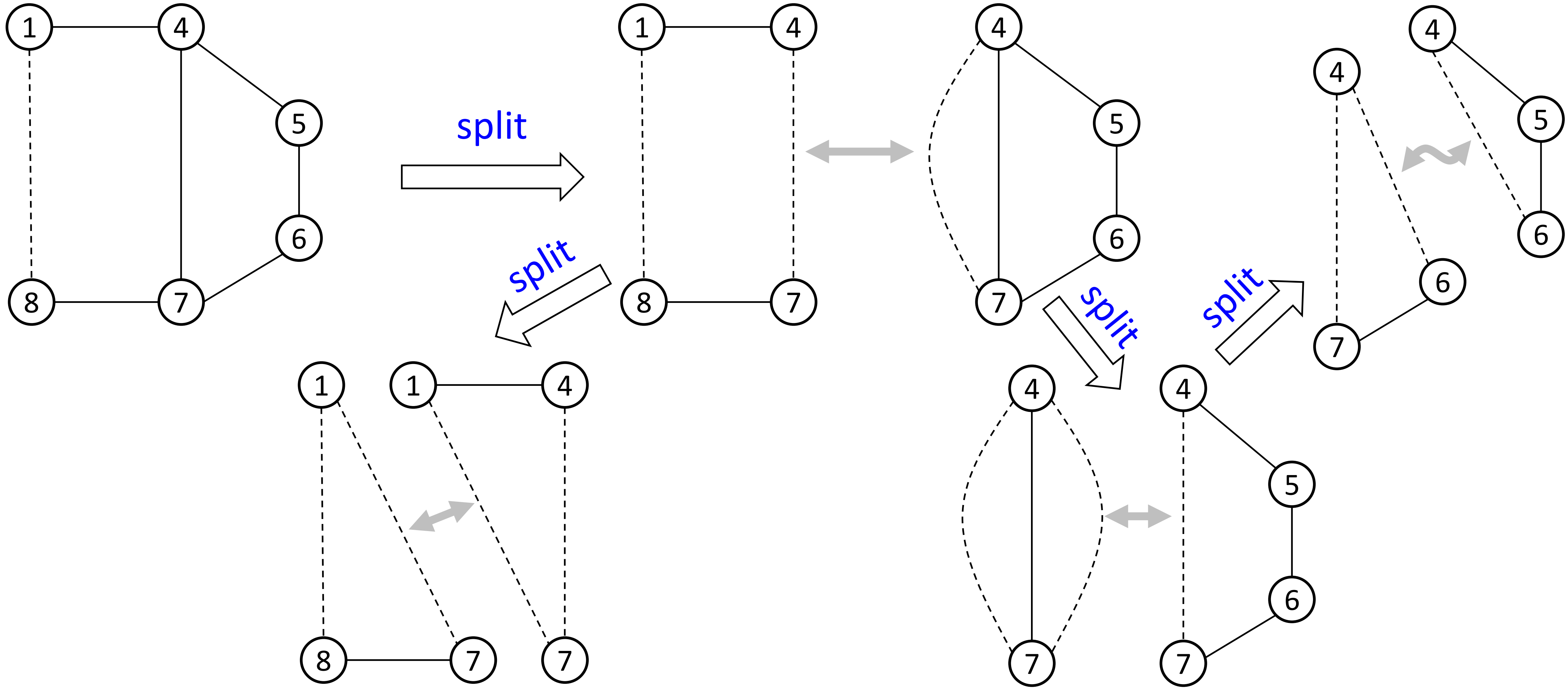
Recursive split operation



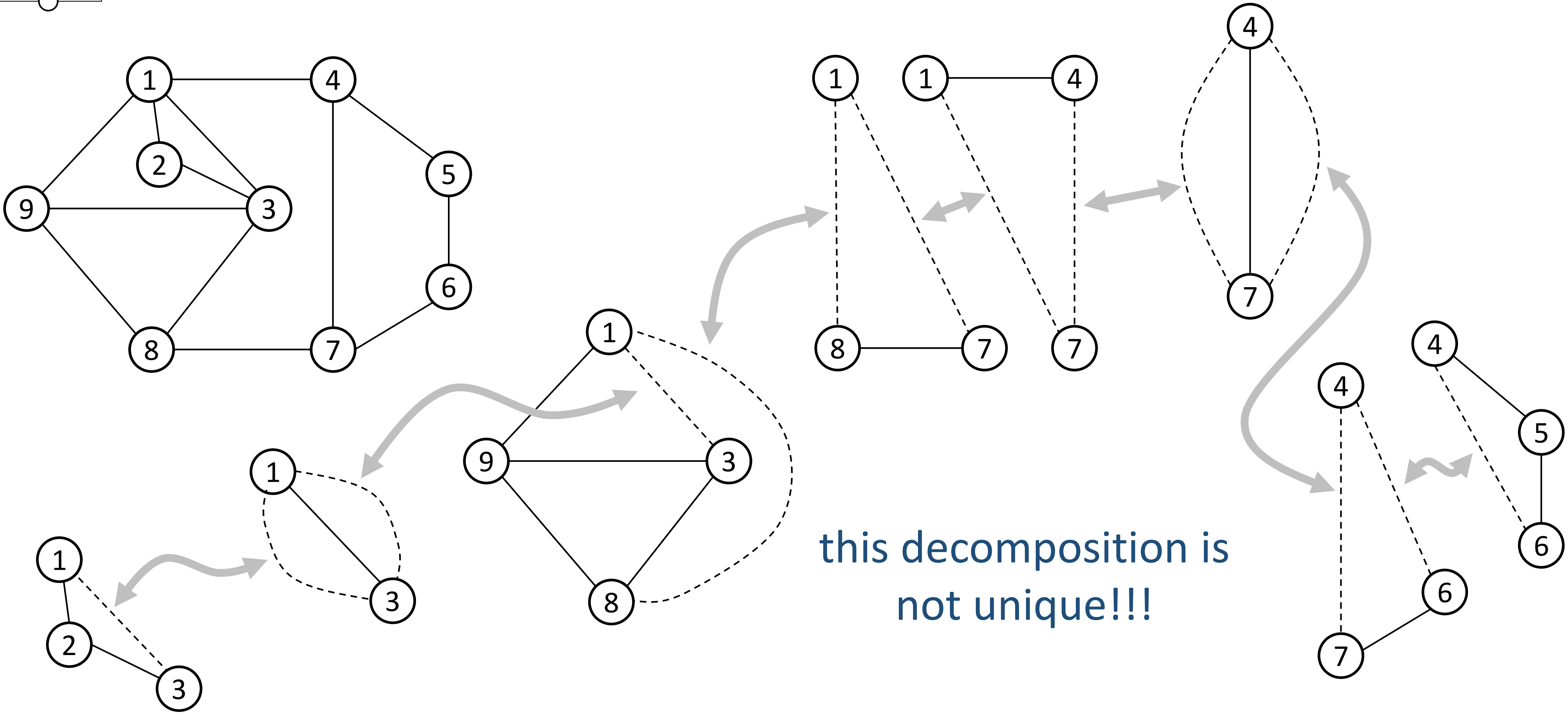
Recursive split operation



Recursive split operation

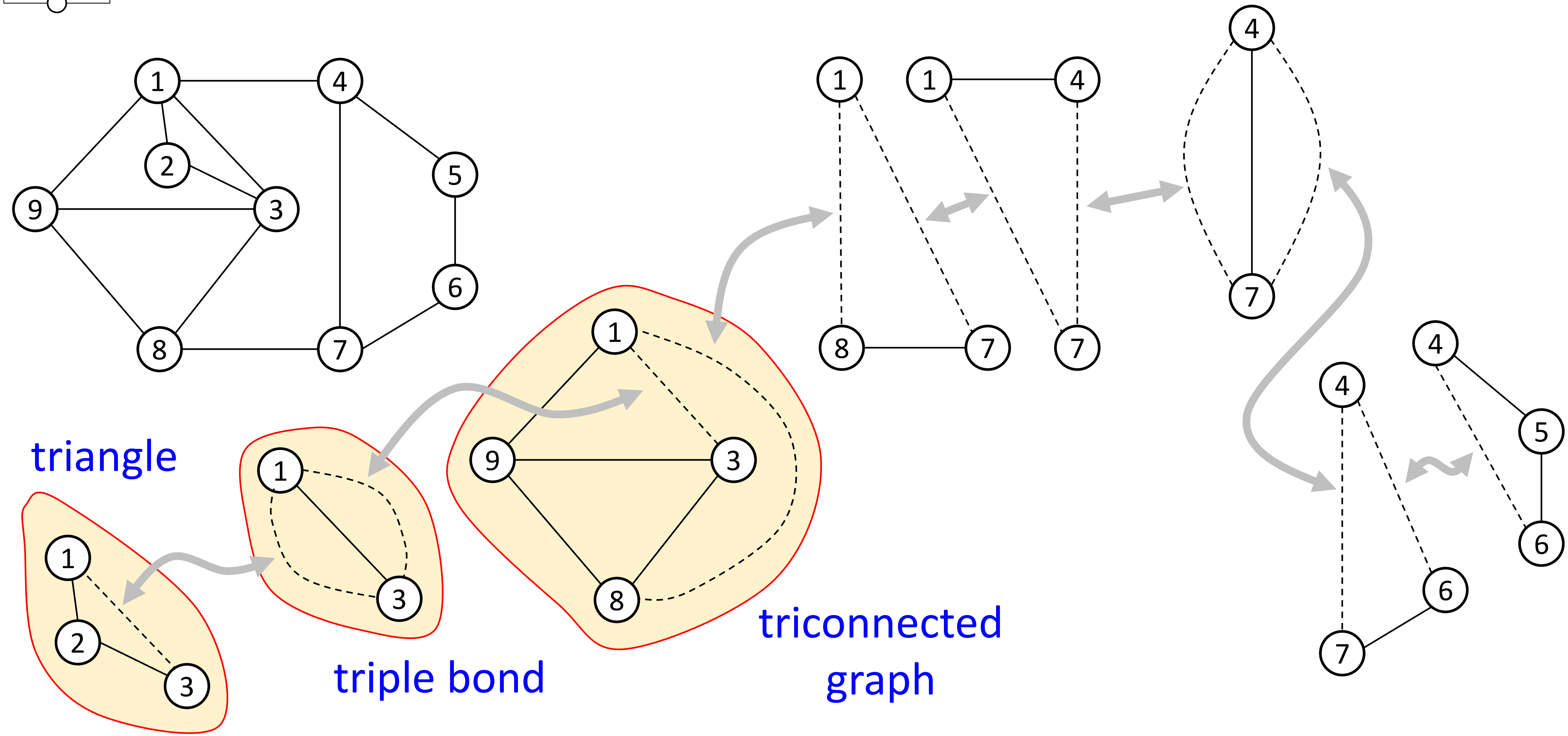


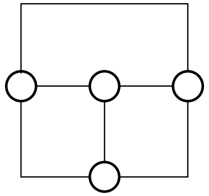
Recursive split operation - output



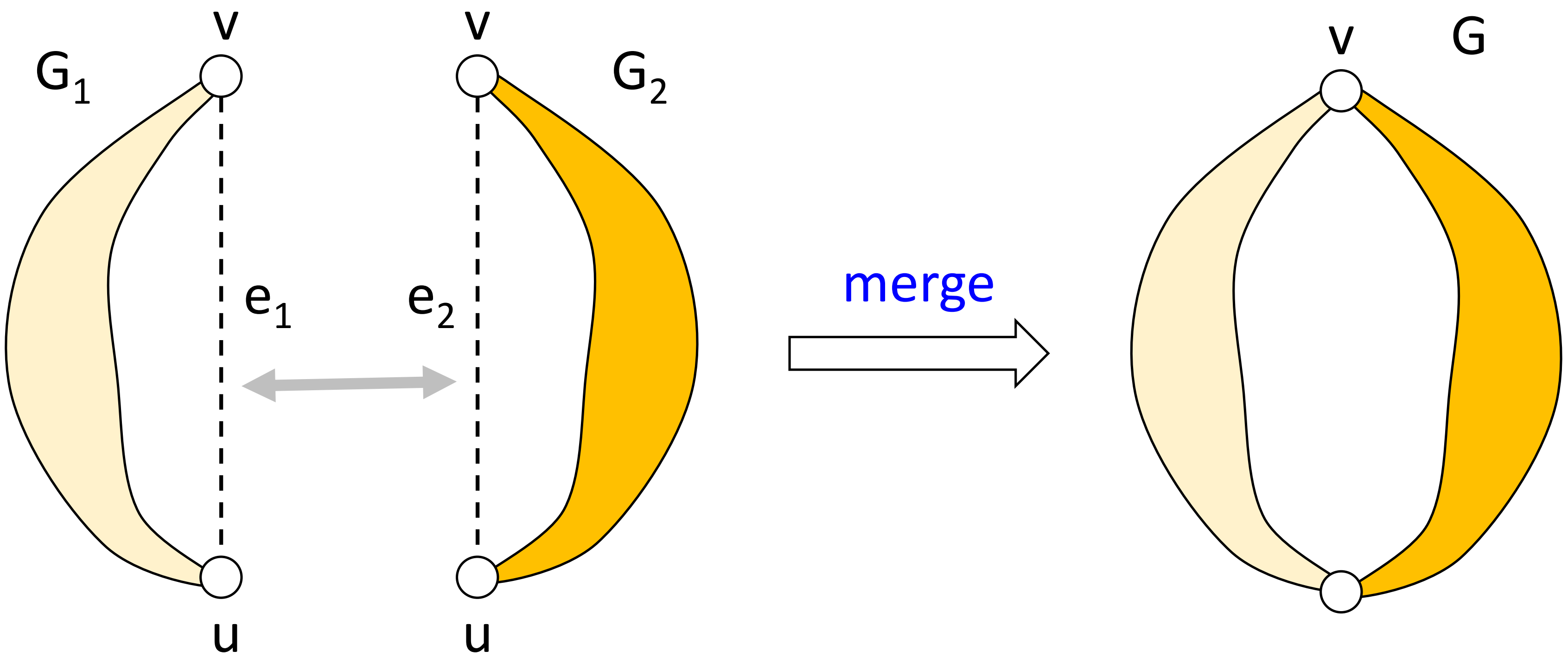
this decomposition is not unique!!!

Recursive split operation - output



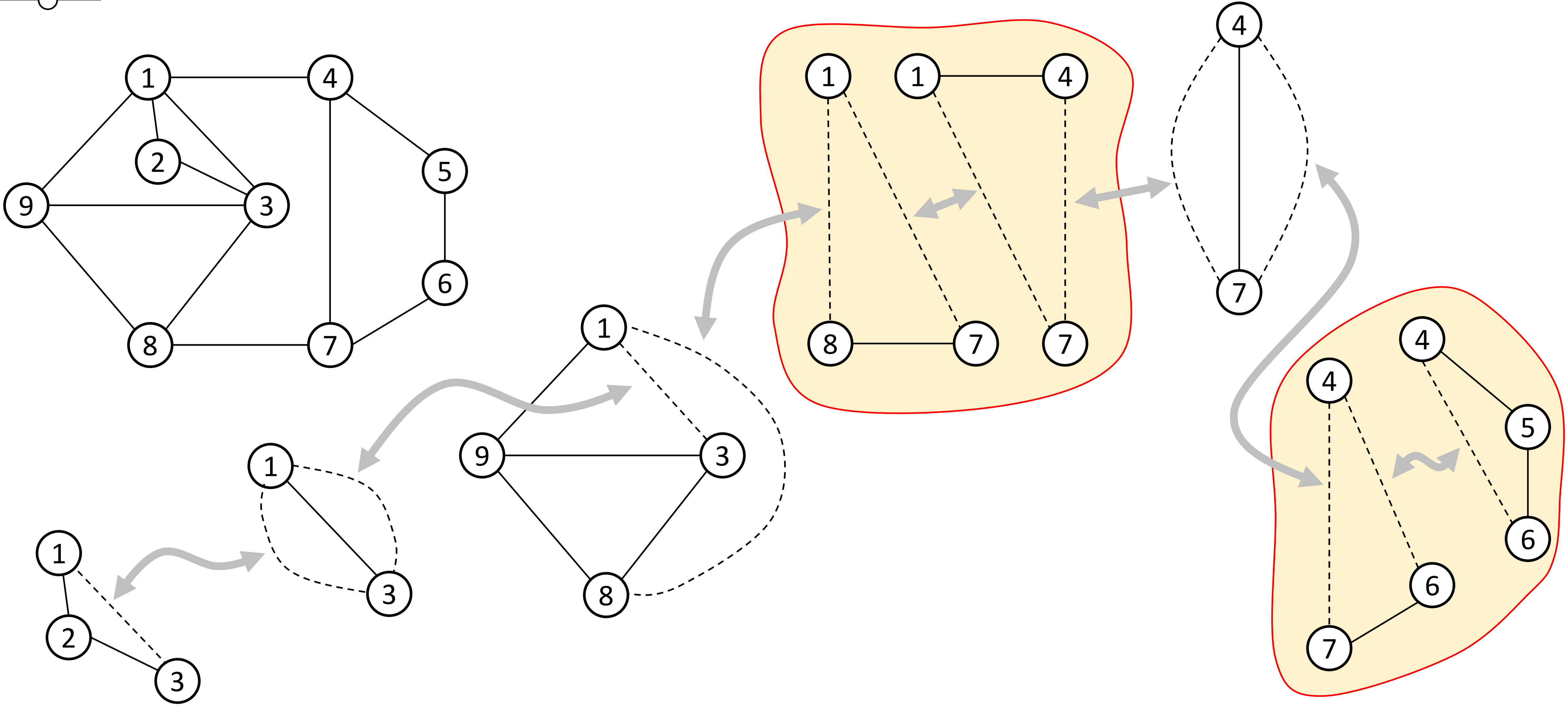


Merge operation

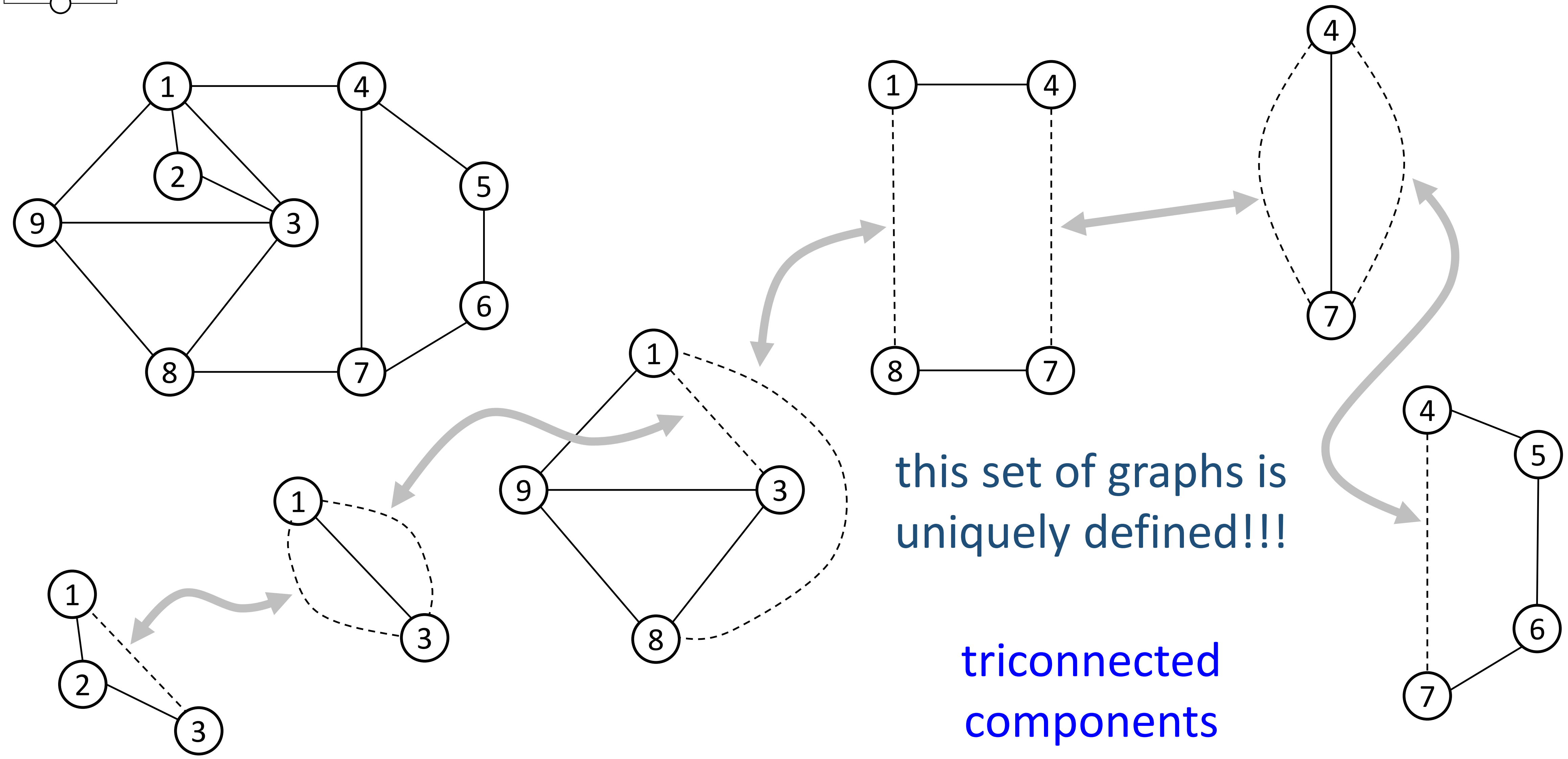


- If each G_i is a triple bond or (more in general) consists of a set of parallel edges only
- If each G_i is a triangle or (more in general) a simple cycle

Recursive merge operation



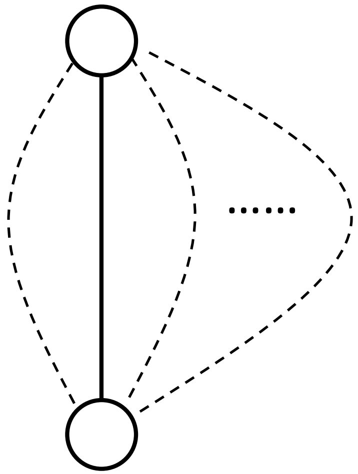
Recursive merge operation – final result



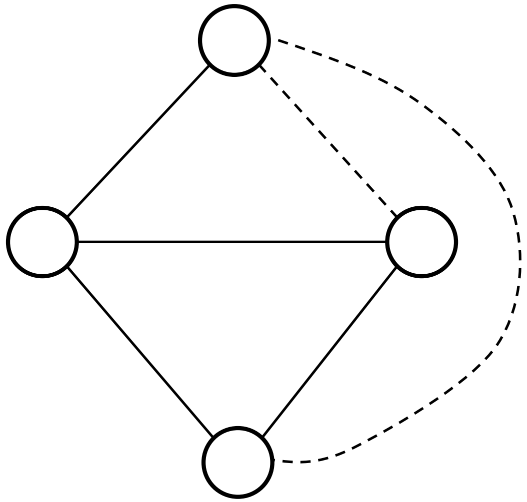
this set of graphs is uniquely defined!!!

triconnected components

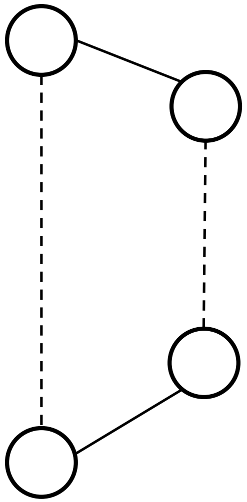
Triconnected components



parallel component

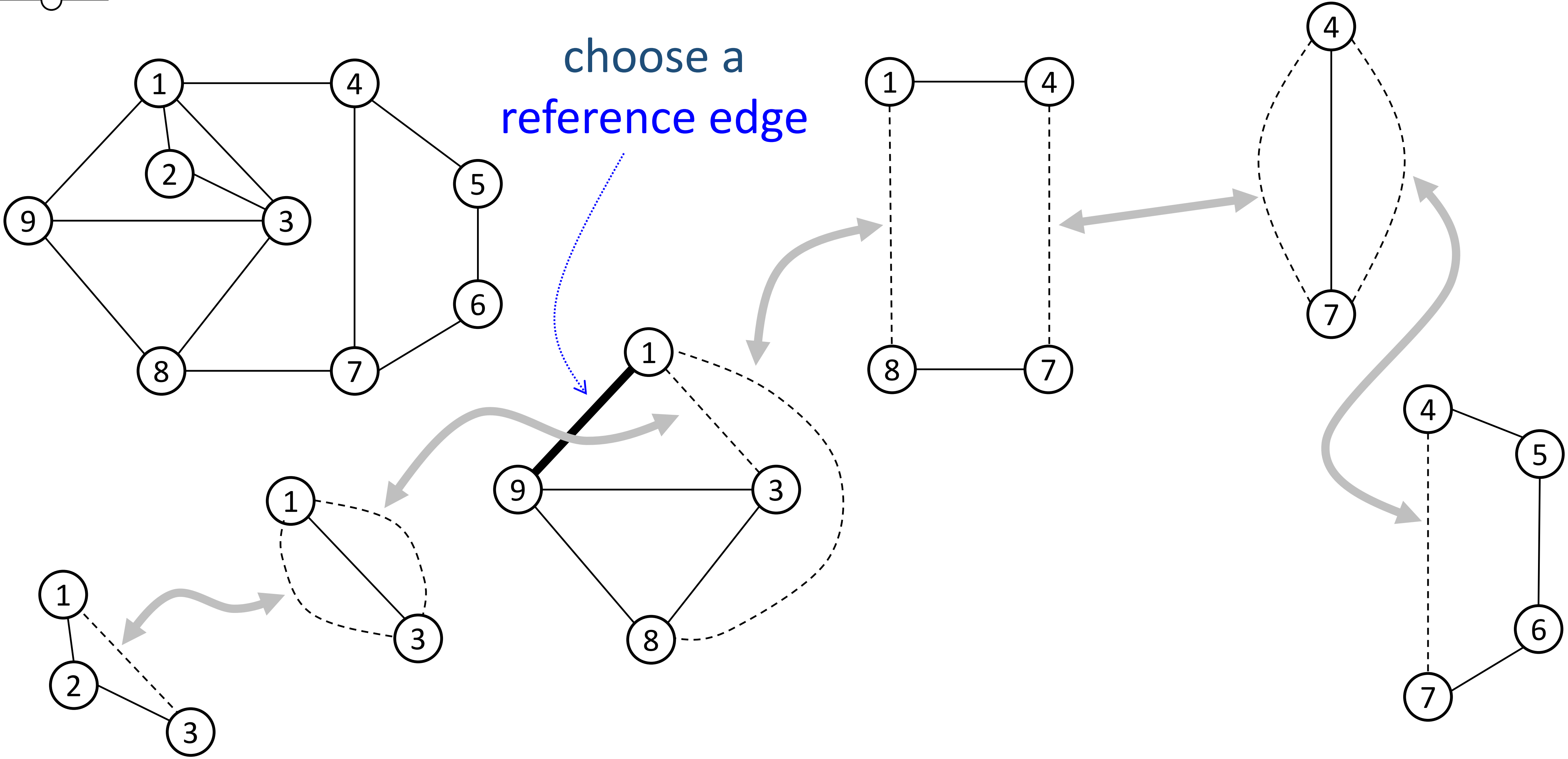


rigid component

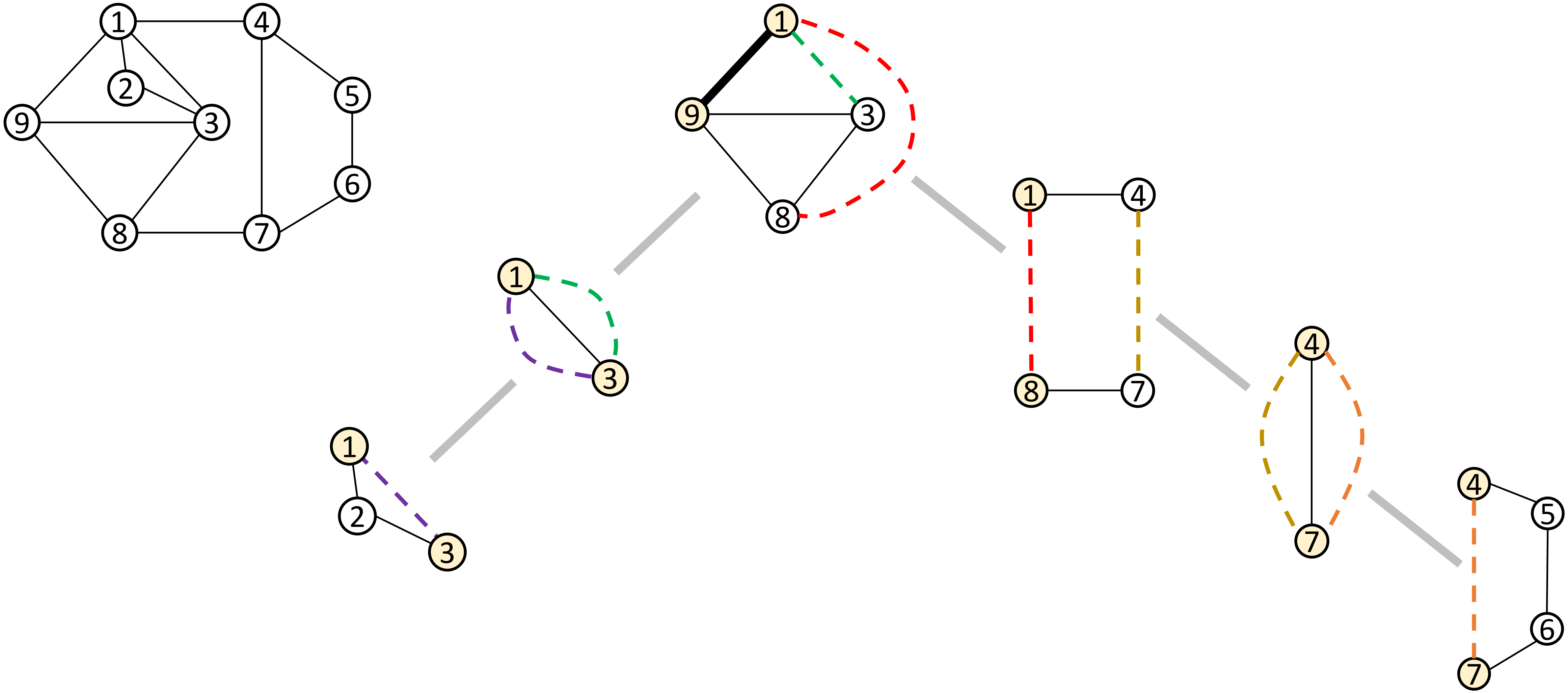


series component

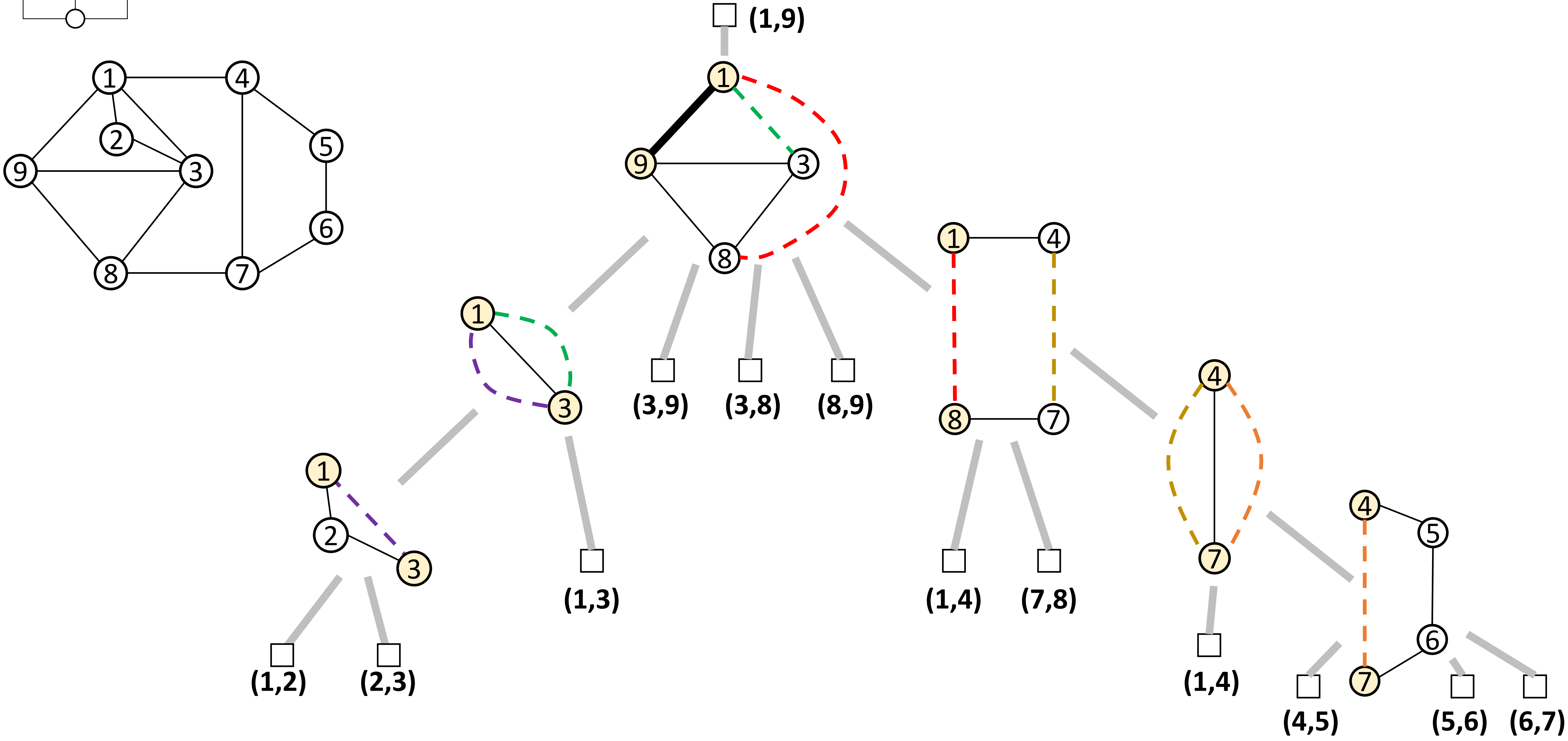
Towards SPQR-trees



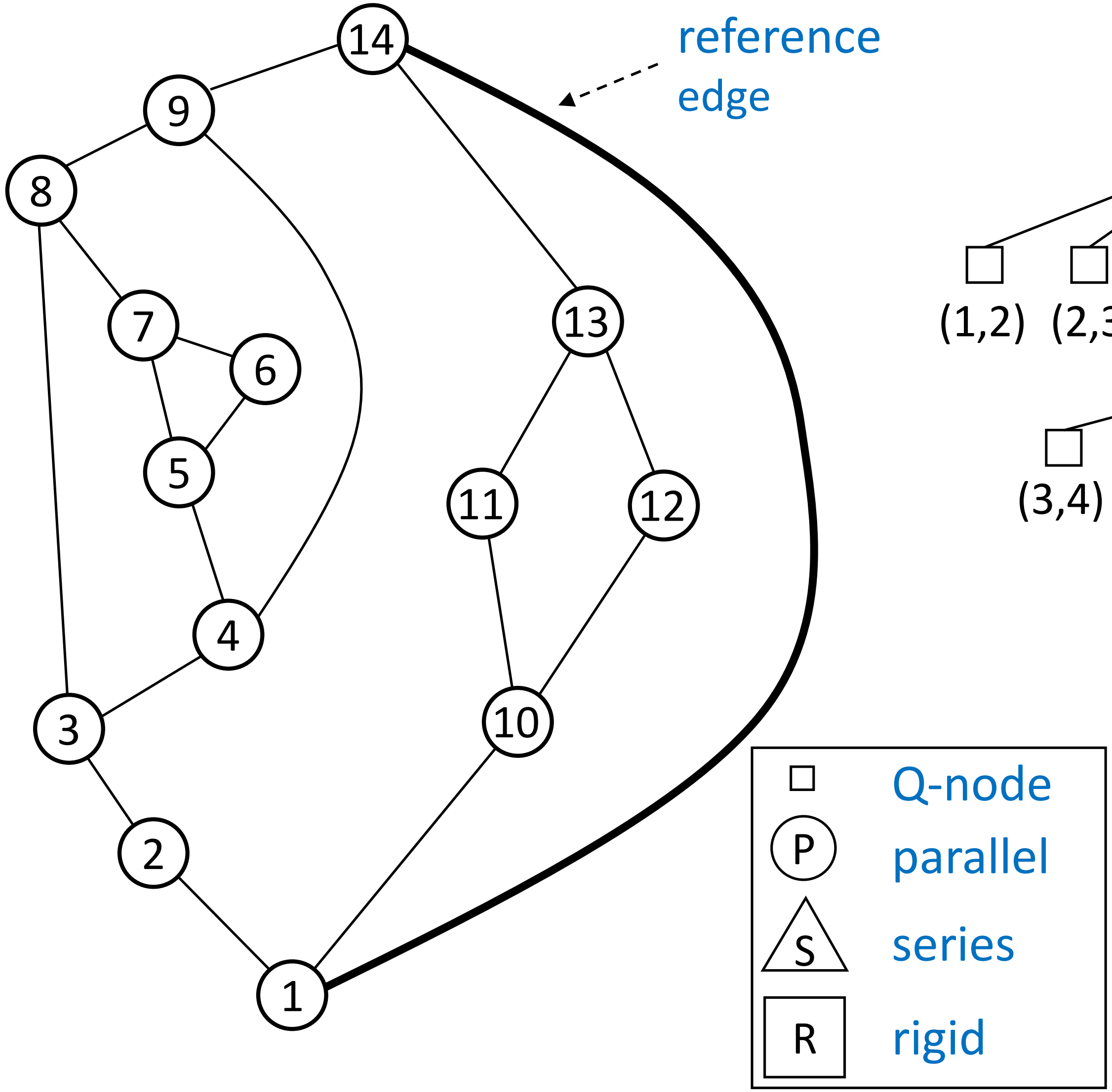
Towards SPQR-trees

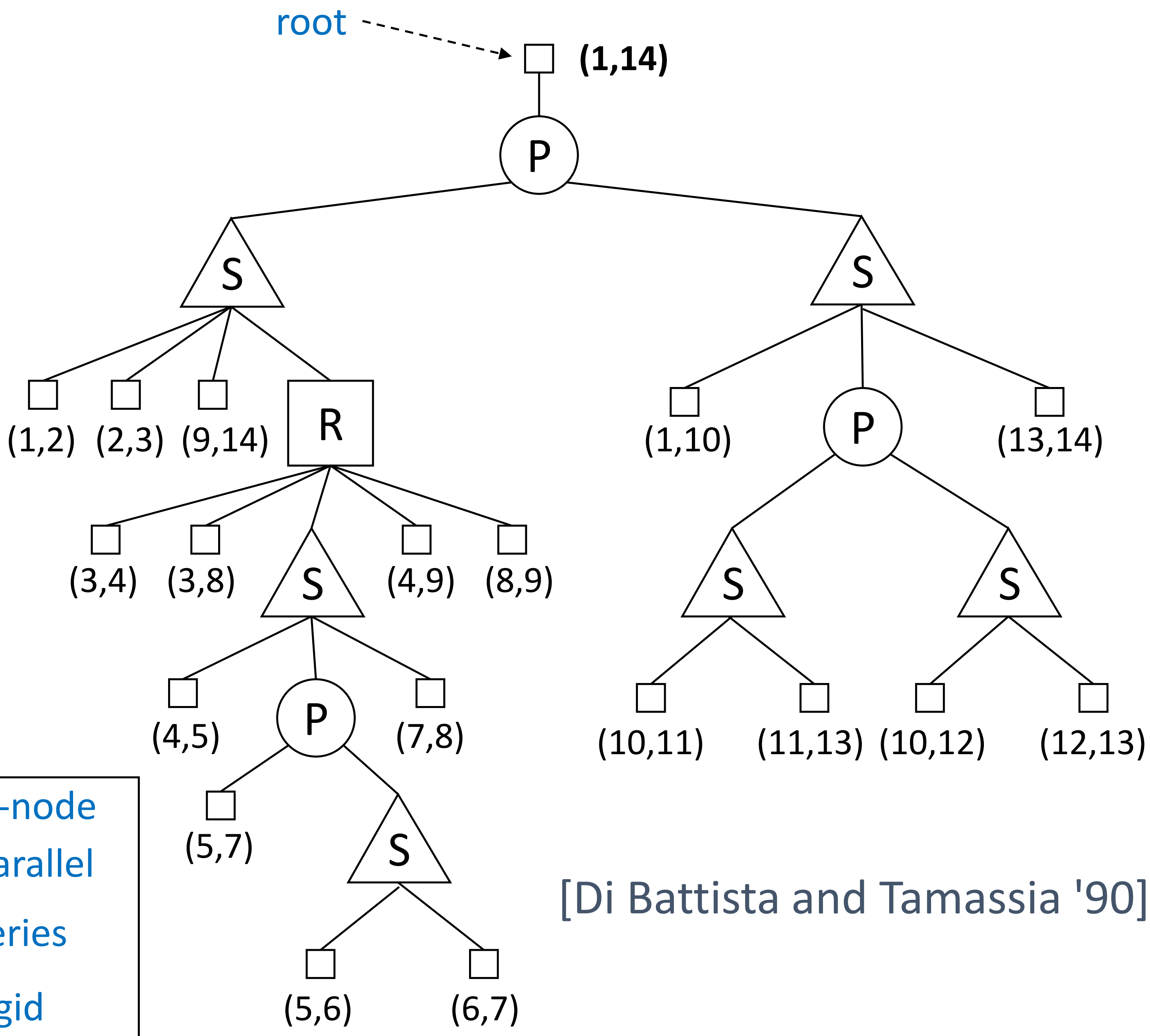


Towards SPQR-trees

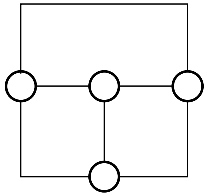


SPQR-trees

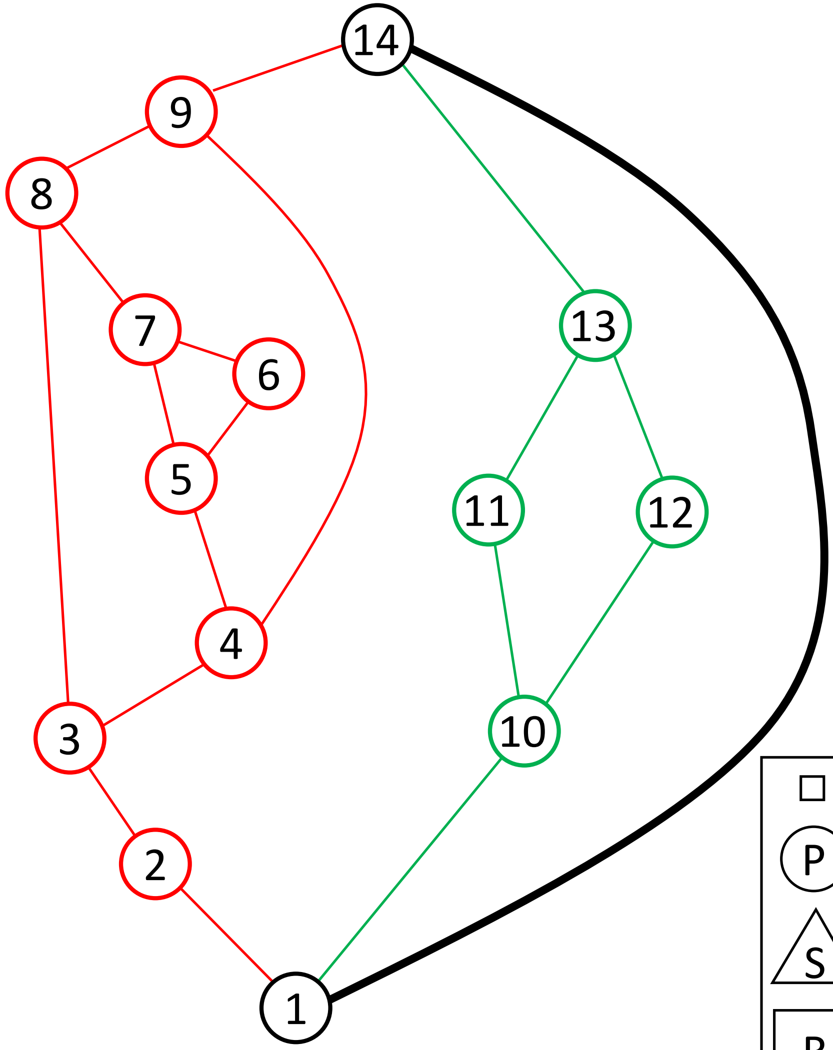




[Di Battista and Tamassia '90]



SPQR-trees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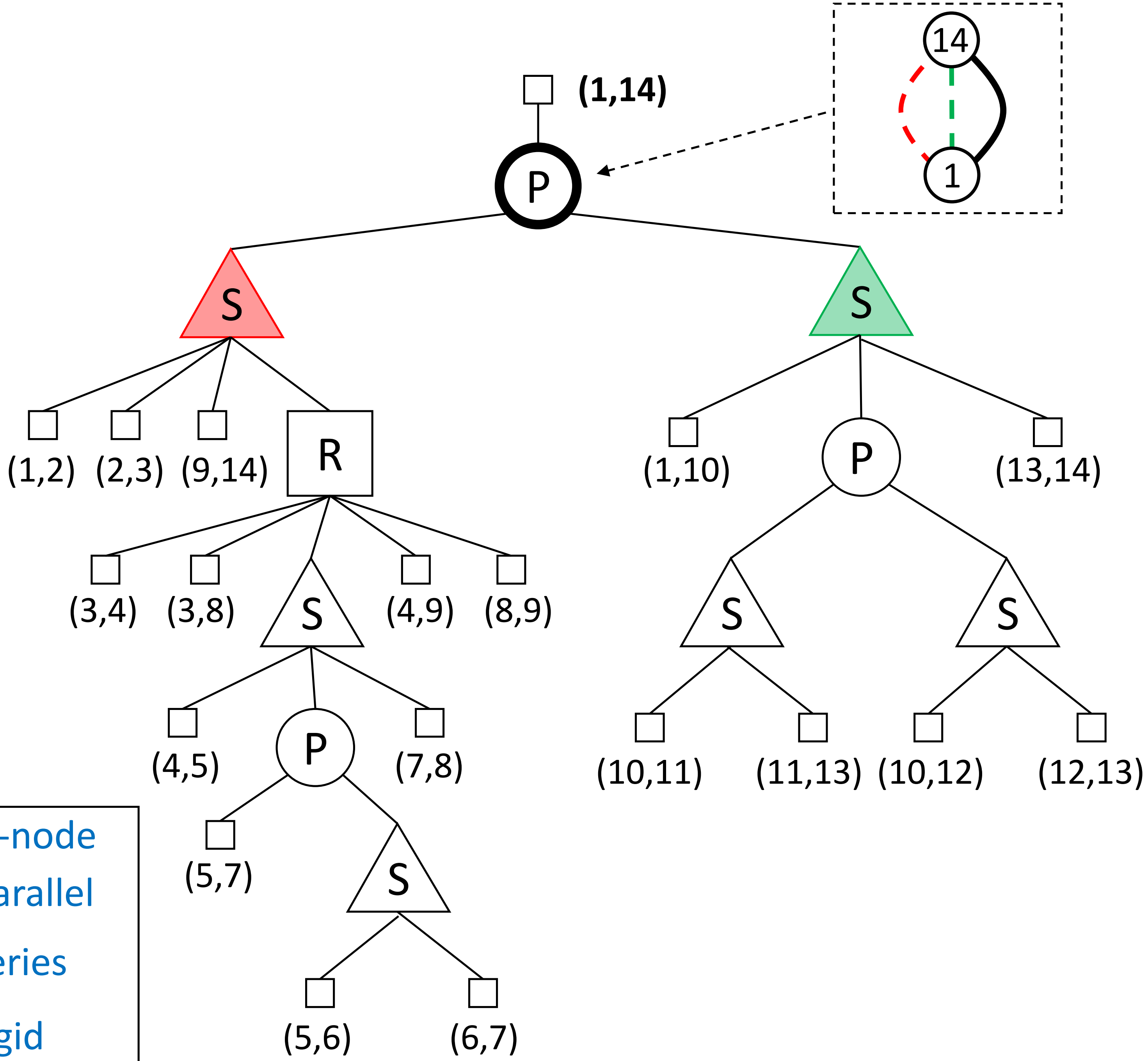
(1,14)
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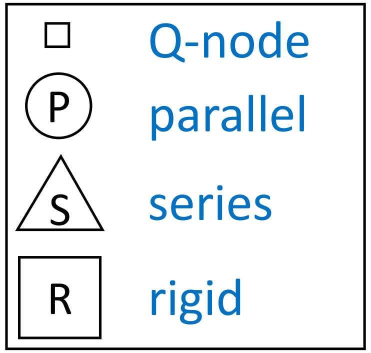
Q-node
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parallel
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series
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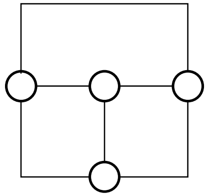
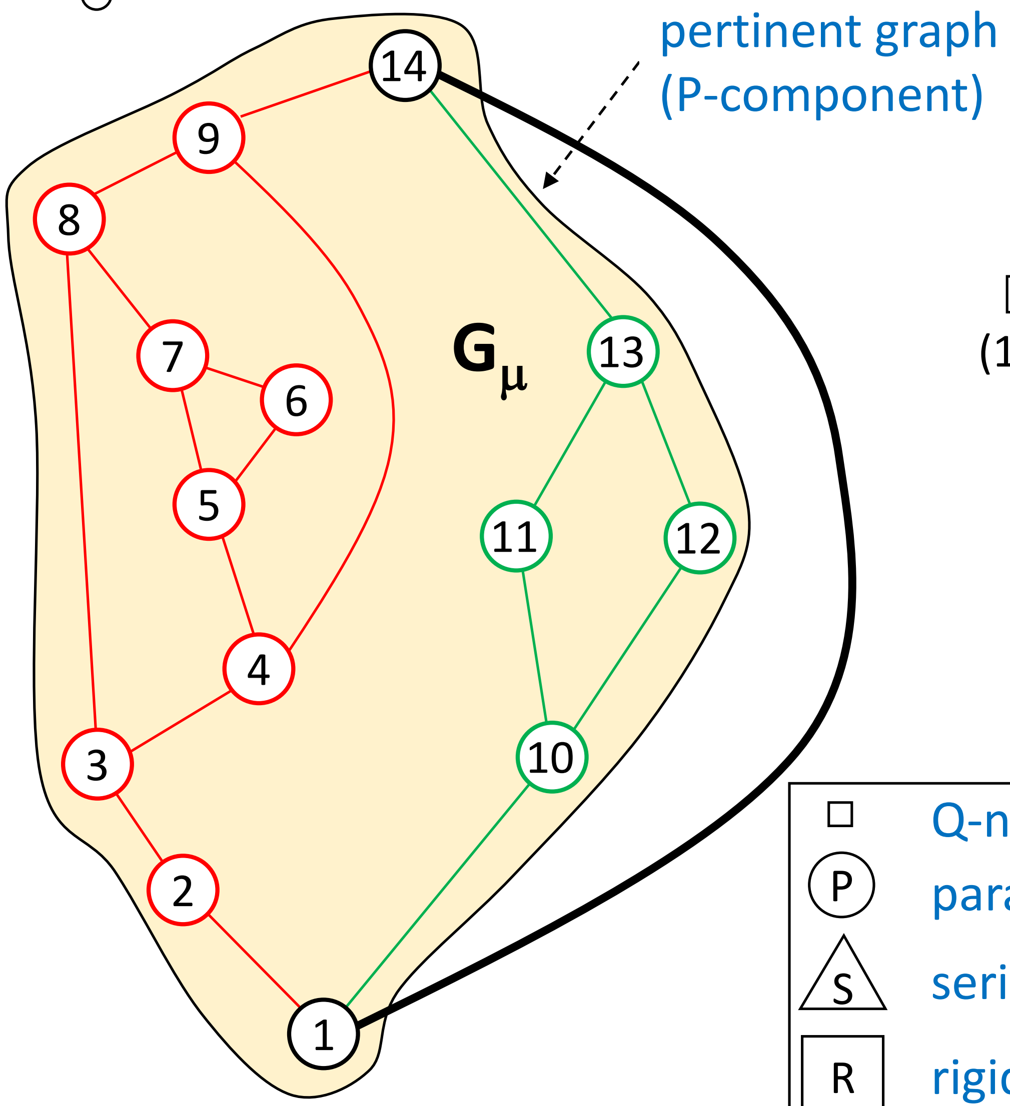
rigid



SPQR-trees



pertinent graph
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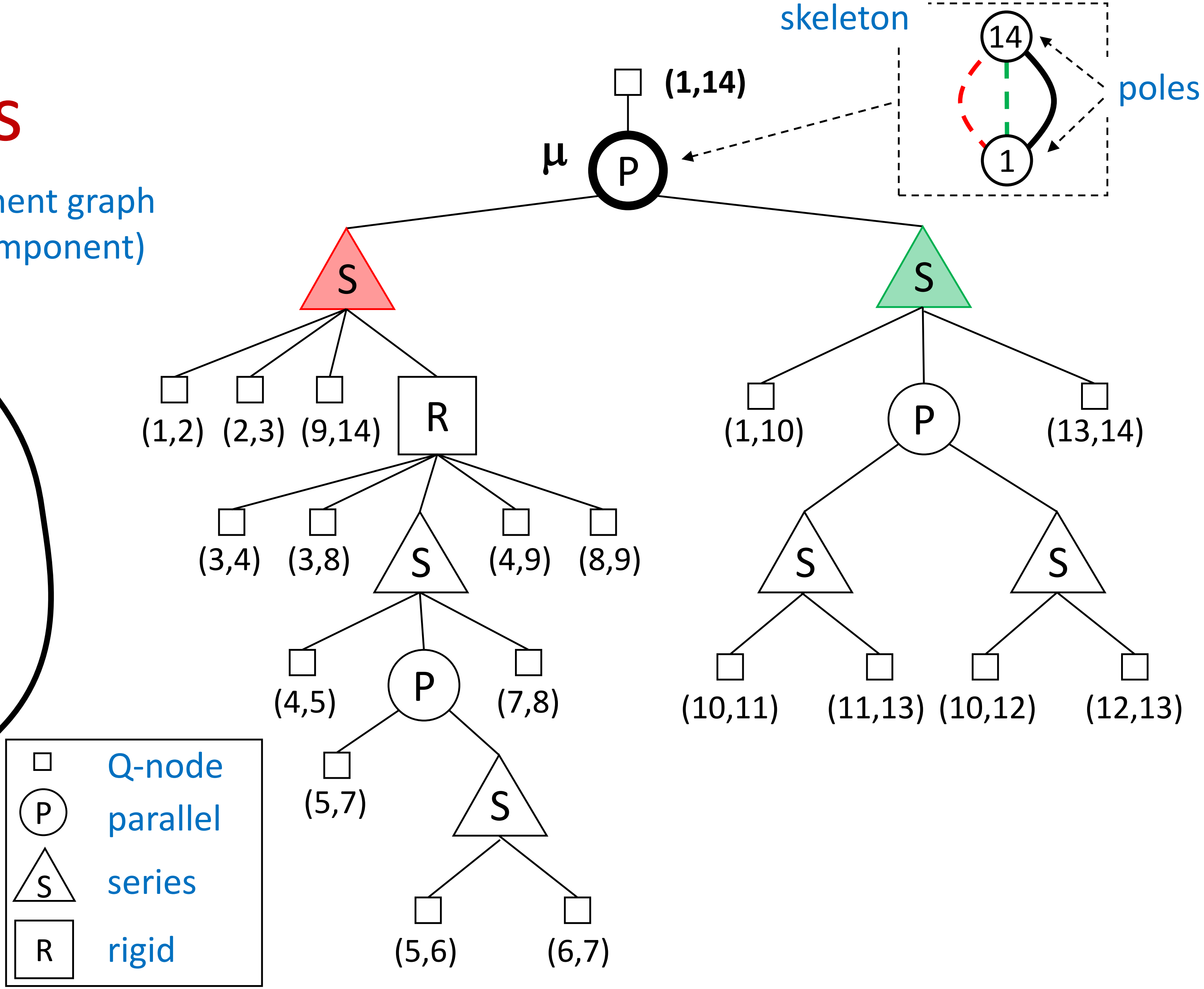

skeleton
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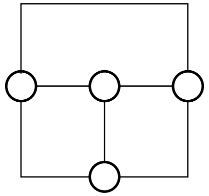
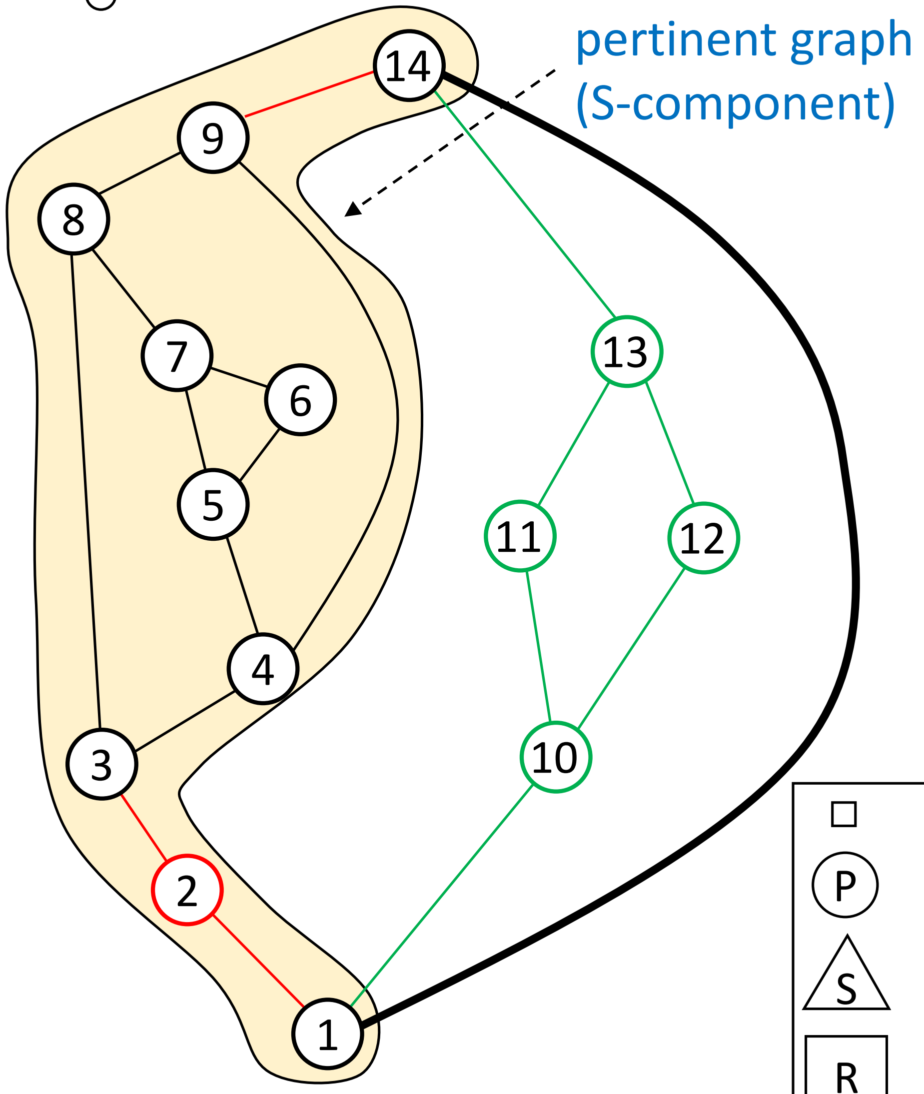
rigid



SPQR-trees



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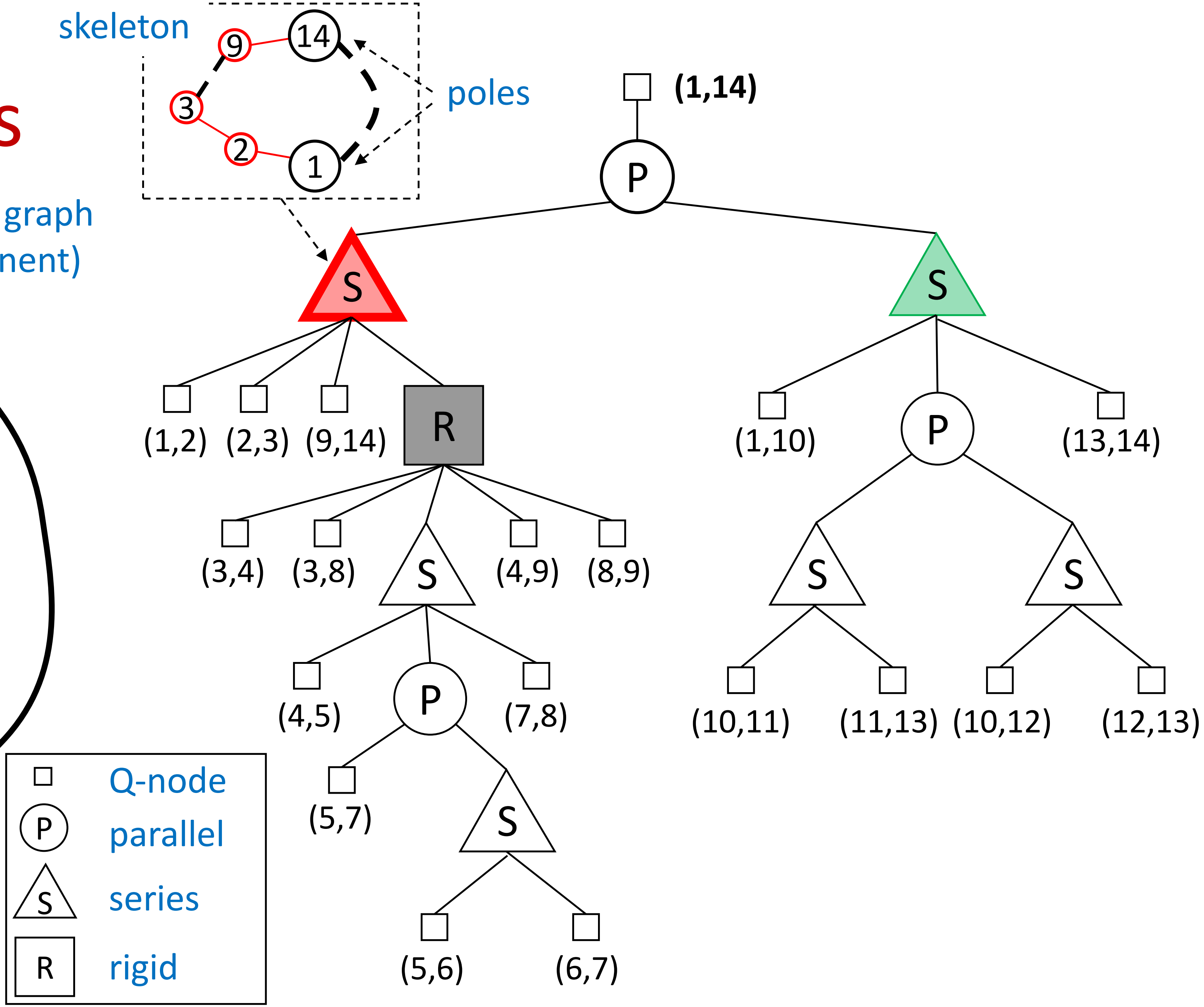
pertinent graph
(S-component)



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Q-node
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parallel
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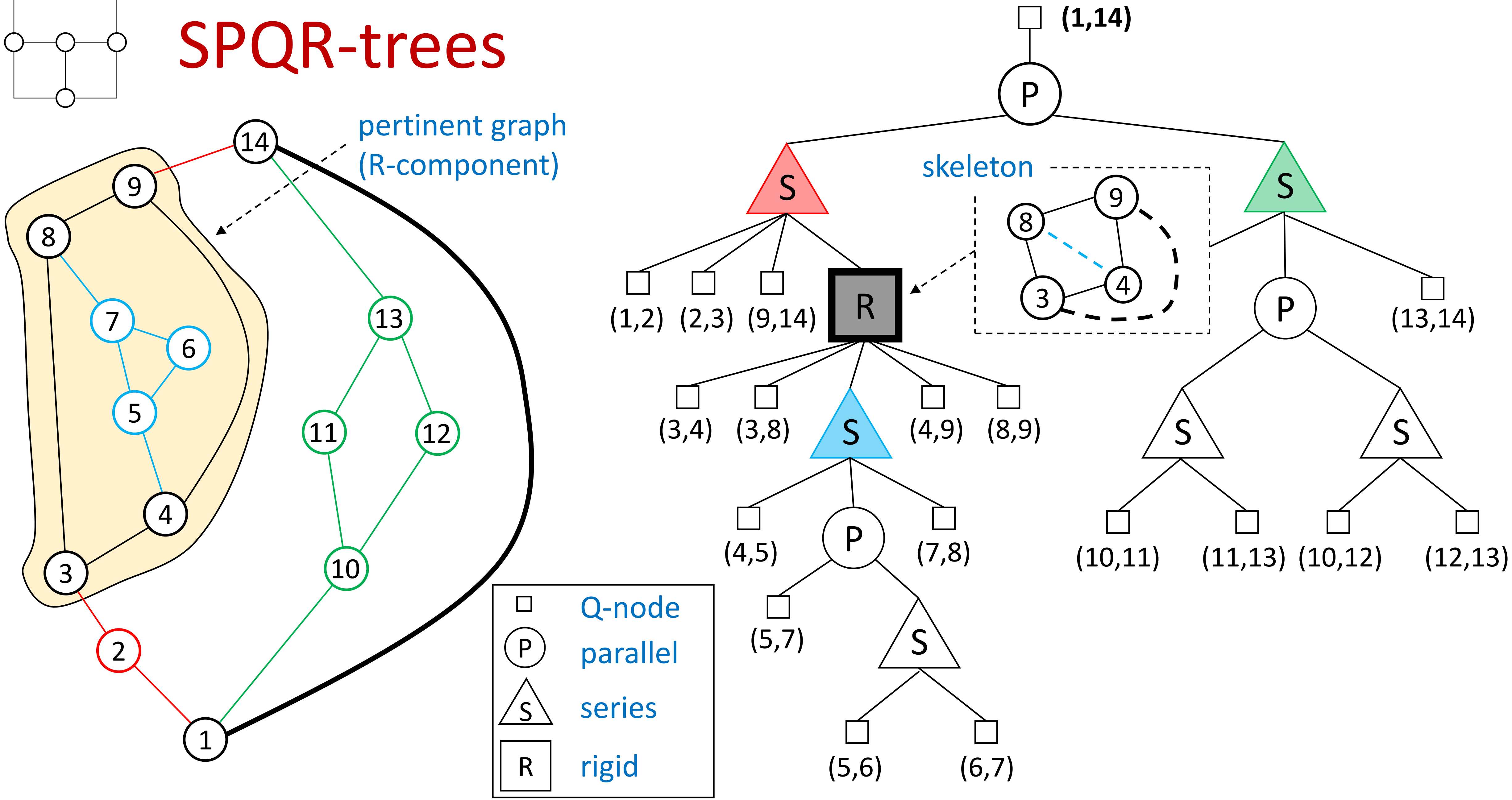
series
R
rigid

SPQR-trees



pertinent graph
(R-component)
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skeleton
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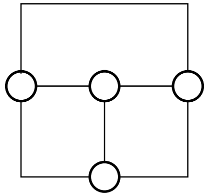
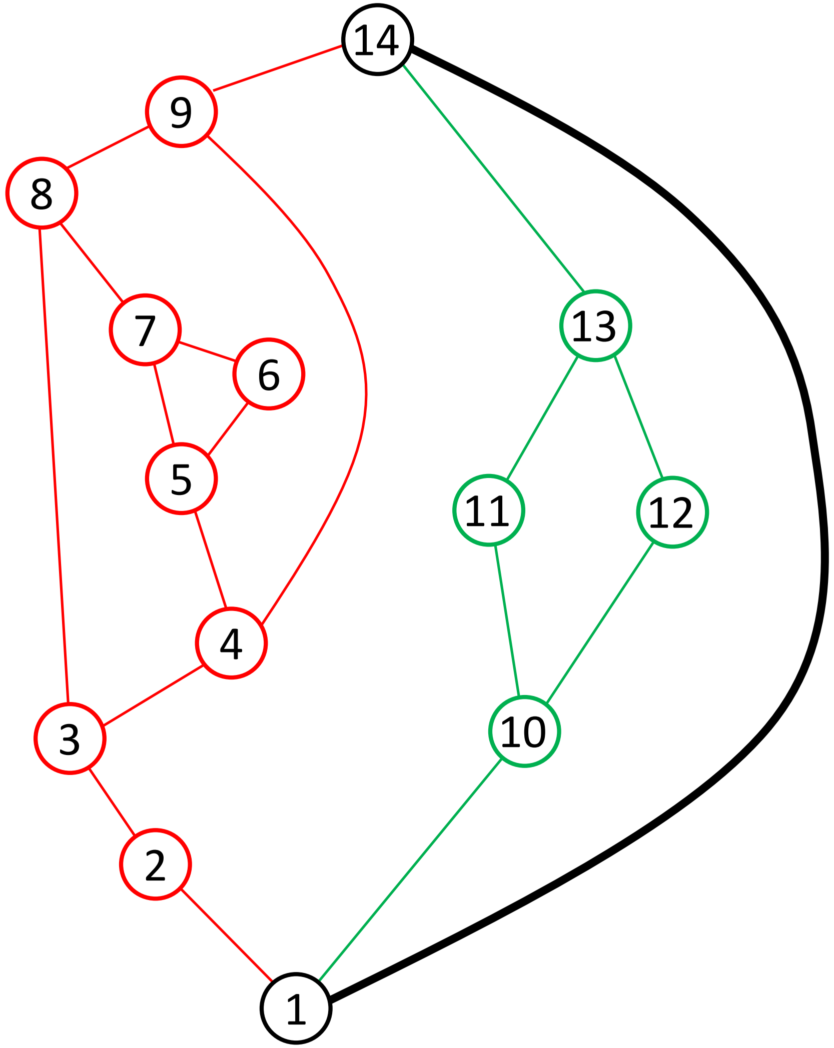
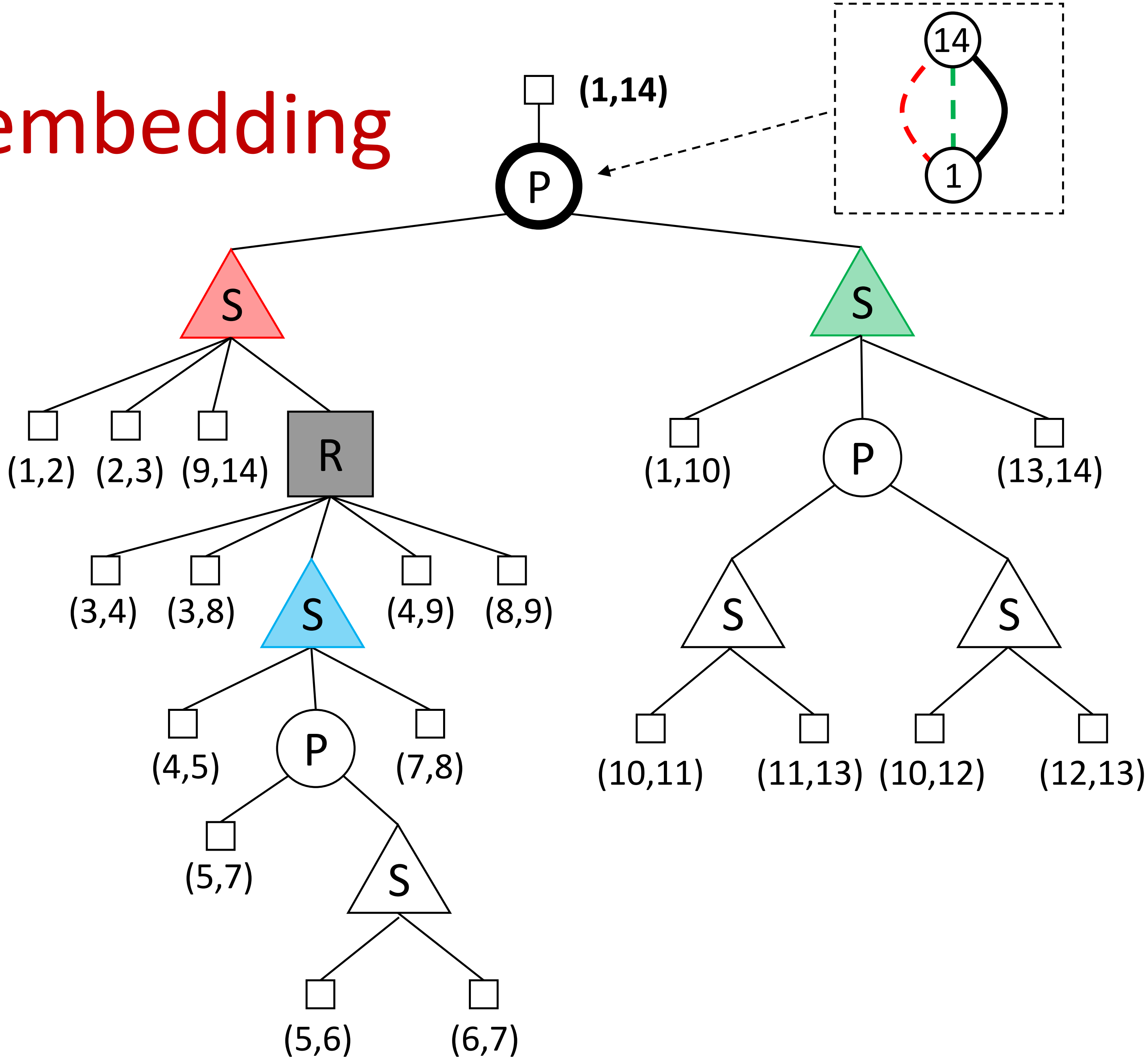
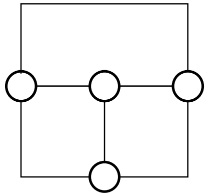
Q-node
P parallel
S series
R rigid



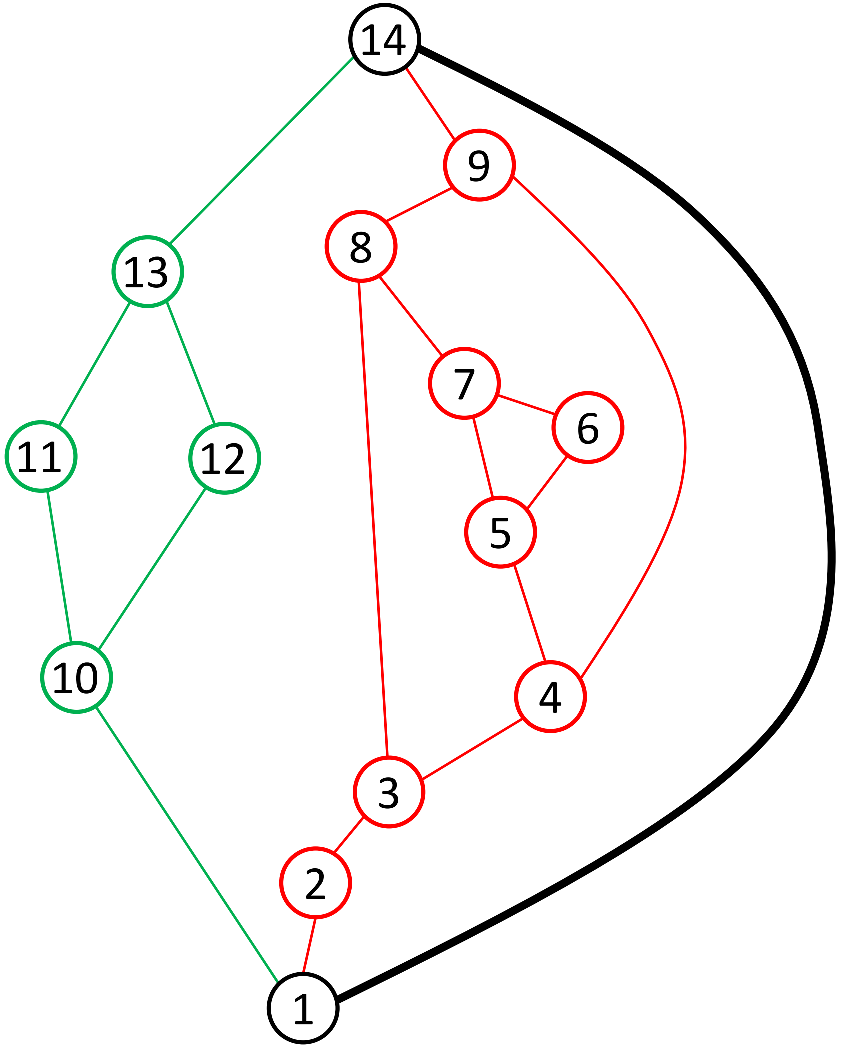
Changing the embedding

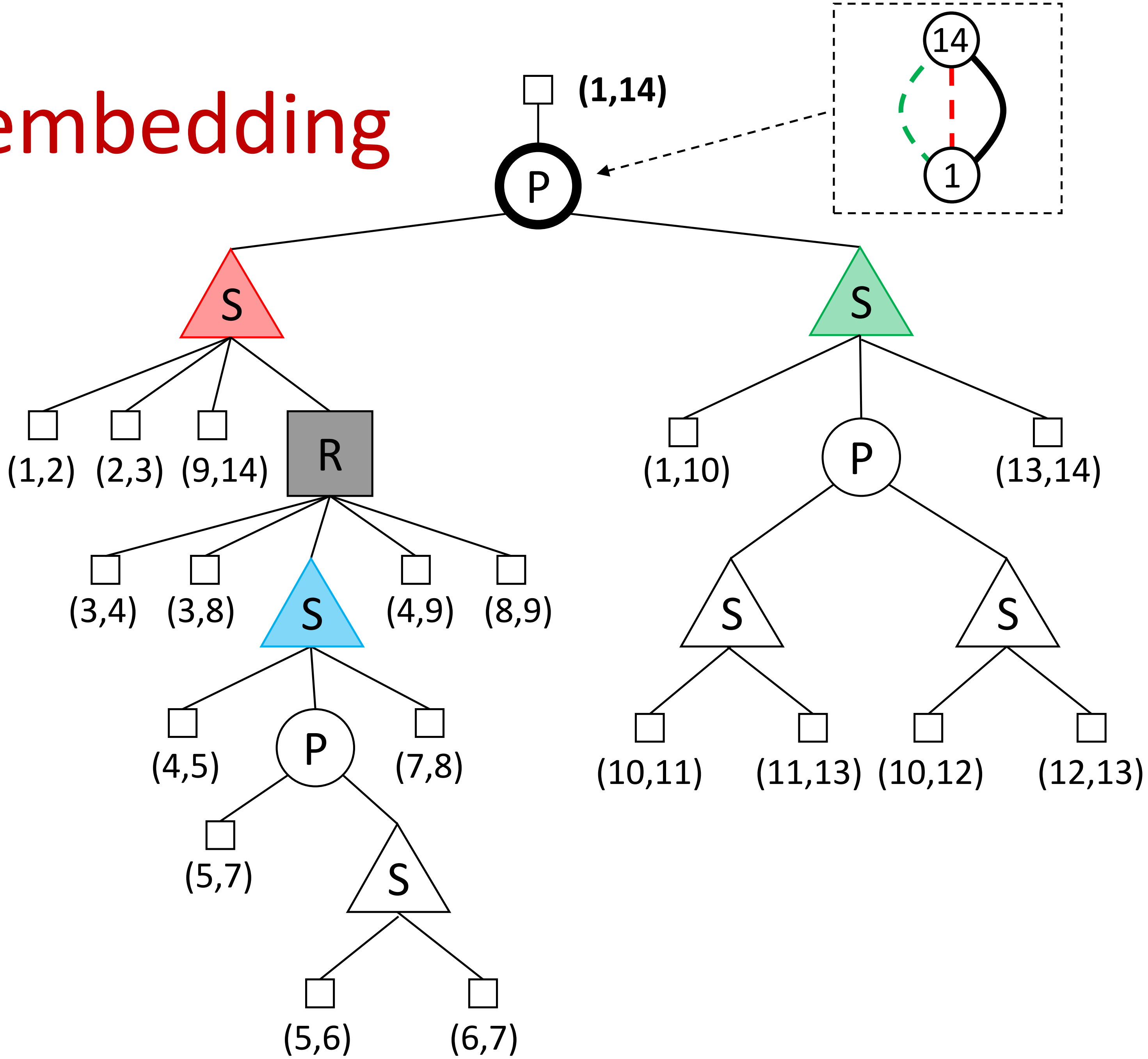




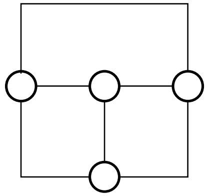


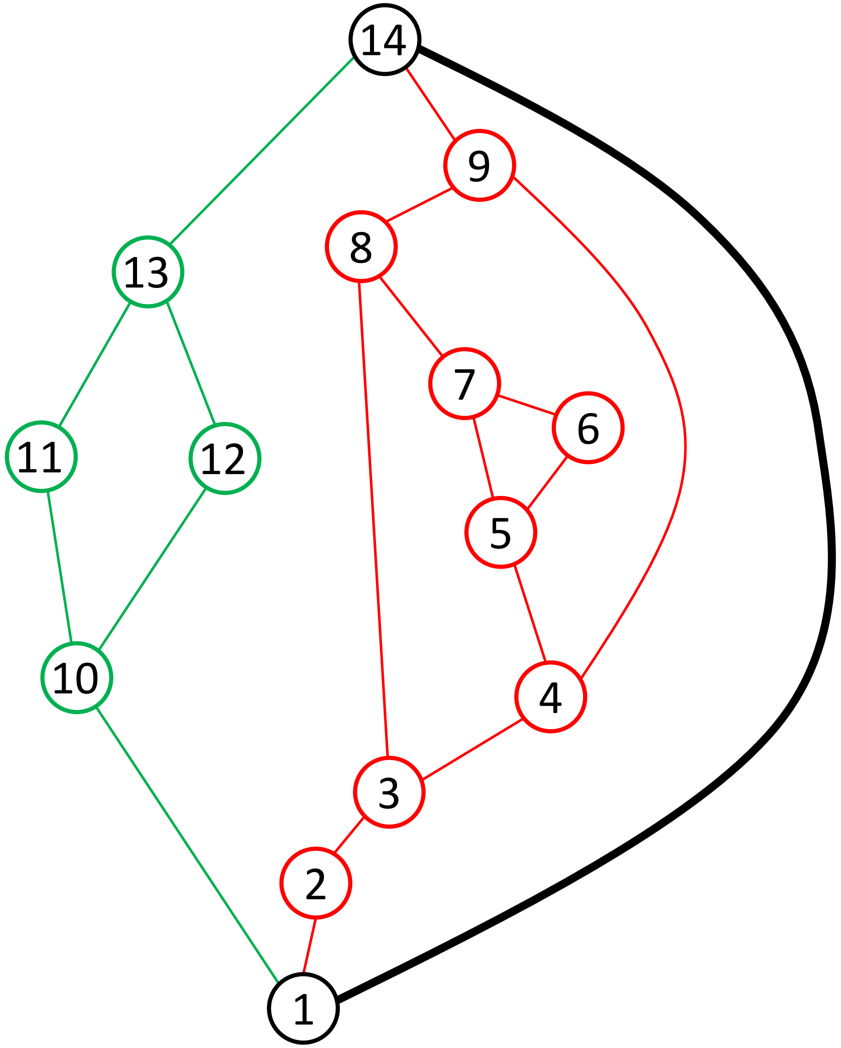
Changing the embedding

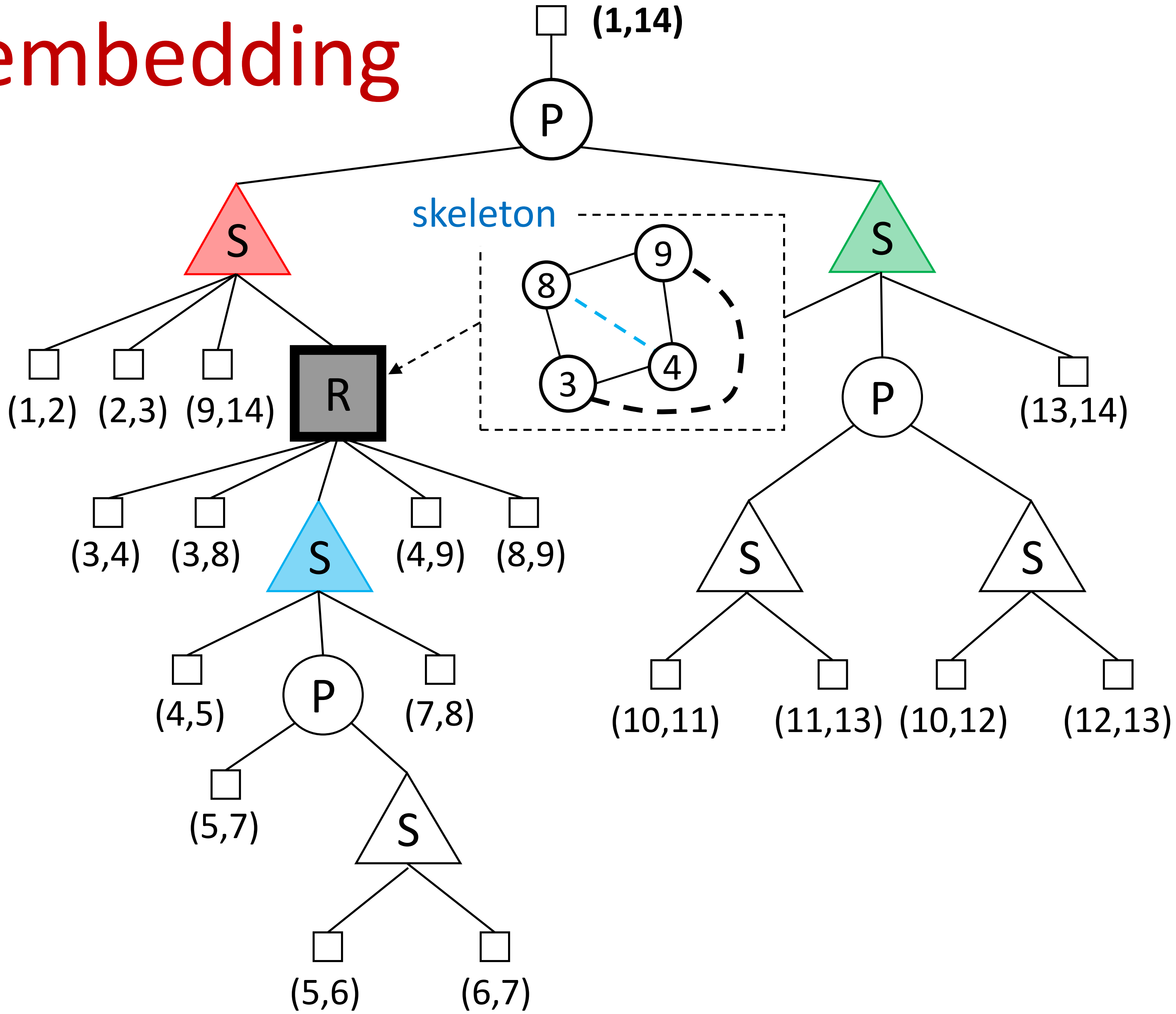




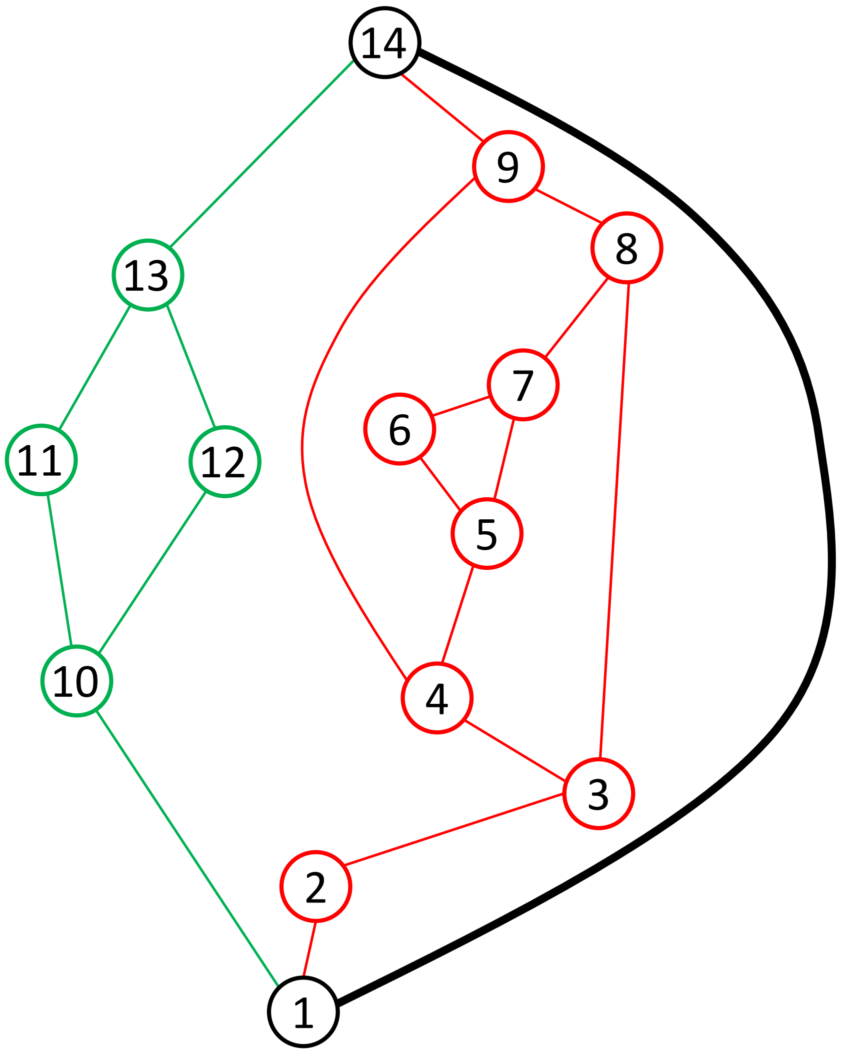
Changing the embedding

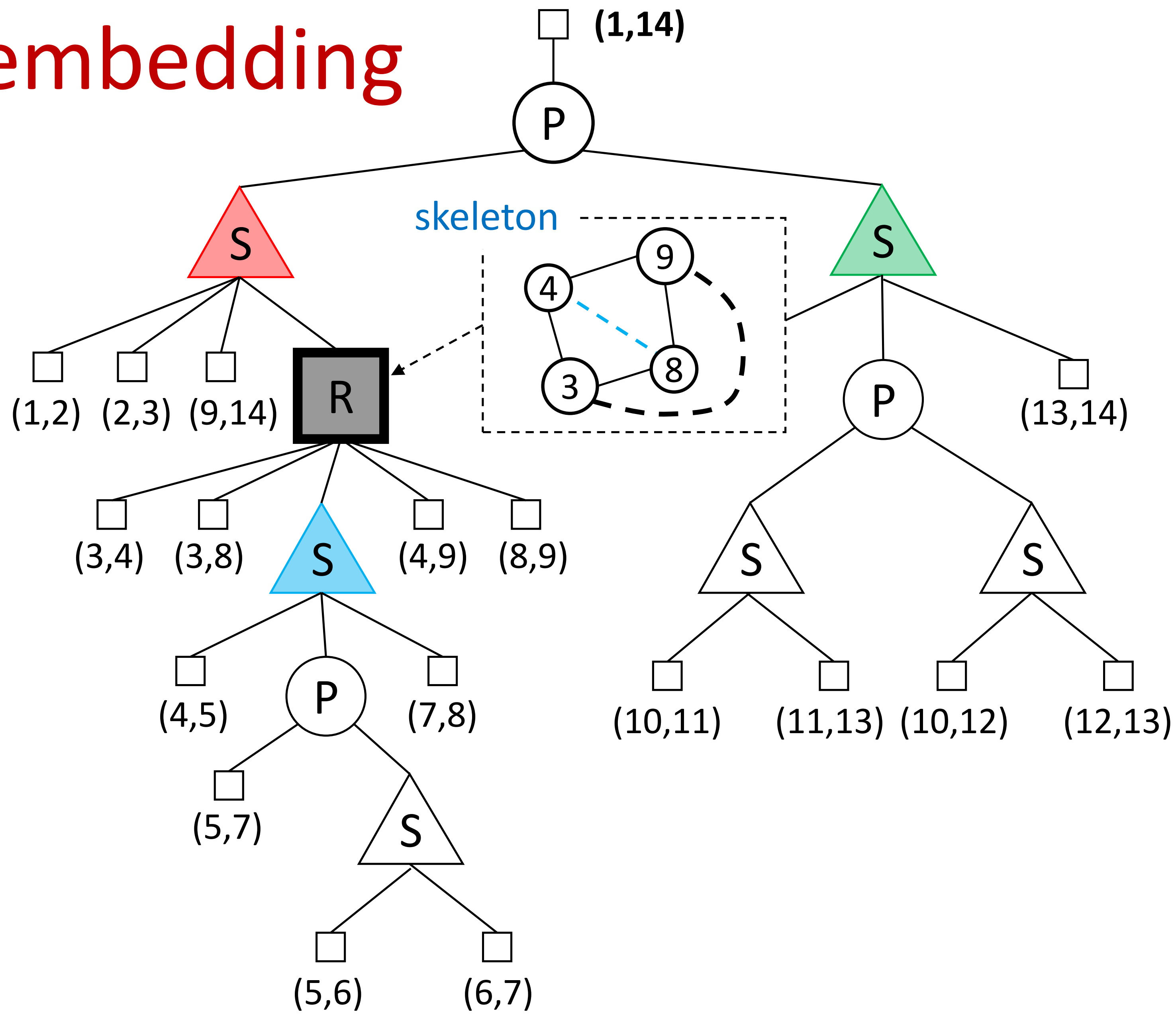


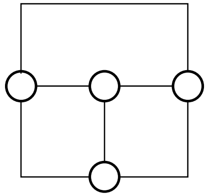




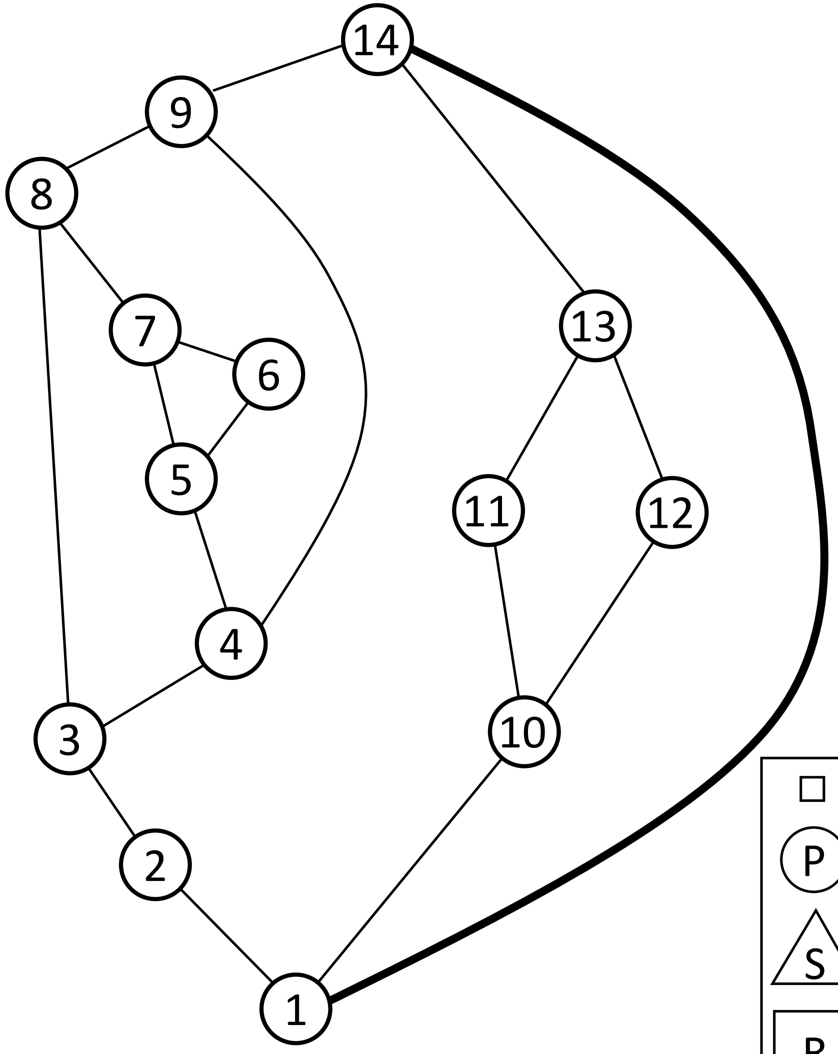
Changing the embedding







SPQR-trees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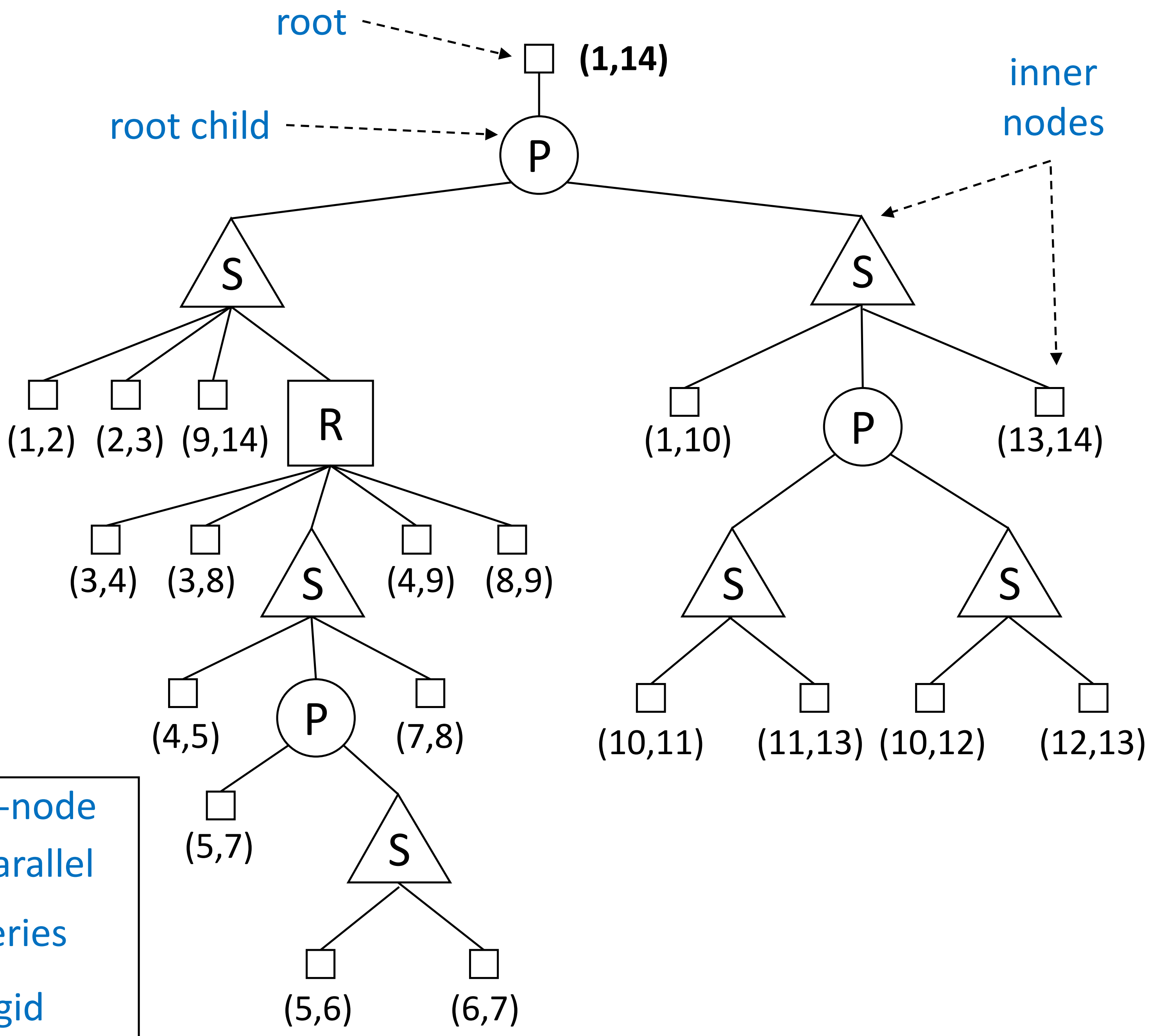
root
(1,14)
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nodes
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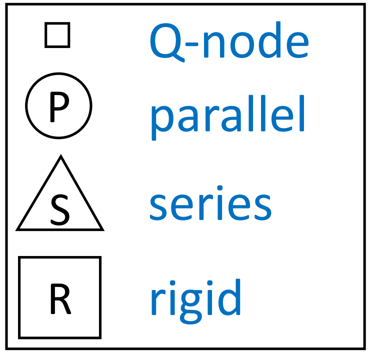
Q-node
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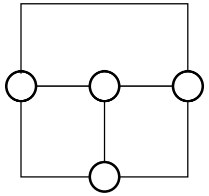
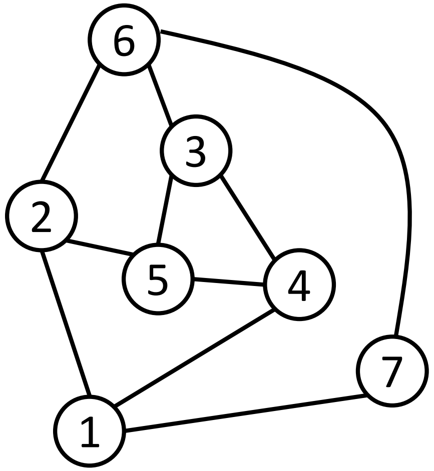


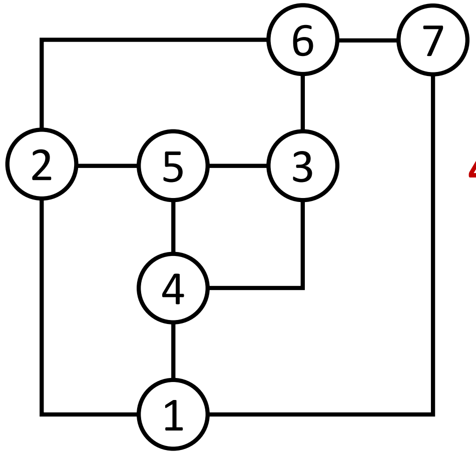
Bend-minimum orthogonal drawings of planar 3-graphs

The problem

Problem: planar **3-graph** ⟹ planar **bend-minimum** orthogonal drawing

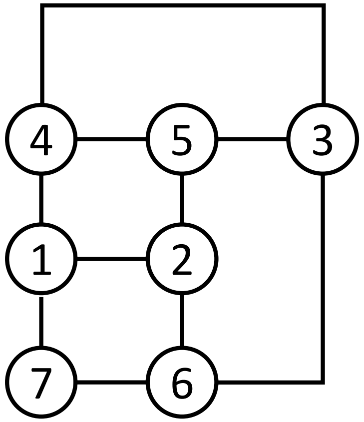


plane 3-graph



4 bends

bend-min orthogonal drawing
(**fixed embedding**)



3 bends

bend-min orthogonal drawing
(**variable embedding**)

History reminder

Bend-min orthogonal drawings: **fixed embedding**

- plane 4-graphs
  - $O(n^2 \log n)$ [*Tamassia* (1987)]
  - $O(n^{7/4} \sqrt{\log n})$ [*Garg, Tamassia* (2001)]
  - $O(n^{1.5})$ [*Cornelsen, Karrenbauer* (2011)]

based on
min-cost flow

- plane 3-graphs

  $O(n)$ [*Rahman, Nishizeki* (2002)]

not based on
flow techniques

History reminder

Bend-min orthogonal drawings: **variable embedding**

- planar 4-graphs: NP-hard [Garg, Tamassia (2001)]
- planar 3-graphs

1998

$O(n^5 \log n)$
Di Battista-Liotta-Vargiu

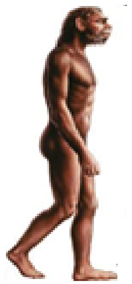
2011

$O(n^{4.5})$
consequence of Cornelsen-Karrenbauer

2017

$O(n^{2.43} \log^k n)$
Chang and Yen

2018



$O(n^2)$
next slides

?

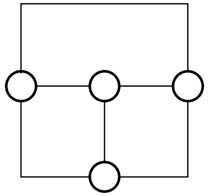
Can we do better?

Result

Theorem. Let G be an n-vertex (simple) planar 3-graph. There exists an $O(n^2)$-time algorithm that computes a bend-minimum orthogonal drawing of G, with at most two bends per edge.

P. S. the algorithm takes $O(n)$ time if we require that a prescribed edge of G is on the external face

W. Didimo, G. Liotta, M. Patrignani: Bend-Minimum Orthogonal Drawings in Quadratic Time. Graph Drawing 2018: 481-494



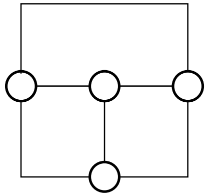
General strategy for biconnected graphs

input: G biconnected planar 3-graph with n vertices

output: bend-min orthogonal drawing Γ of G

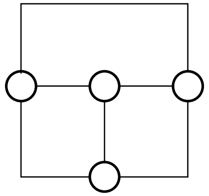
- for each edge *e* of G
  - $\Gamma_e \leftarrow$ bend-min orthogonal drawing of G with *e* on the external face
- return $\Gamma \leftarrow$ min-bends $\{\Gamma_e\}$

Γ_e is computed in O(n) time



Strategy for the linear-time algorithm

- Incremental construction of Γ_e
  1. bottom-up visit of the SPQR-tree + *orthogonal spirality*
     - similar to [*G. Di Battista, G. Liotta, F. Vargiu*: Spirality and optimal orthogonal drawings, SIAM J. Comput., 27 (1998)]
  2. new properties of bend-min orthogonal drawings of planar 3-graphs
  3. non-flow based computation of bend-min orthogonal drawings for the rigid components



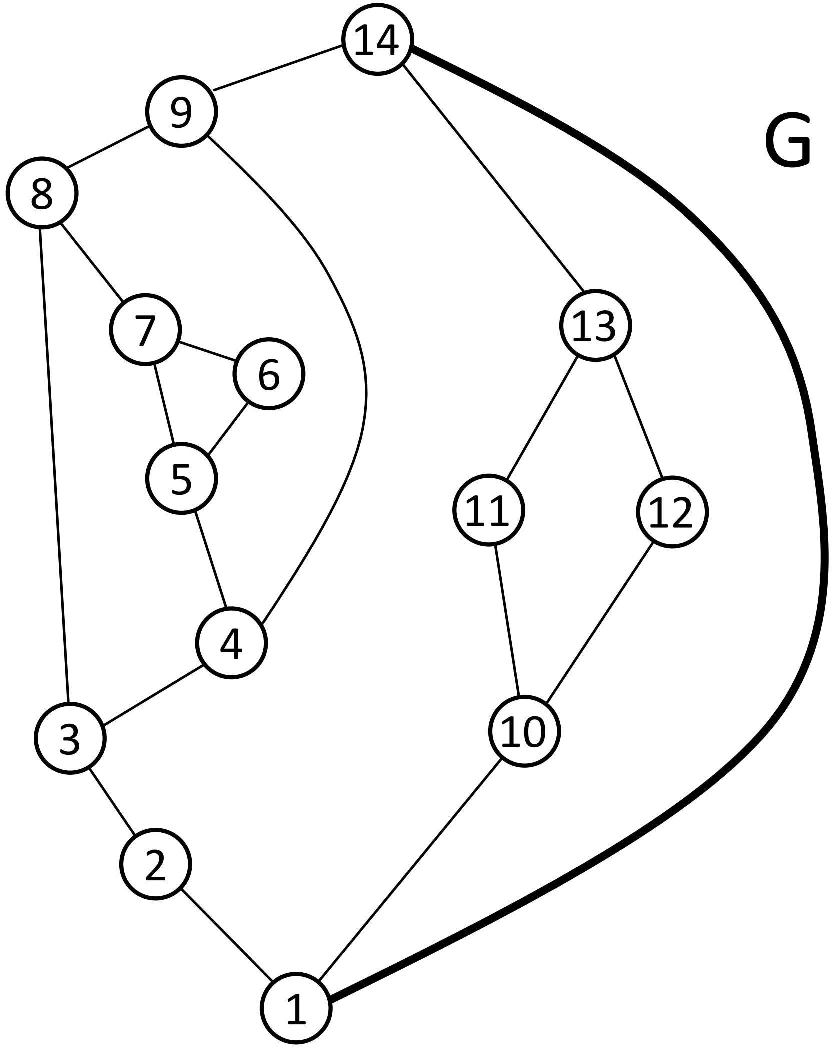
Orthogonal representations: reminder

orthogonal representation = equivalence class of orthogonal drawings with the same vertex angles and the same sequence of bends along the edges

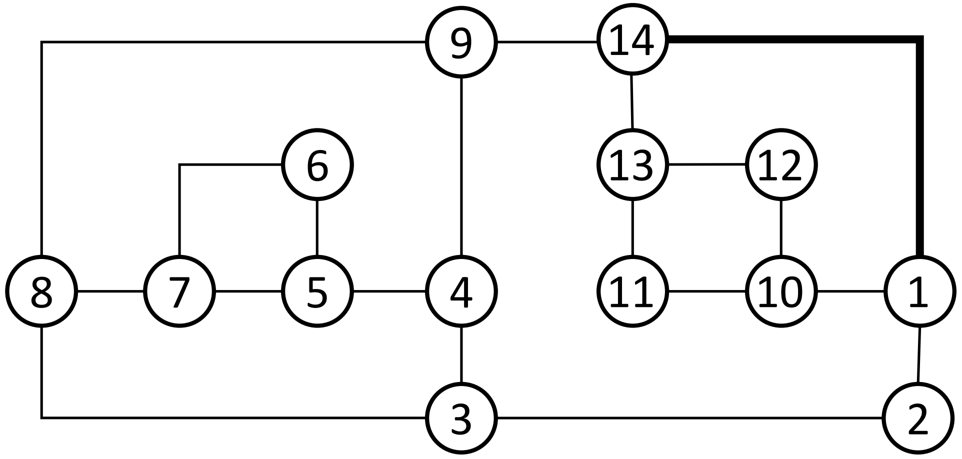
- a drawing of an orthogonal representation can be computed in linear time

orthogonal component = orthogonal representation H_μ of a component G_μ

Orthogonal components: example

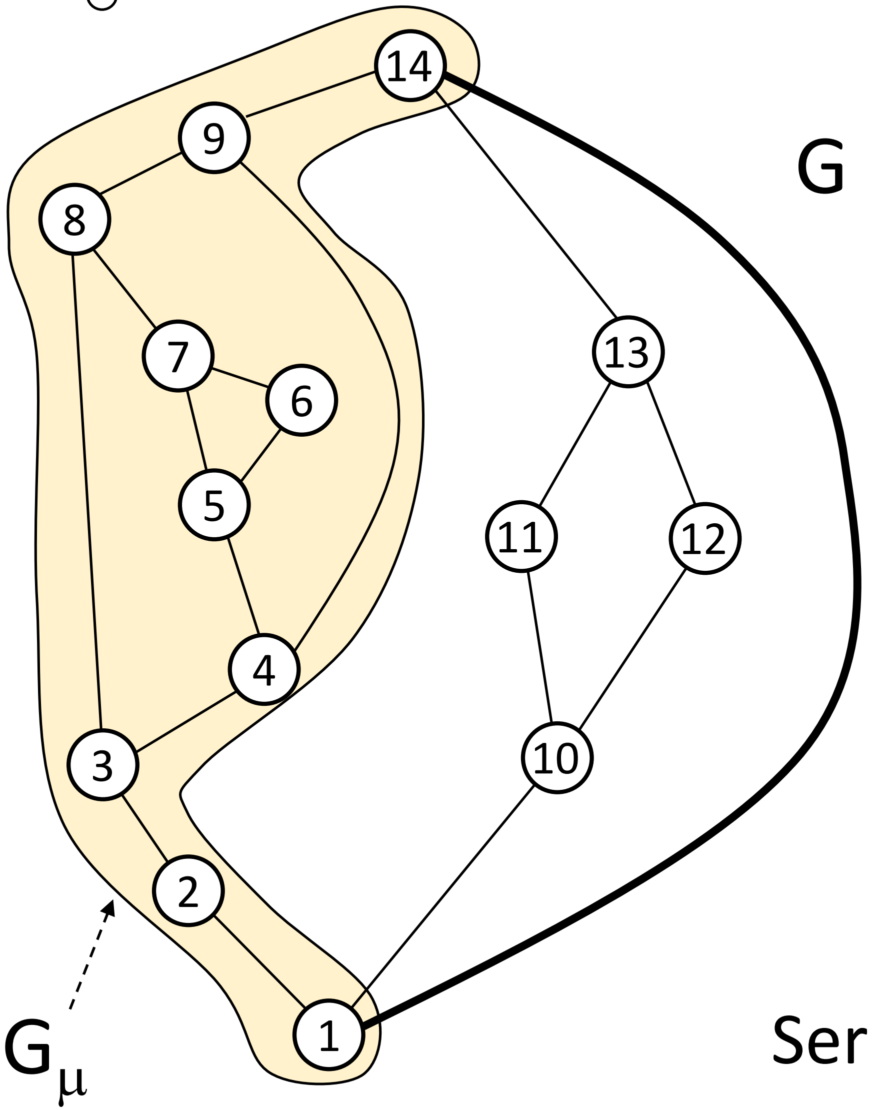


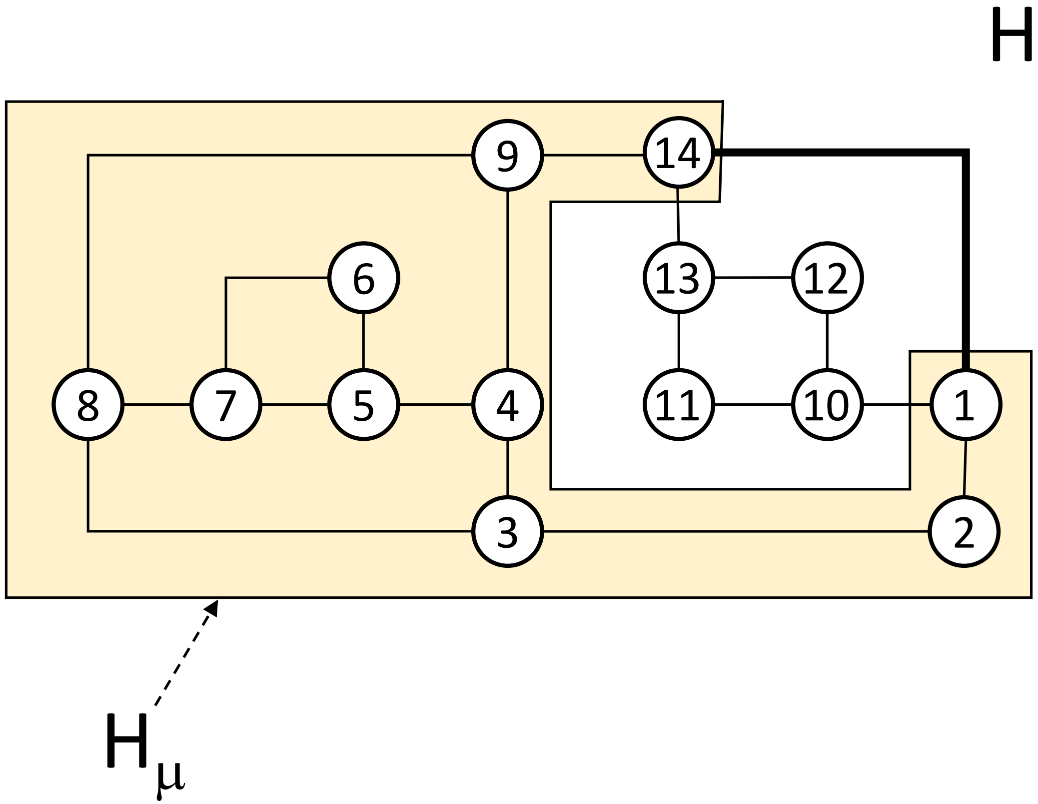
G



H

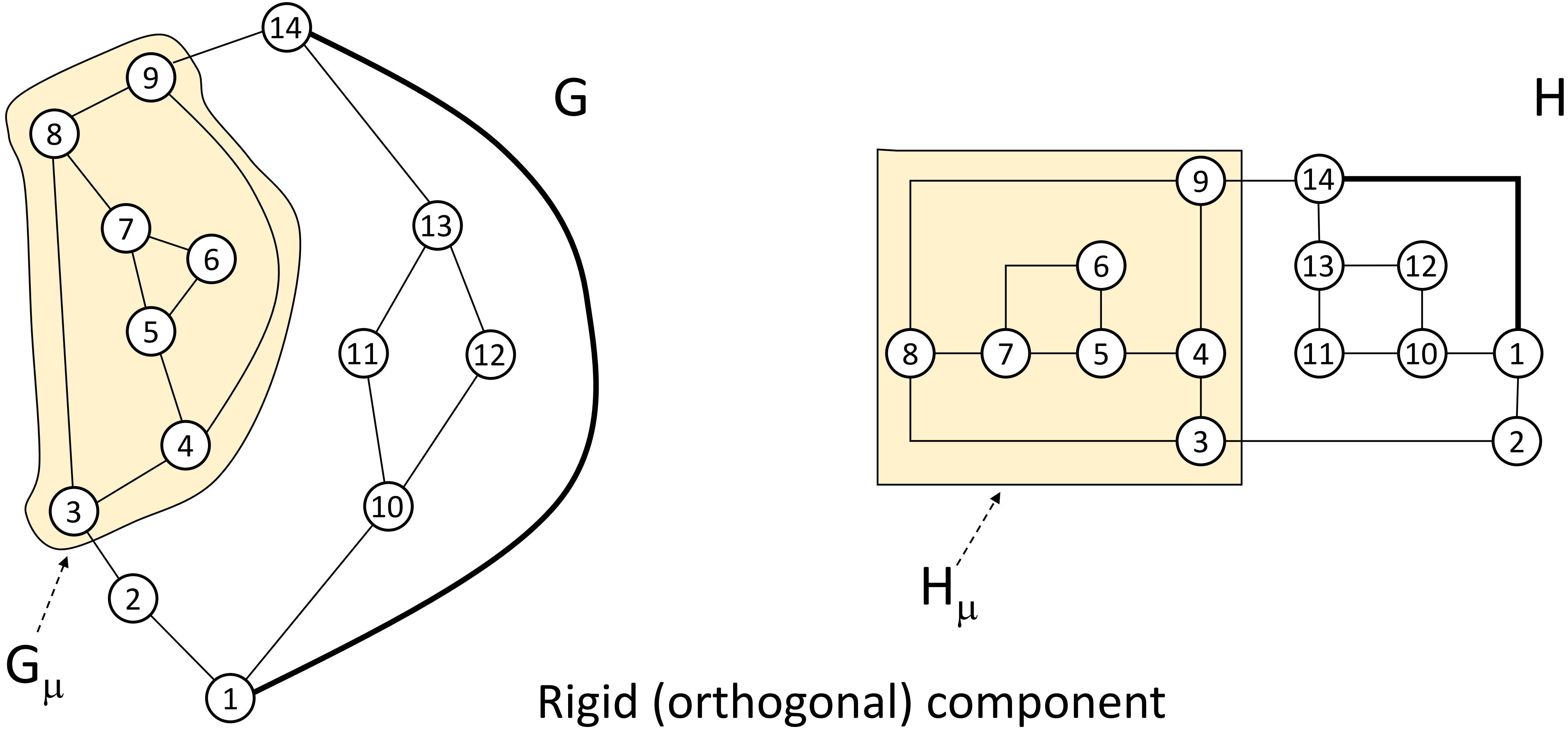
Orthogonal components: examples





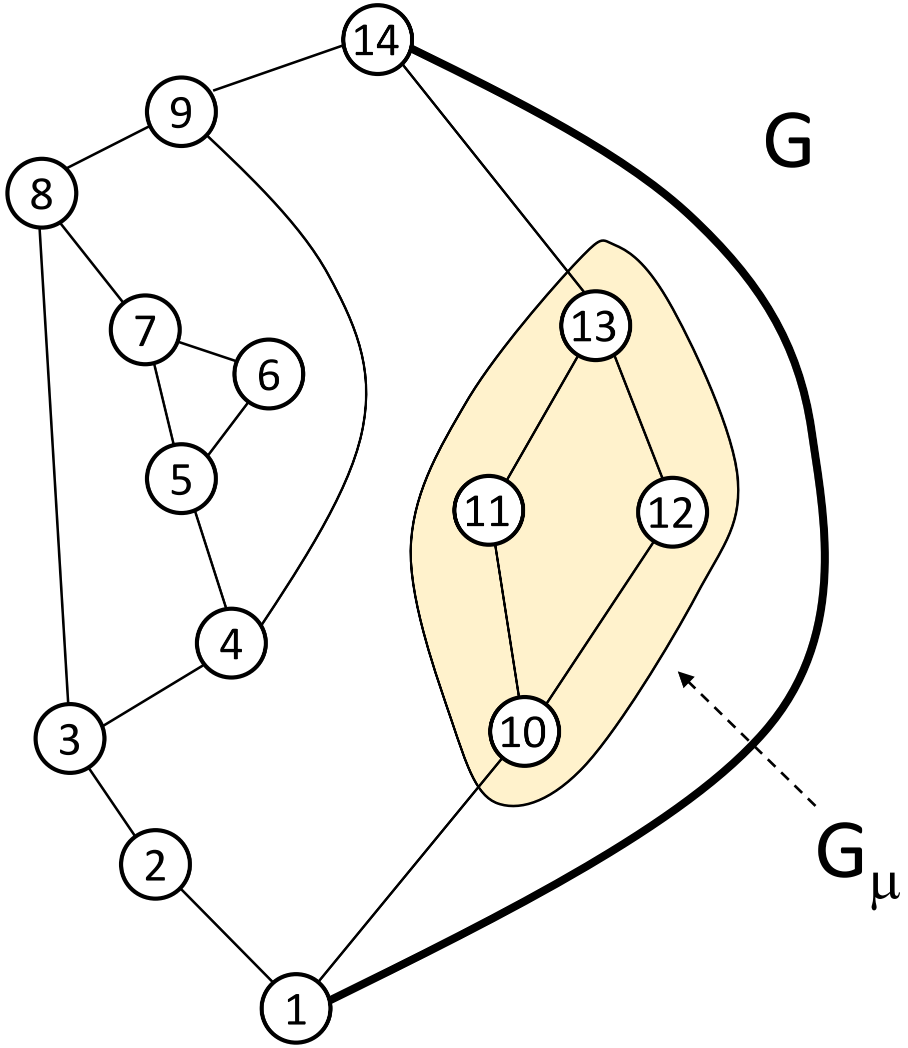
Series (orthogonal) component

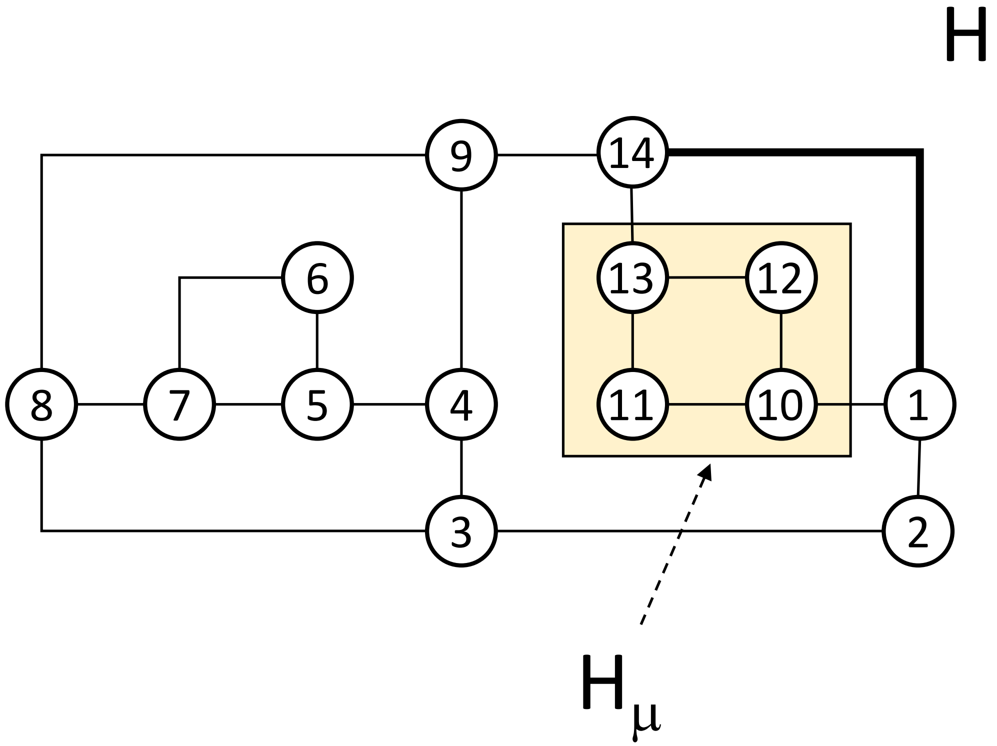
Orthogonal components: examples



Rigid (orthogonal) component

Orthogonal components: examples

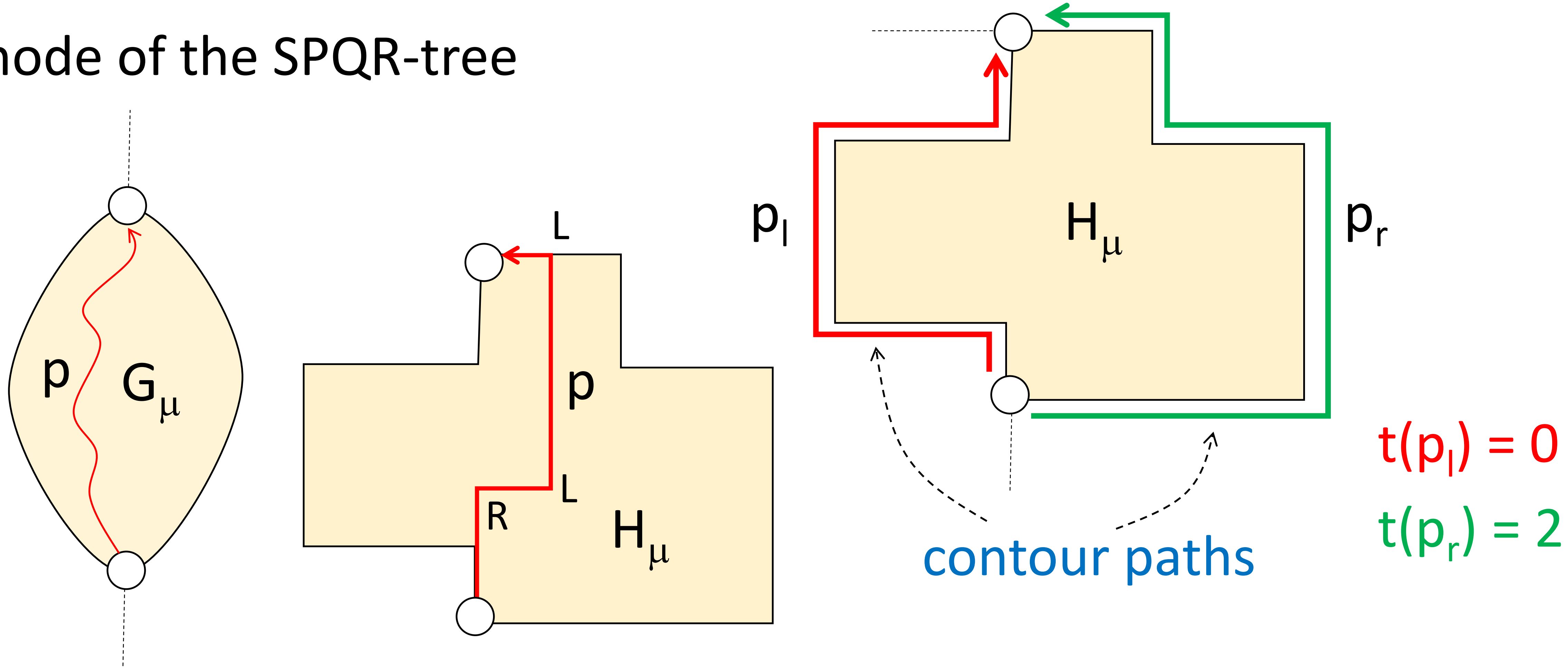




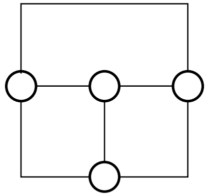
Parallel (orthogonal) component

Turn number and contour paths

μ = node of the SPQR-tree



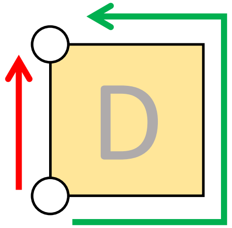
t(p) = turn number = |#left turns – # right turns| (along p)



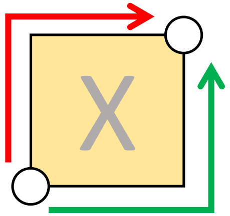
P- and R-components: Representative shapes

μ = P-node or R-node

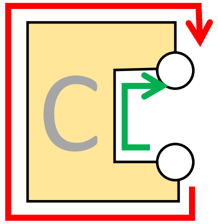
H_μ is D-shaped $\Leftrightarrow$ $t(p_l) = 0$ and $t(p_r) = 2$ or vice versa



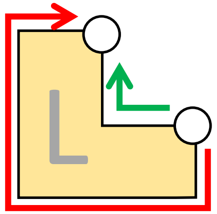
H_μ is X-shaped $\Leftrightarrow$ $t(p_l) = t(p_r) = 1$

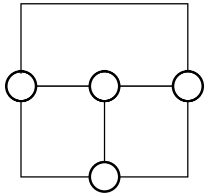


H_μ is C-shaped $\Leftrightarrow$ $t(p_l) = 4$ and $t(p_r) = 2$ or vice versa



H_μ is L-shaped $\Leftrightarrow$ $t(p_l) = 3$ and $t(p_r) = 1$ or vice versa

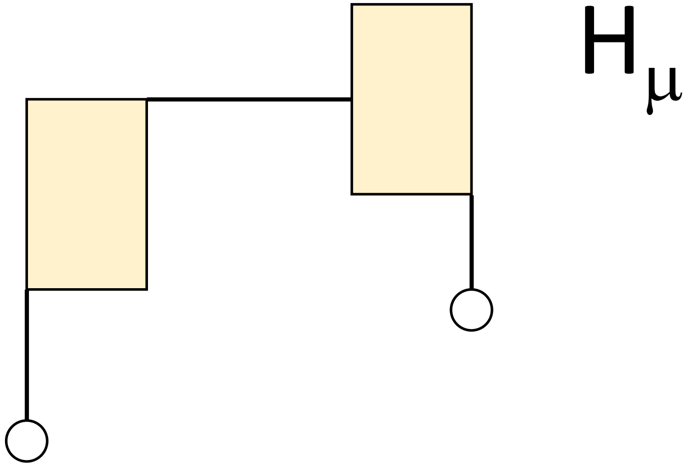


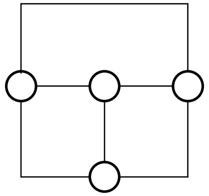


Inner S-components: spirality

μ = inner S-node

Lemma. All paths between the poles of an orthogonal component H_μ have the same turn number





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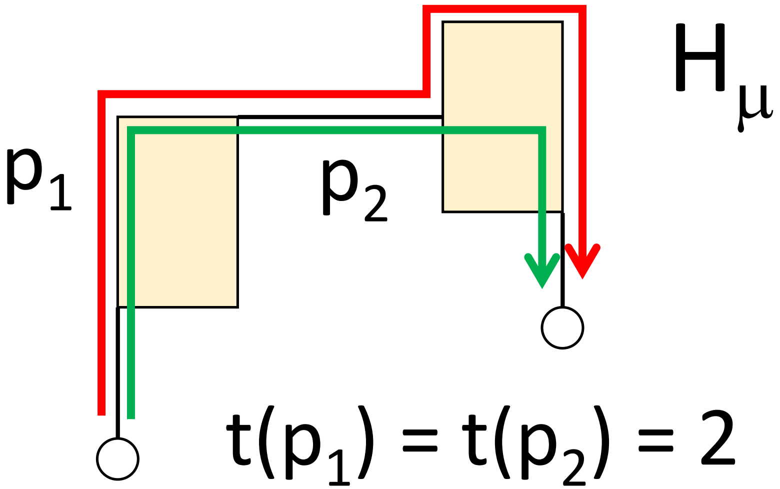
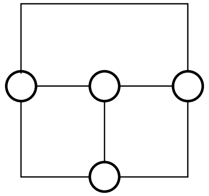


$t(p_1) = t(p_2) = 2$

$t(p) = k$

H_μ is k-spiral

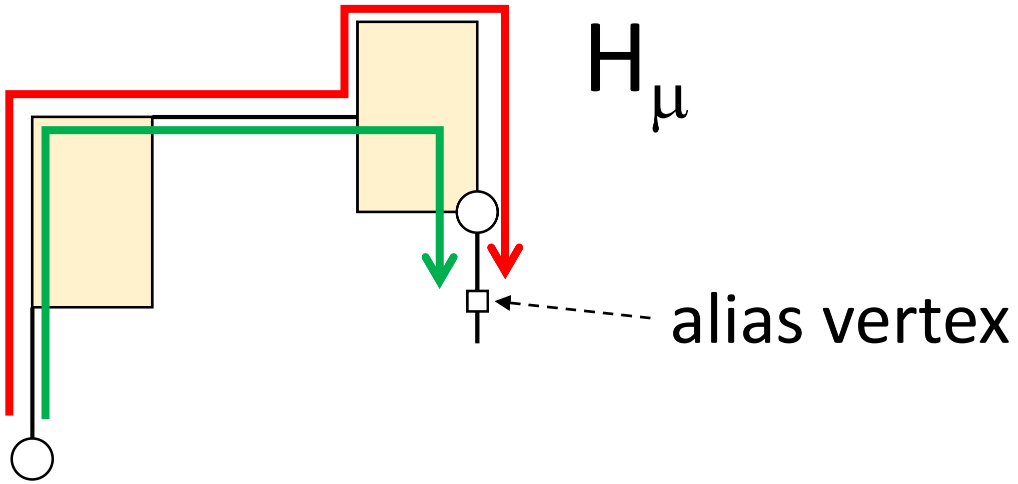
H_μ has spirality k

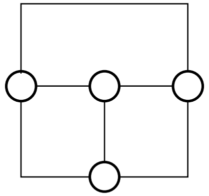


Root child S-components: spirality

μ = root child S-node

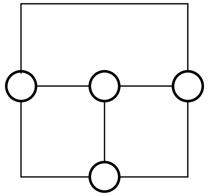
The definition of k-spiral and the lemma are extended by considering an external alias vertex in place of a pole with in-degree 2





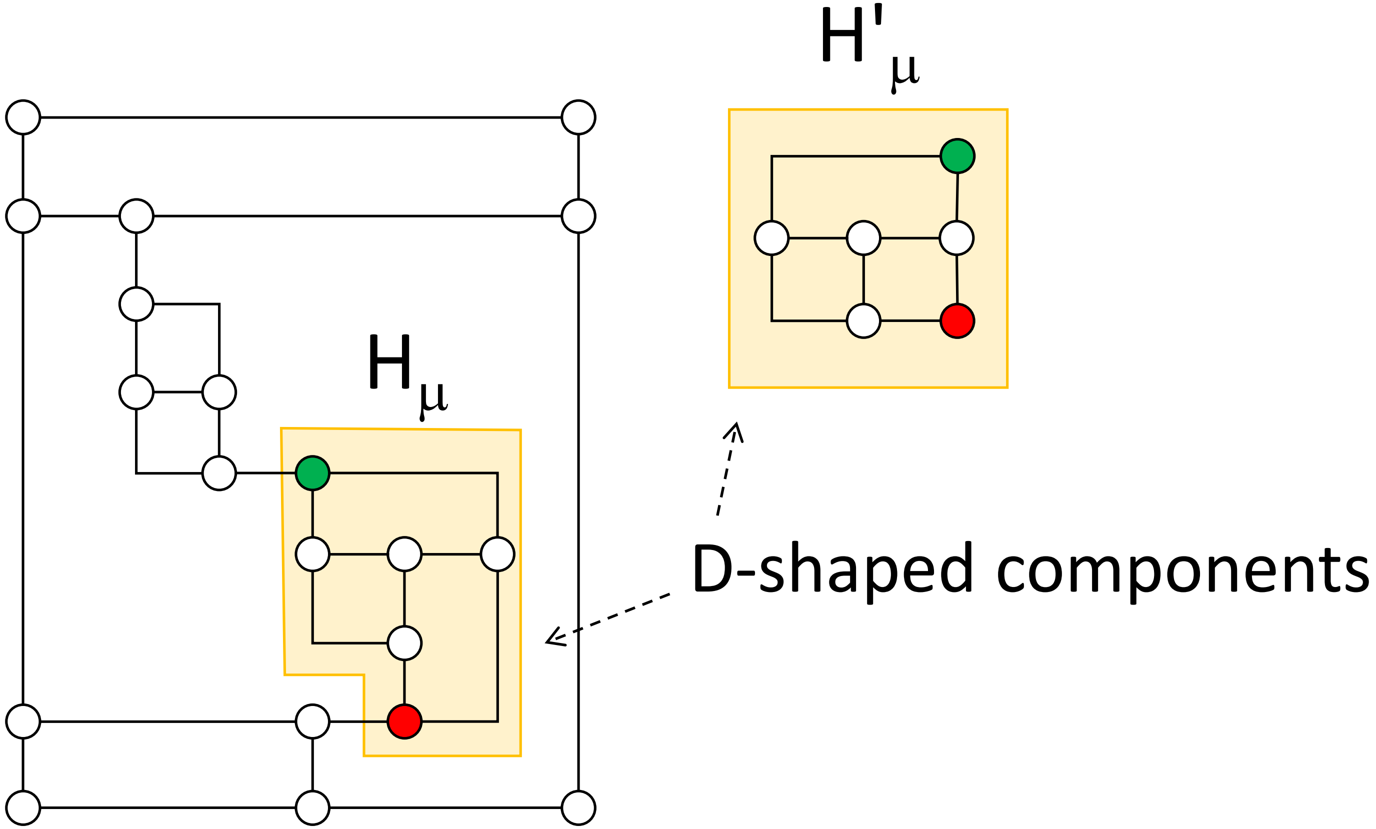
Equivalent orthogonal components

- H_μ and H'_μ = two distinct orthogonal representations of G_μ
- H_μ and H'_μ are equivalent if:
  - μ is a P- or an R-node and H_μ, H'_μ have the same representative shape
  - μ is an S-node and H_μ, H'_μ have the same spirality



Equivalent orthogonal components

Theorem (substitution). Equivalent orthogonal components are interchangeable



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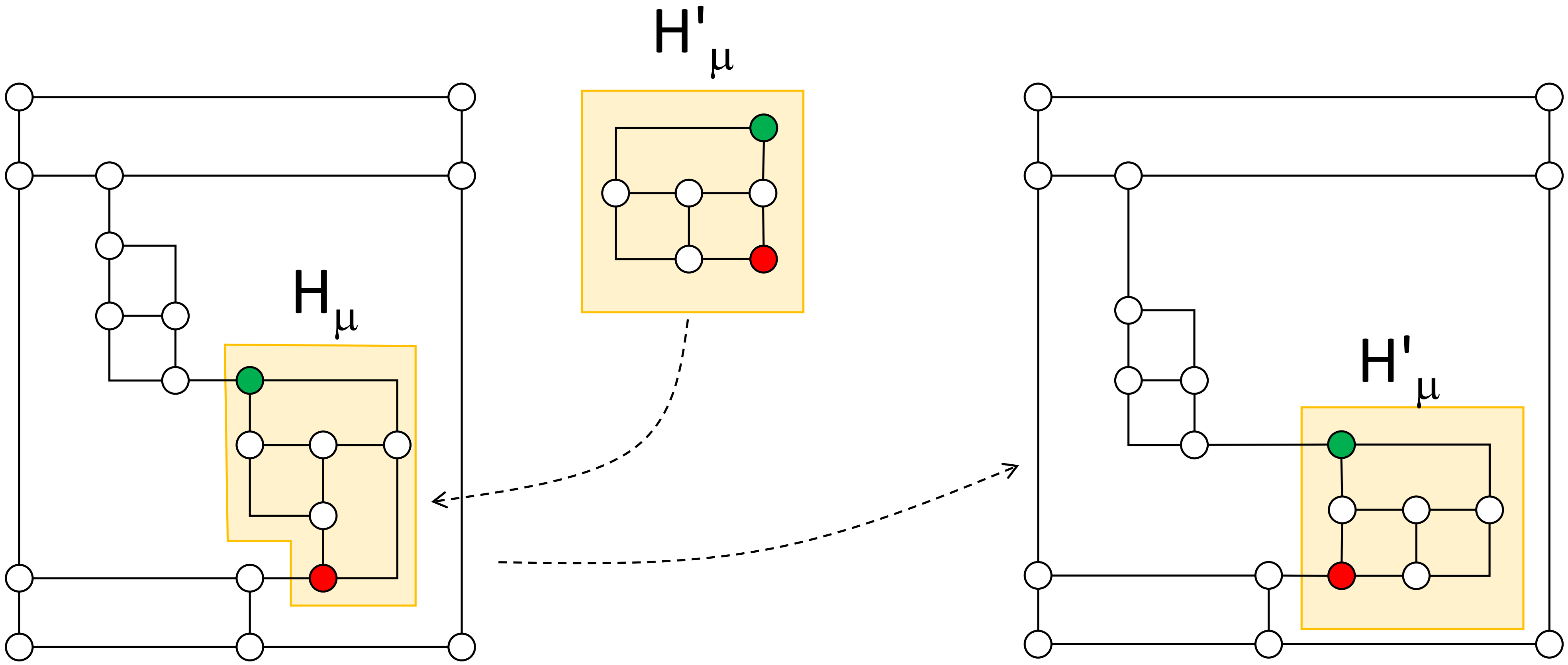

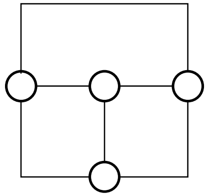

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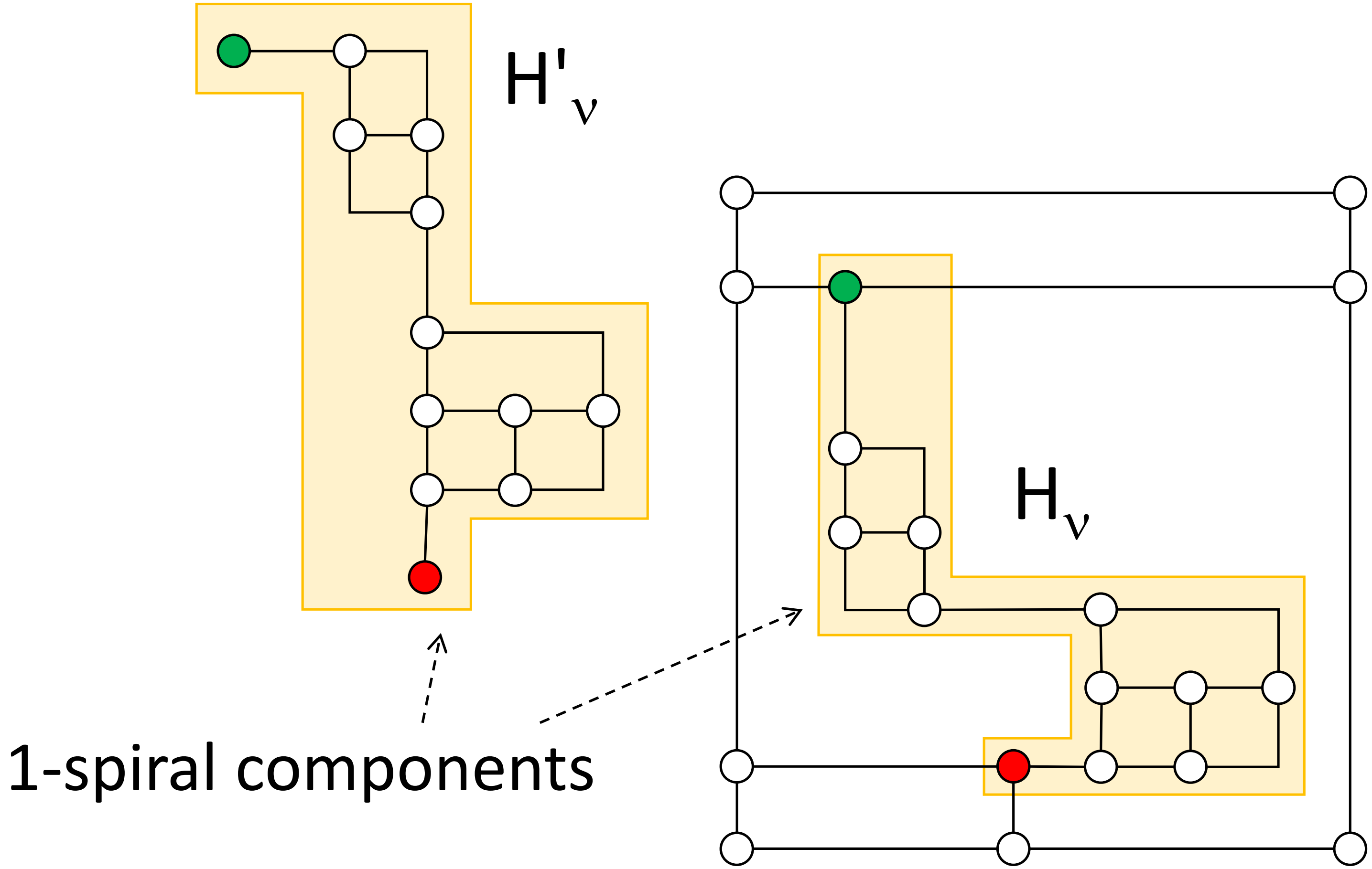

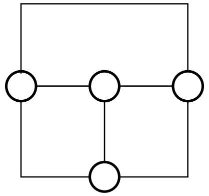

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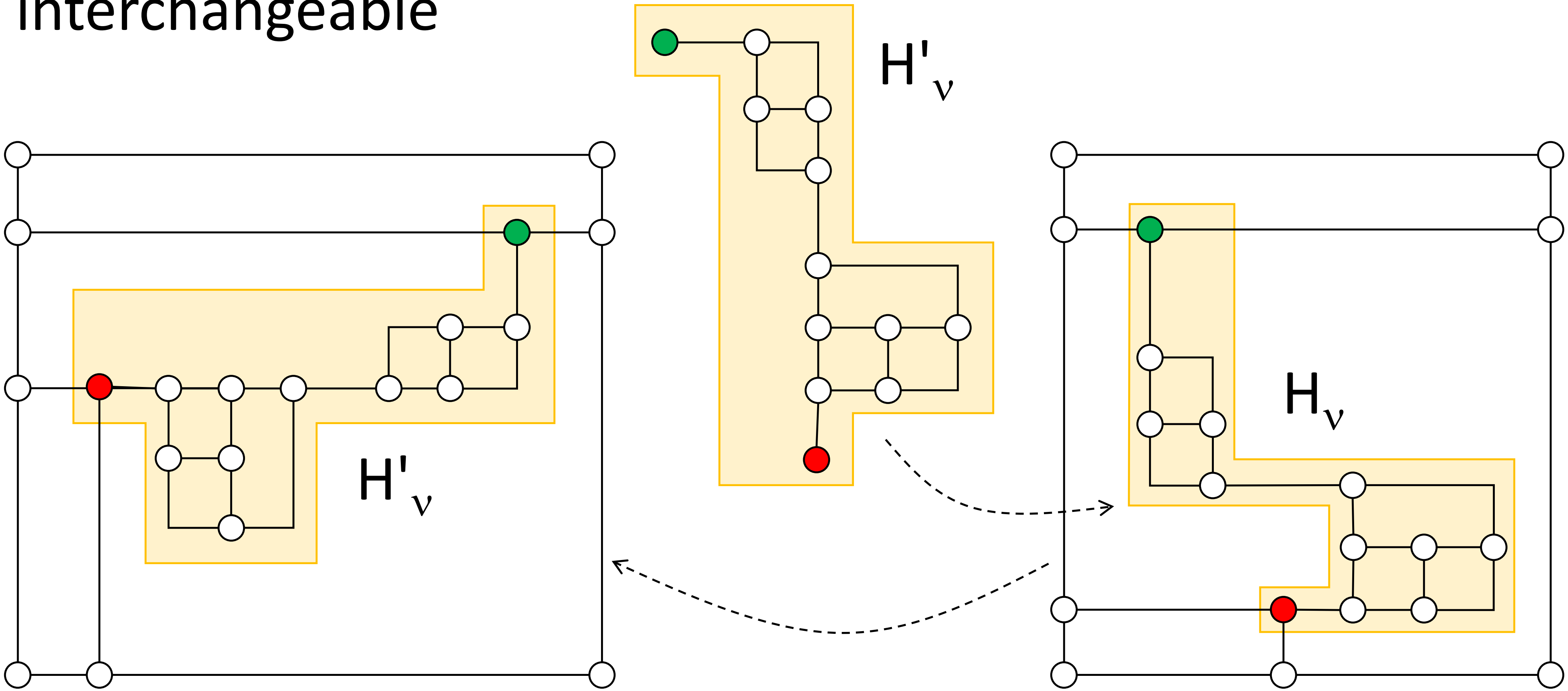


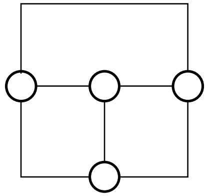
Key lemma

Key-Lemma. Every biconnected planar 3-graph with a given edge *e* admits a bend-min orthogonal representation with *e* on the external face such that:

O1. every edge has at most two bends

O2. every inner P- or R-component is D- or X-shaped; if the root child is a P- or an R-component, it is either D-, C-, or L-shaped

O3. every S-component has spirality at most 4



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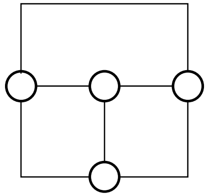
Proof ingredients: partially based on a characterization of no-bend orthogonal representations [Rahman, Nishizeki, Naznin 2003]



Key lemma: Consequence

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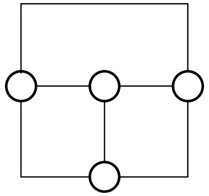
Consequence: we can restrict our algorithm to search for a bend-min representation that satisfies O1, O2, and O3.



\begin{Characterization of no-bend drawings}

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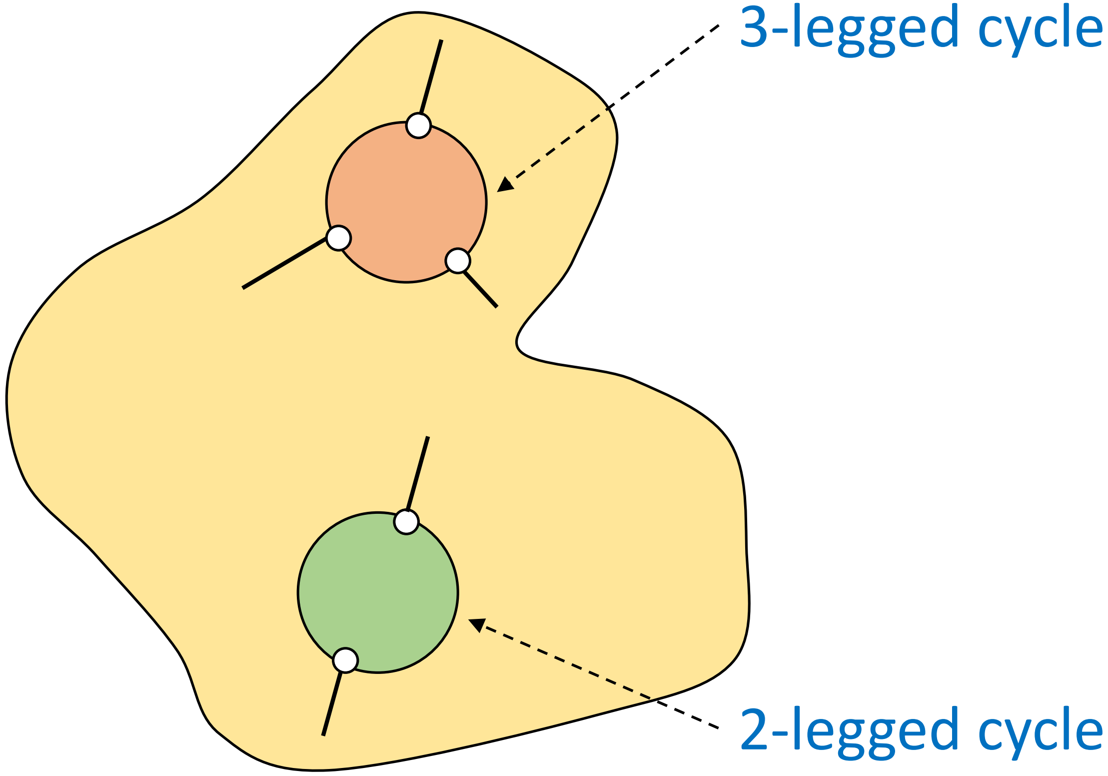
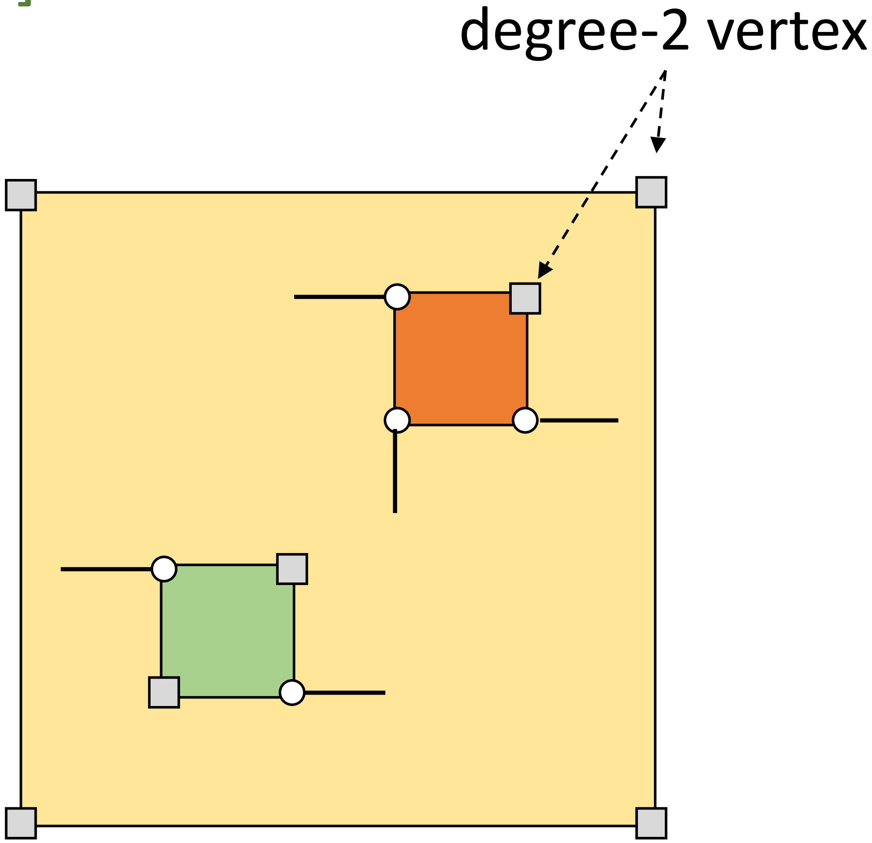
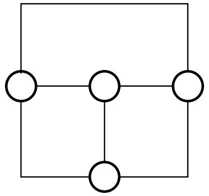
[Rahman, Nishizeki, Naznin, JGAA 2003] = [RNN'03]



biconnected plane 3-graph



no-bend orthogonal drawing of G



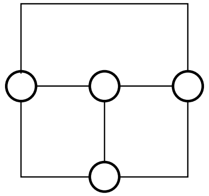
Characterization of no-bend drawings

Theorem [RNN'03]. Let G be a biconnected plane 3-graph. G admits a **no-bend** orthogonal drawing ⇔

(i) the external cycle of G has at least 4 degree-2 vertices

(ii) each k-legged cycle of G has at least (4-k) degree-2 vertices

Definition: we call **bad** a 2-legged or a 3-legged cycle that does not satisfy (ii)



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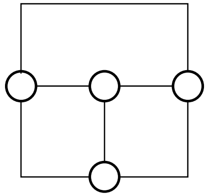

Key-Lemma: O1

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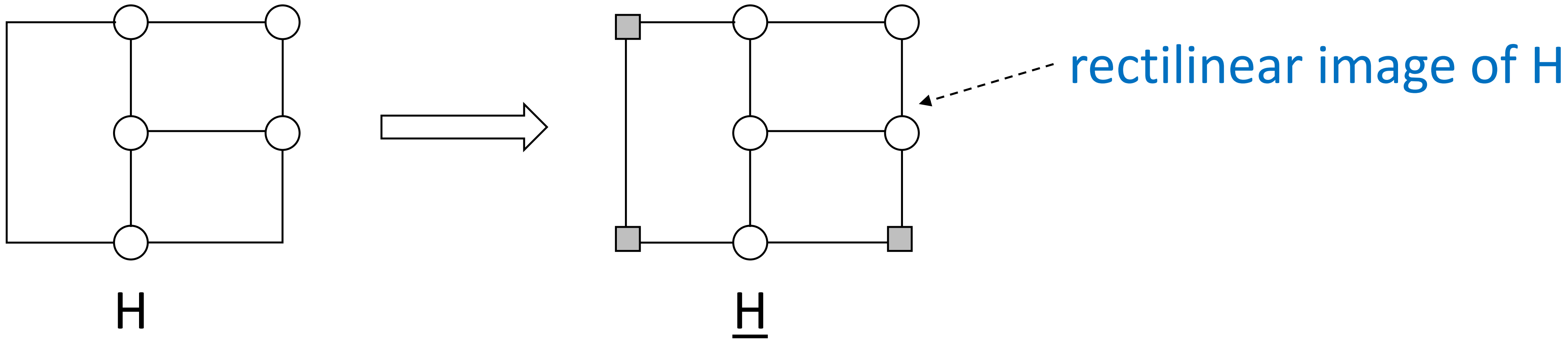
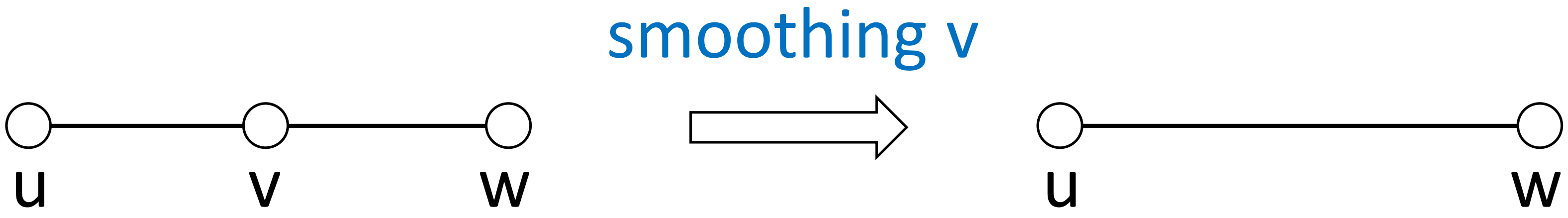
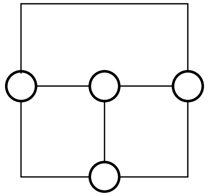
Key-Lemma: O1

Proof of O1 (at most two bends per edge)

Notation







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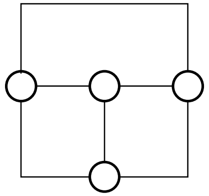
Proof of O1 (at most two bends per edge)

- H = bend-min representation of G with *e* on the external face
- g = edge of H with (at least) three bends



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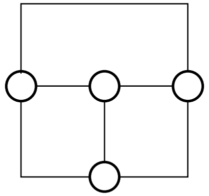

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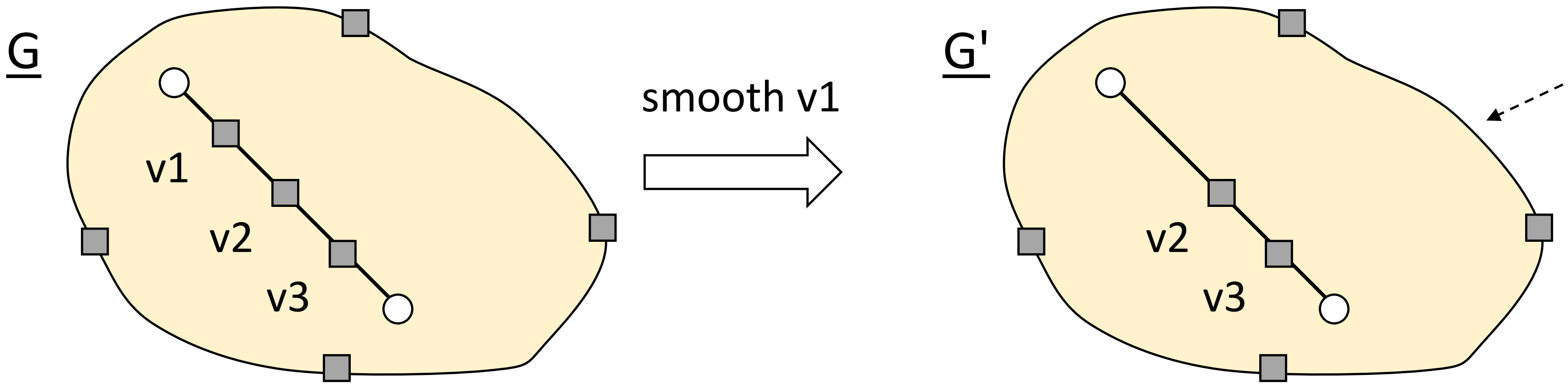

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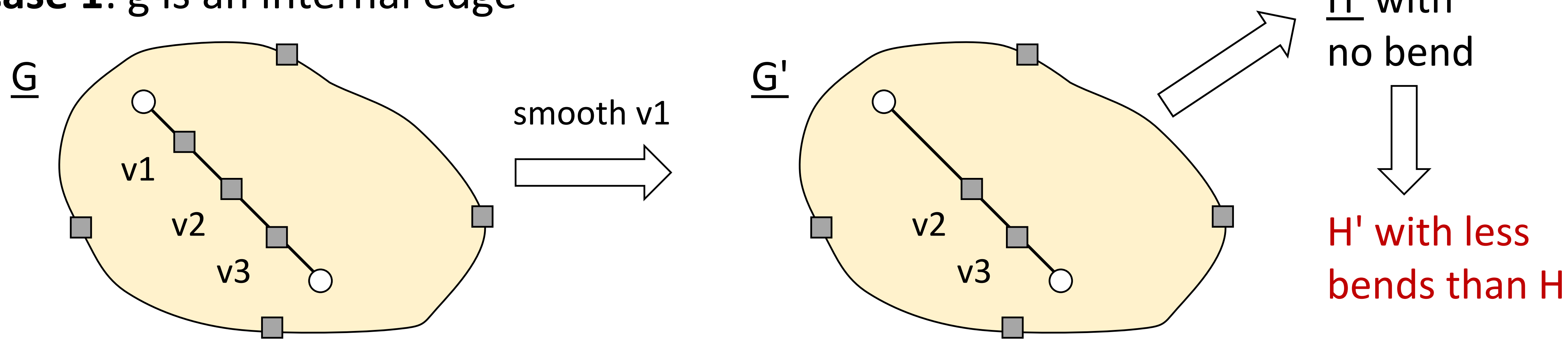

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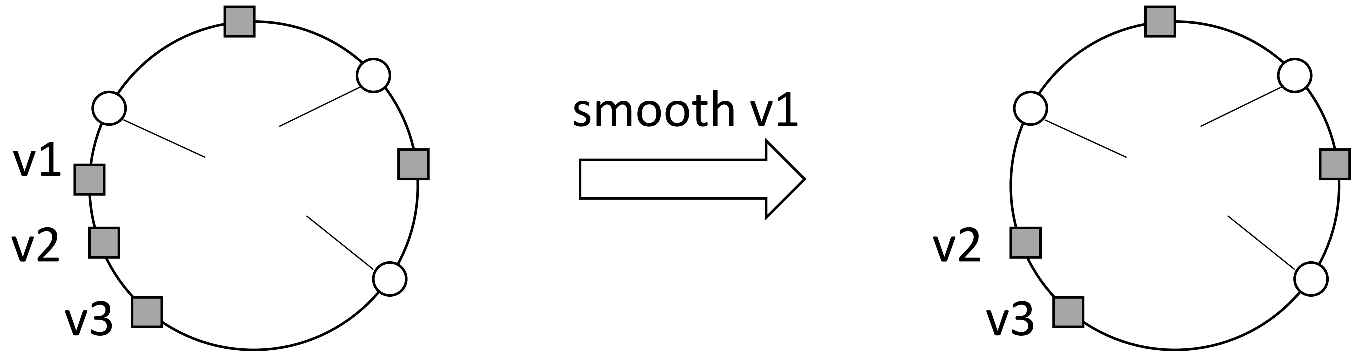
Case 2: g is an external edge (call $C_0(G)$ the external boundary of G)

- **Case 2.1.** $C_0(\underline{G})$ has more than 4 degree-2 vertices



contradiction
as before

Key-Lemma: O1

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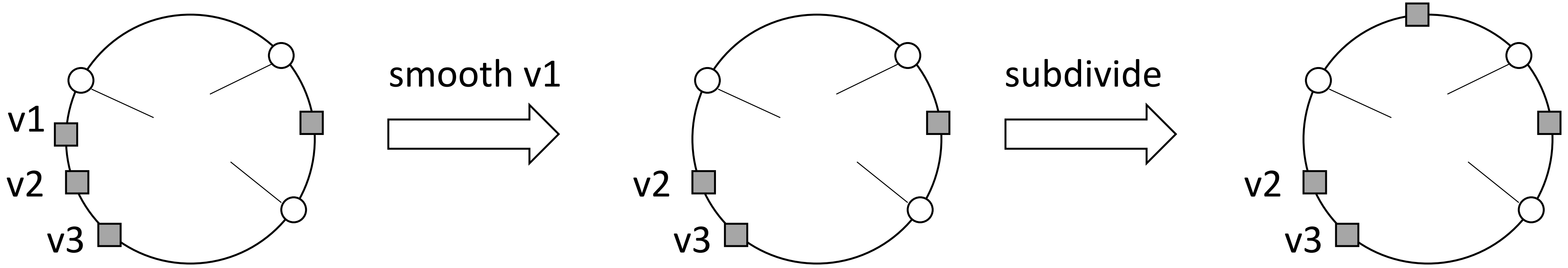
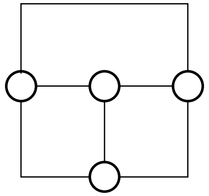
Case 2: g is an external edge (call $C_0(G)$ the external boundary of G)

- **Case 2.2.** $C_0(\underline{G})$ has exactly 4 degree-2 vertices





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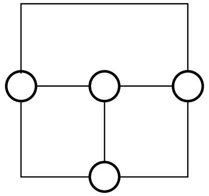

Key-Lemma: O2

Proof of O2 (inner P- or R-components are D- or X-shaped)

- H = bend-min representation of G with *e* on the external face and property O1
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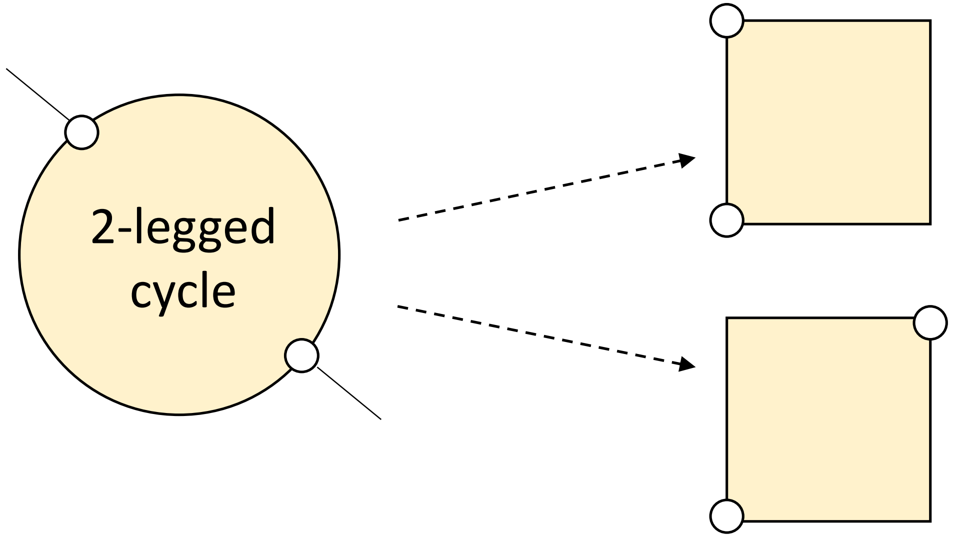
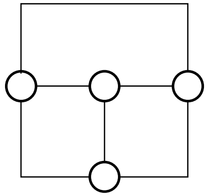
- [RNN'03] gives an algorithm that computes a no-bend representation H' of G such that every 2-legged (and 3-legged) cycle is either D-shaped or X-shaped





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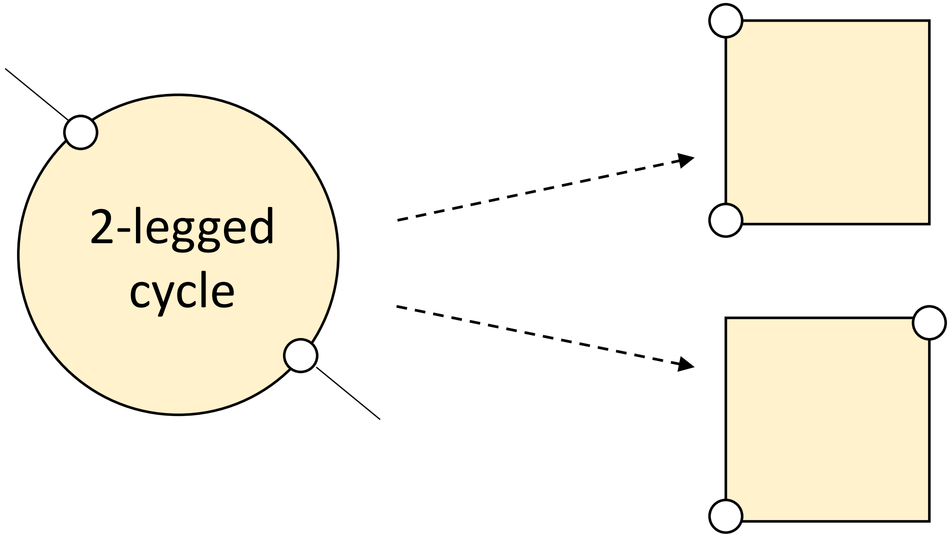
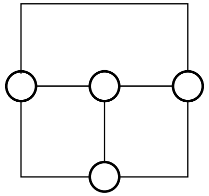


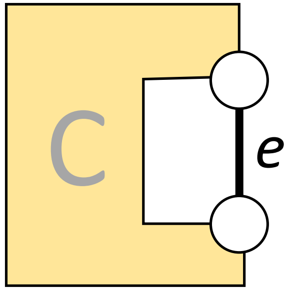
… each inner P- and R-component is a 2-legged cycle in G



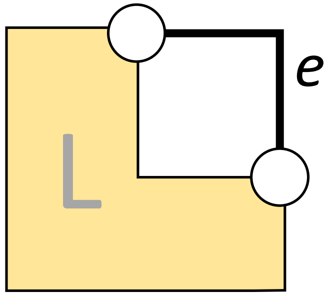
Key-Lemma: O2

Proof of O2 (root child P- or R-components are D-, C-, or L-shaped)

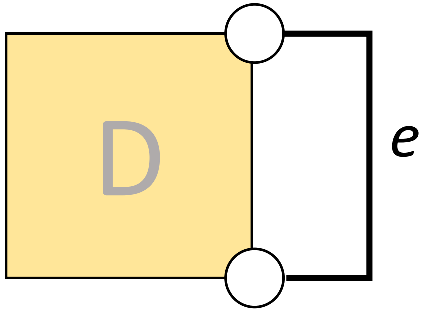
- H = bend-min representation of G with *e* on the external face and property O1



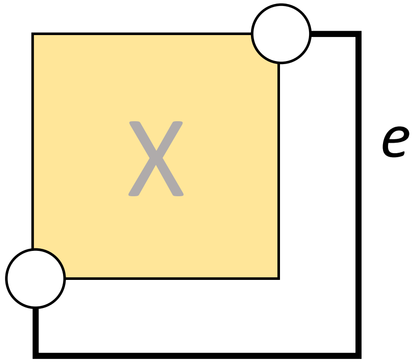
e has 0 bends



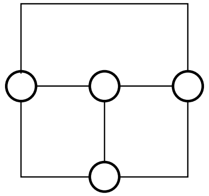
e has 1 bend



e has 2 bends



e has 3 bends



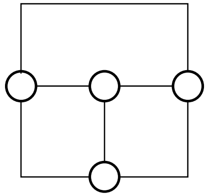
Key-Lemma: O3

Key-Lemma. Let G be a biconnected planar 3-graph with a given edge *e*; G admits a bend-min orthogonal representation with *e* on the external face and having these properties:

O1. at most two bends per edge

O2. every inner P- or R-component is D- or X-shaped; if the root child is a P- or an R-component, it is either D-, C-, or L-shaped

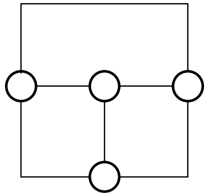
O3. every S-component has spirality at most 4



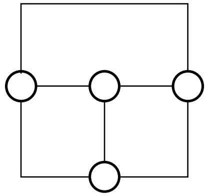
Key-Lemma: O3

Proof of O3 (S-components have spirality at most 4)

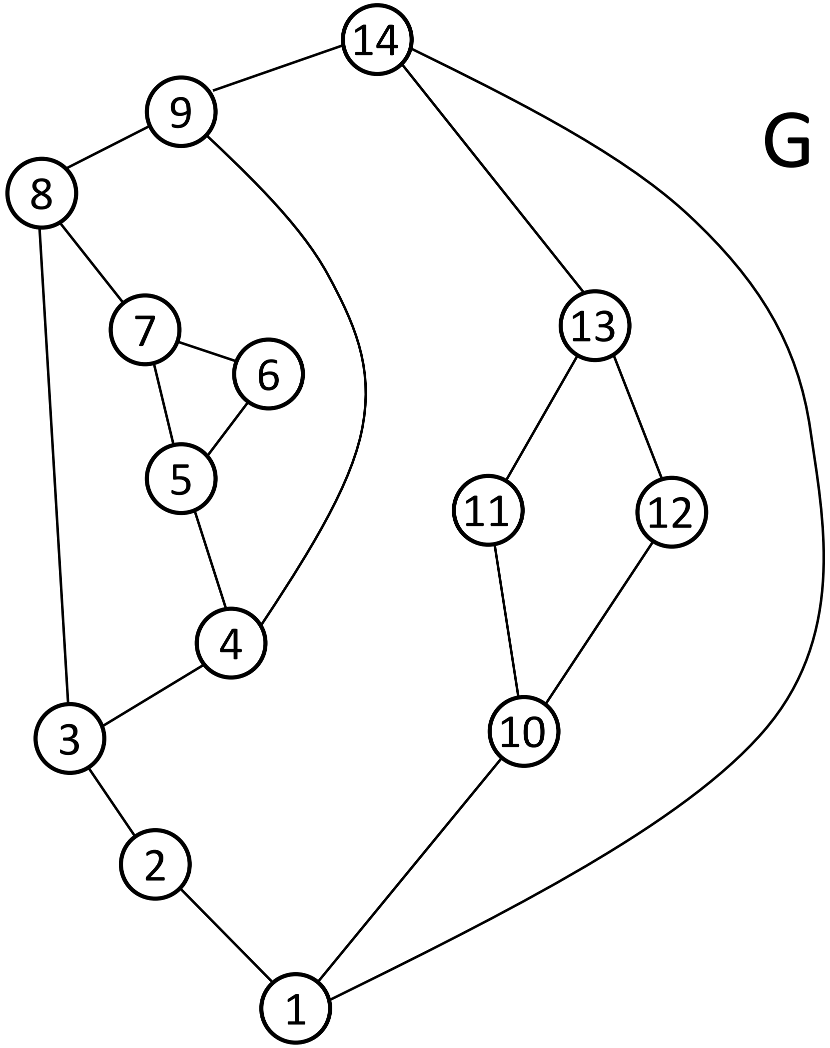
- H = bend-min representation of G with *e* on the external face and properties O1 and O2;
- $\underline{H}$ was computed with the [RNN'03] alg, which we call NoBend-Alg
- we prove that every S-component in $\underline{H}$ (and thus in H) has spirality at most 4



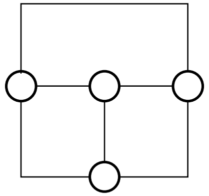
\begin{NoBend-Alg}



Step 1: choose 4 external corners



four vertices of degree 2 are used as corners (in our case, these vertices may be obtained by subdividing edges)



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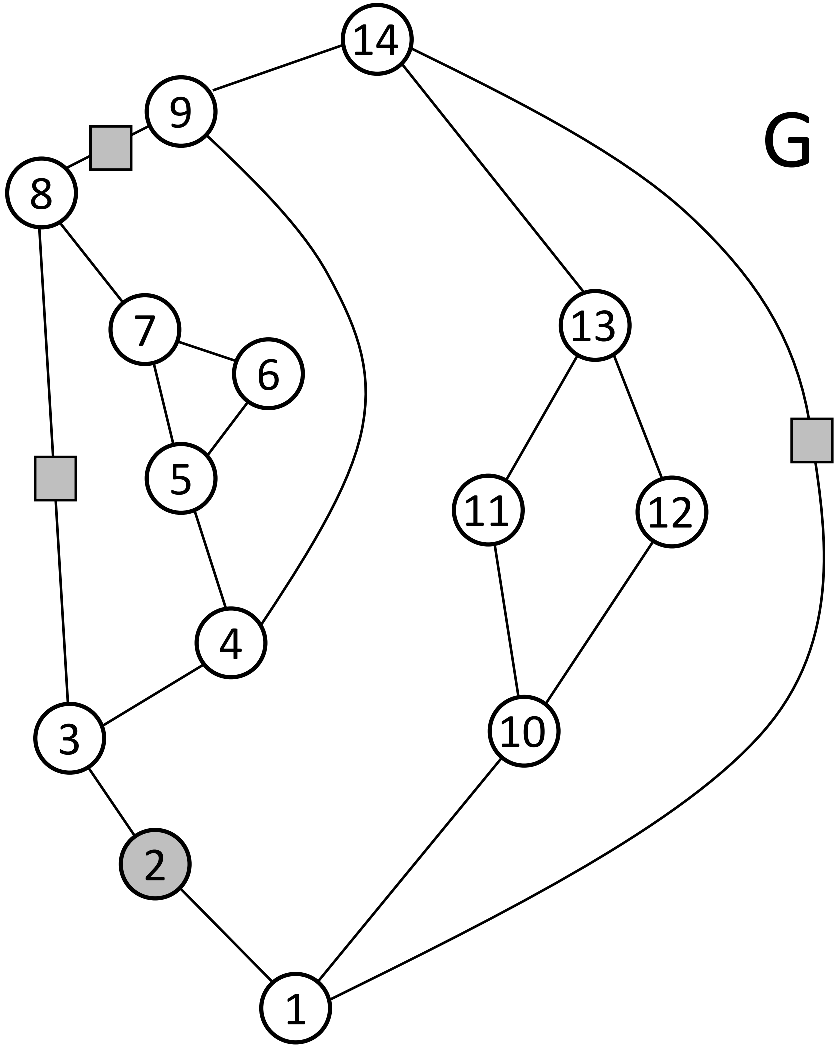

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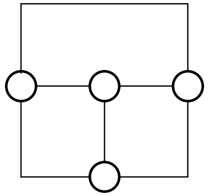
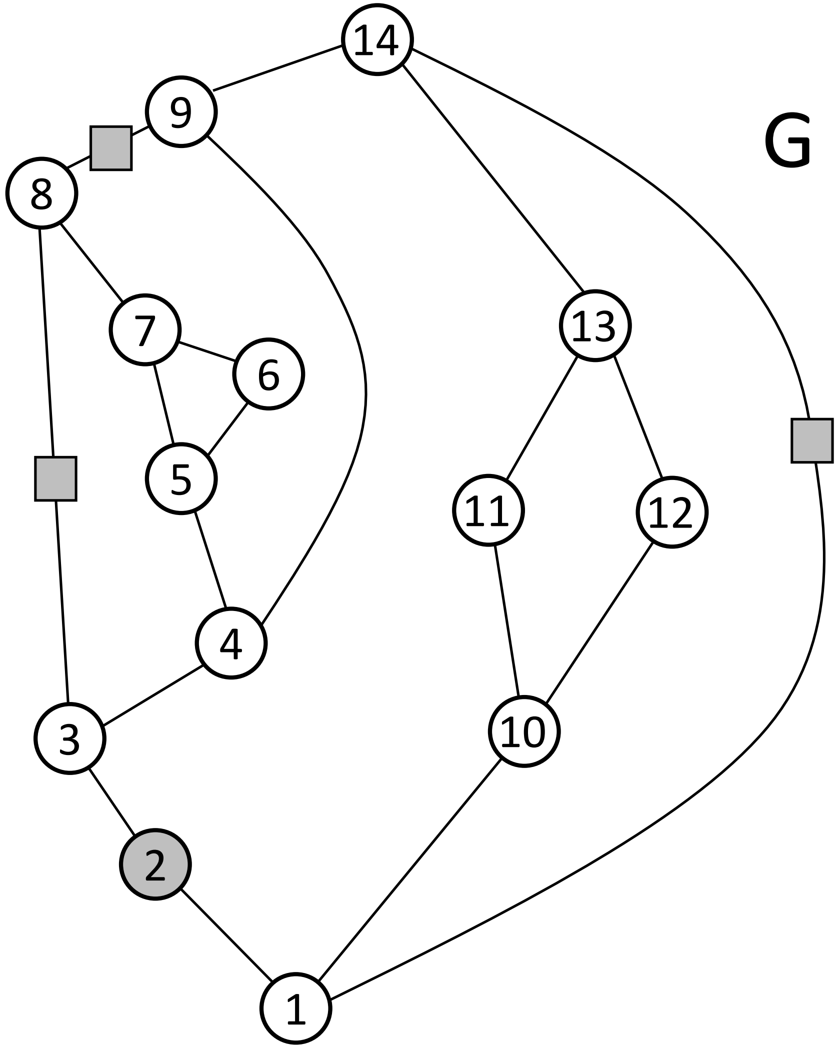
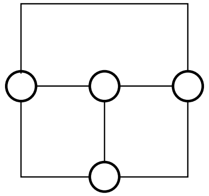


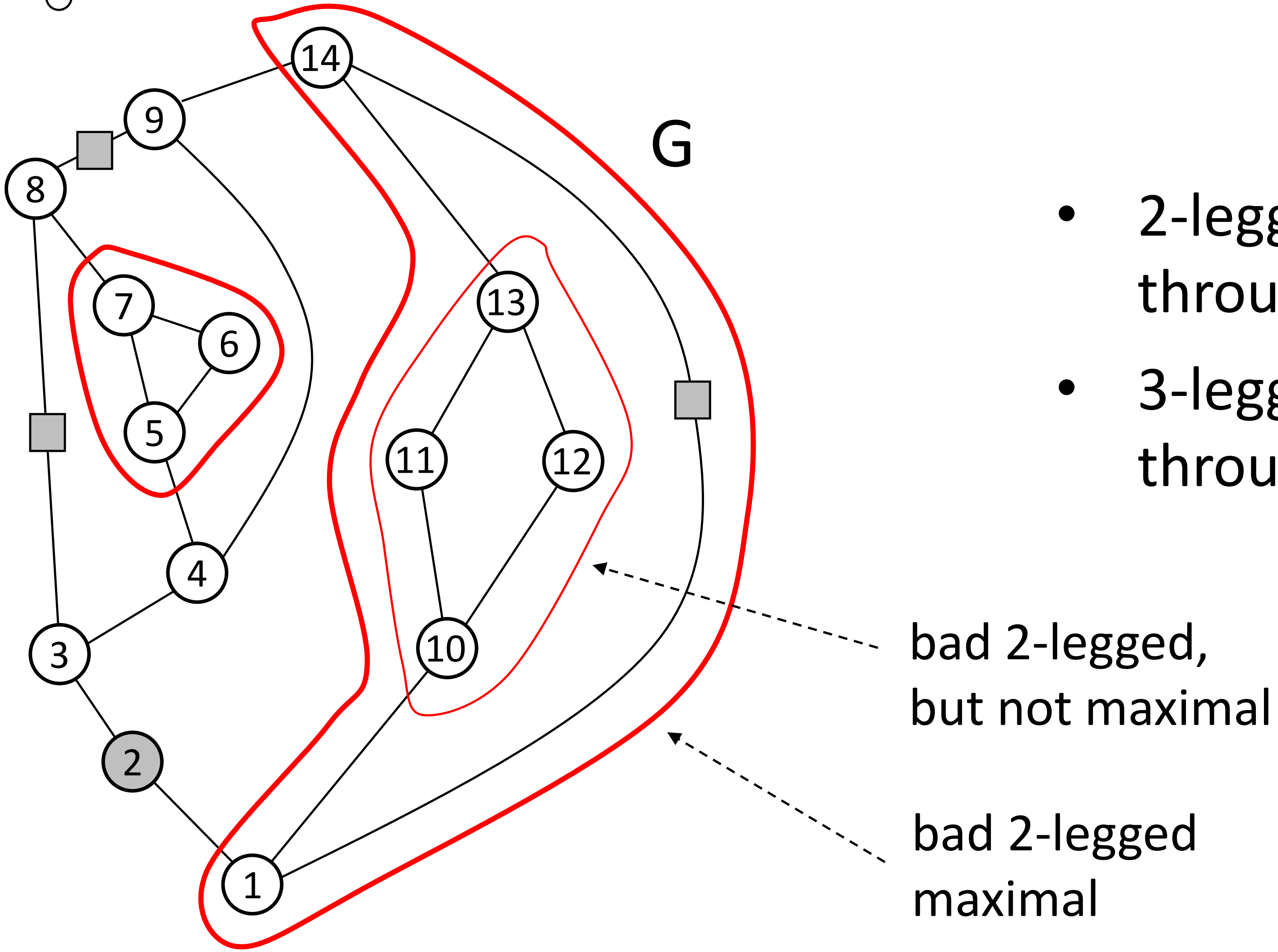
Step 2: find maximal bad cycles w.r.t. the corners



- 2-legged cycles not passing through (at least) 2 corners
- 3-legged cycles not passing through (at least) 1 corner

Step 2: find maximal bad cycles w.r.t. the corners





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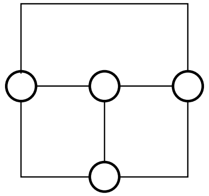
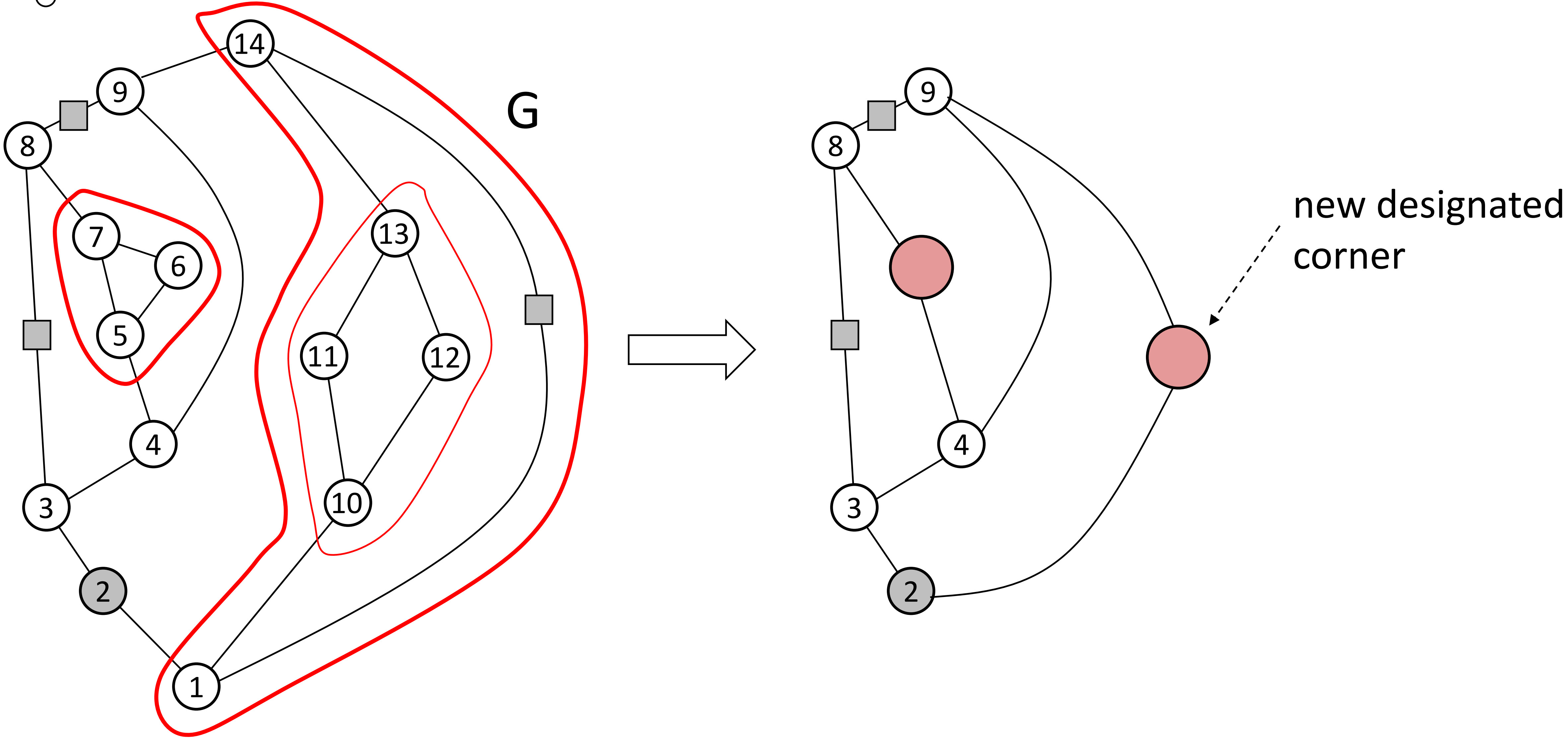
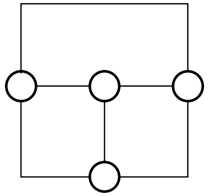
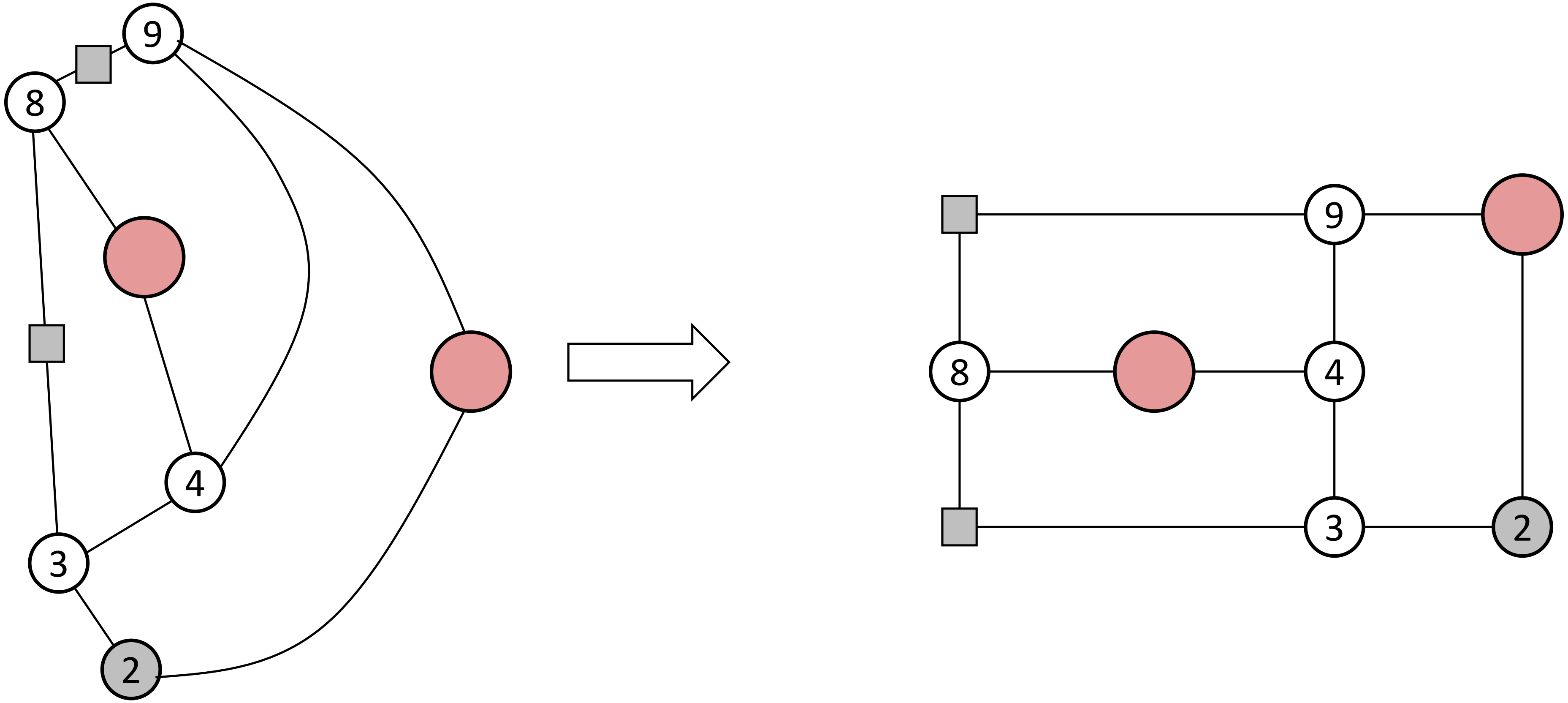


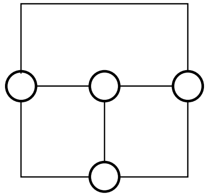
Step 3: collapse maximal bad cycles



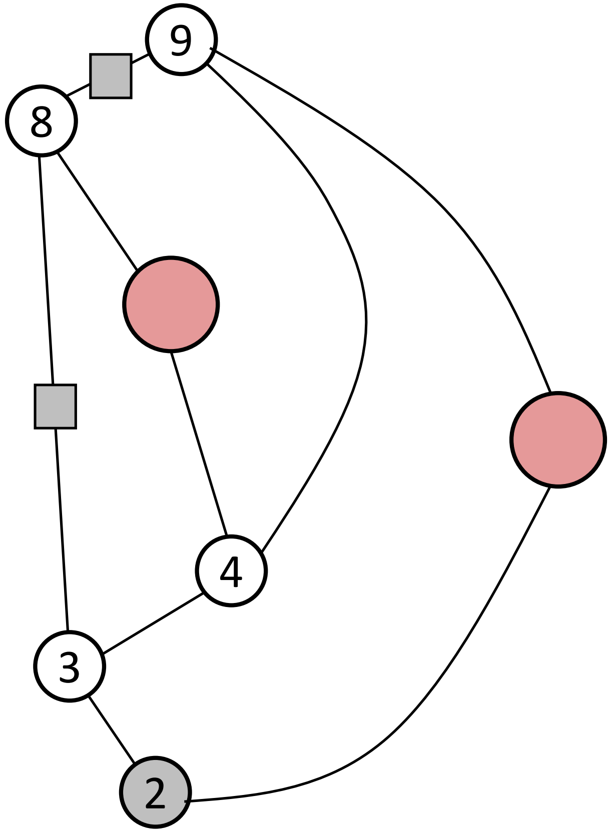


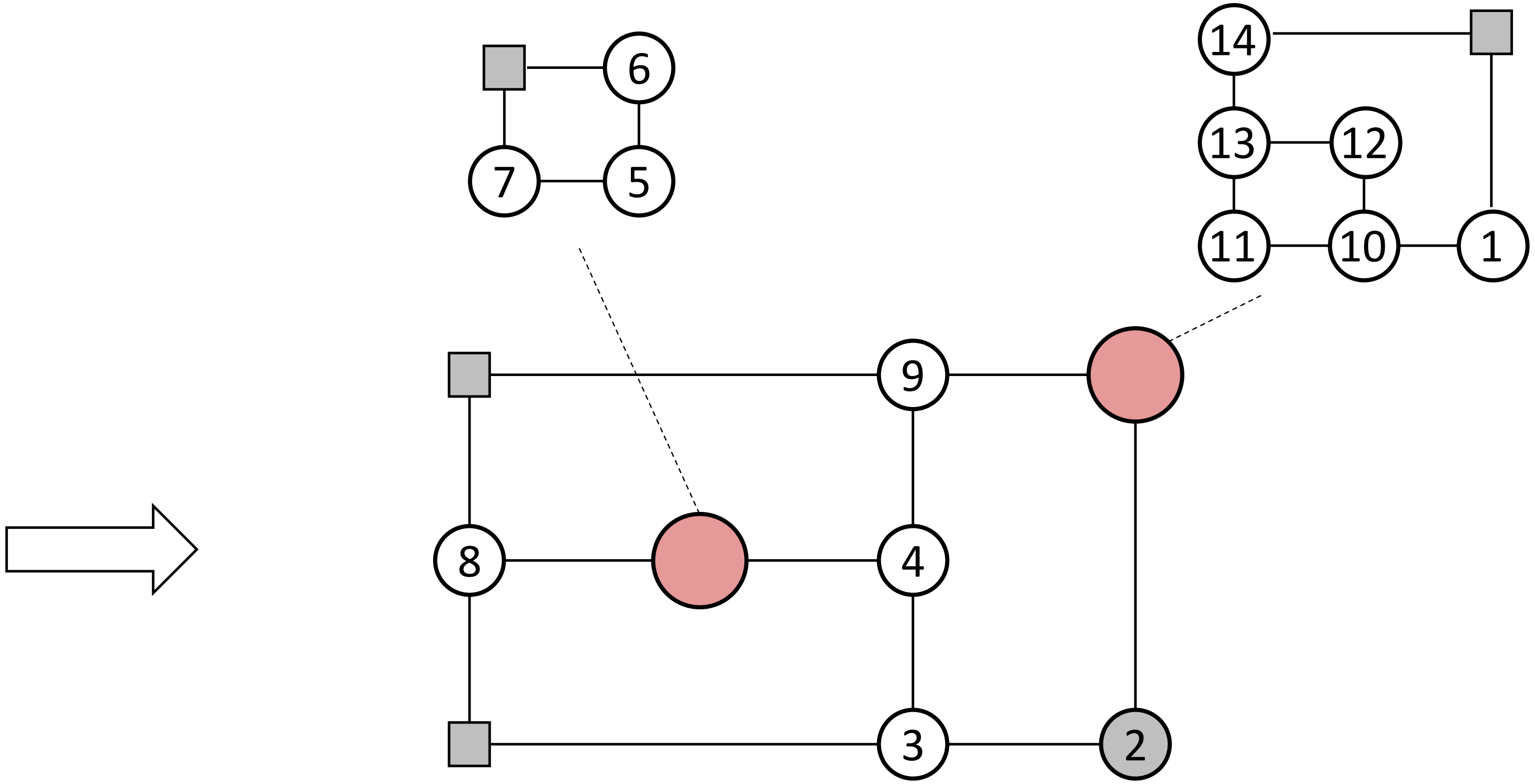
Step 4: compute a rectangular representation

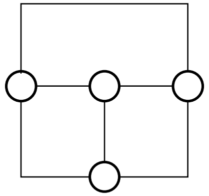




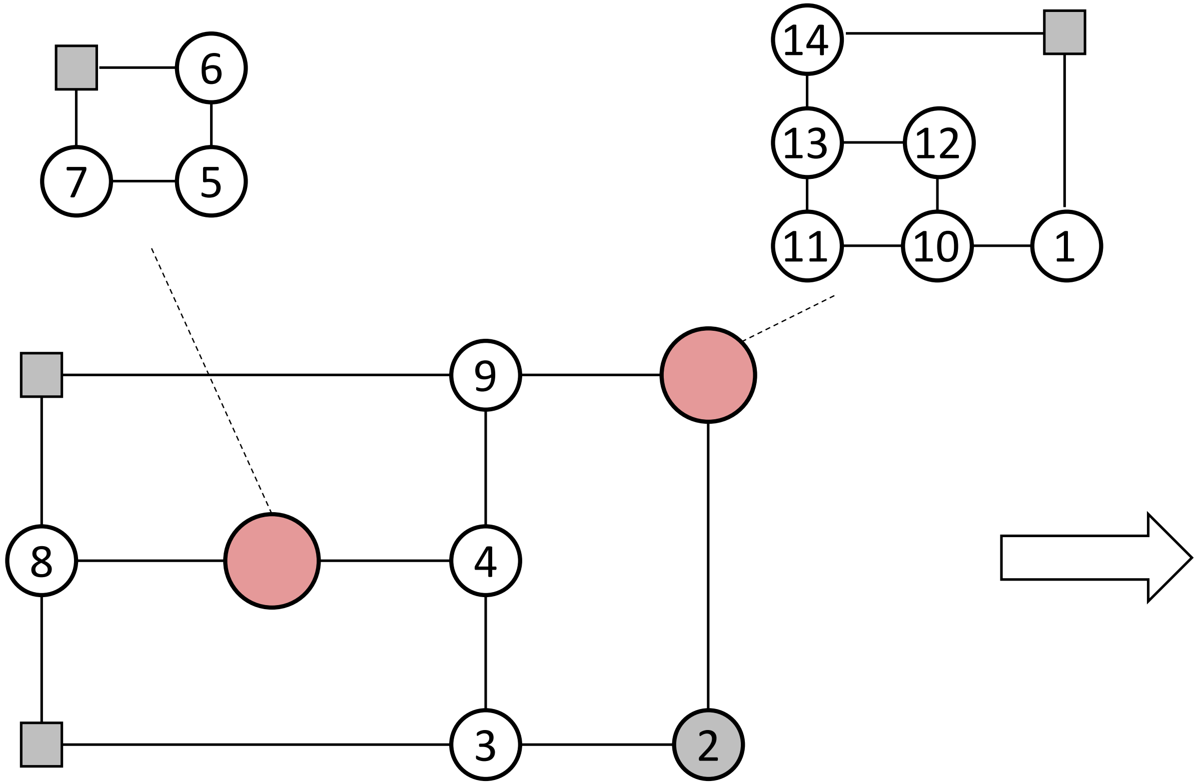
Step 5: recourse into the collapsed nodes

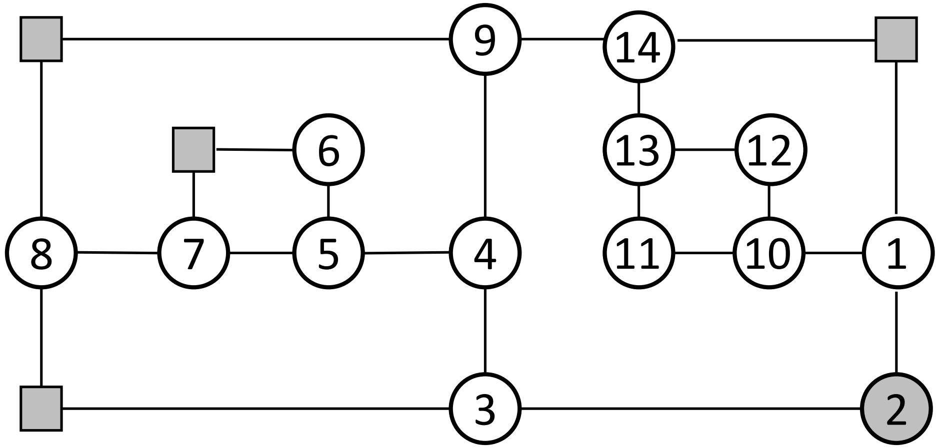


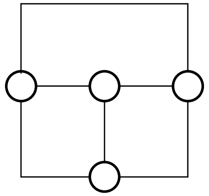




Step 6: … and plug the components





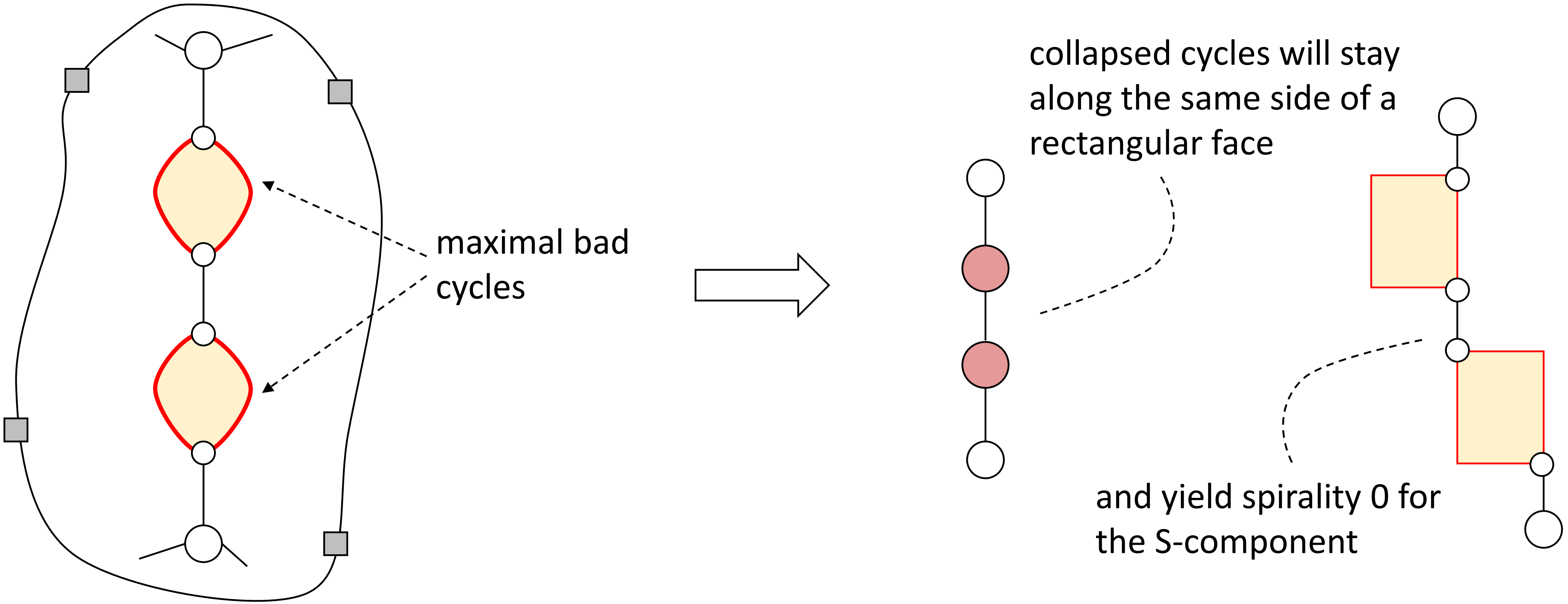


\end{NoBend-Alg}

Key-Lemma: O3

Proof of O3 (<u>inner</u> S-components have spirality at most 4)

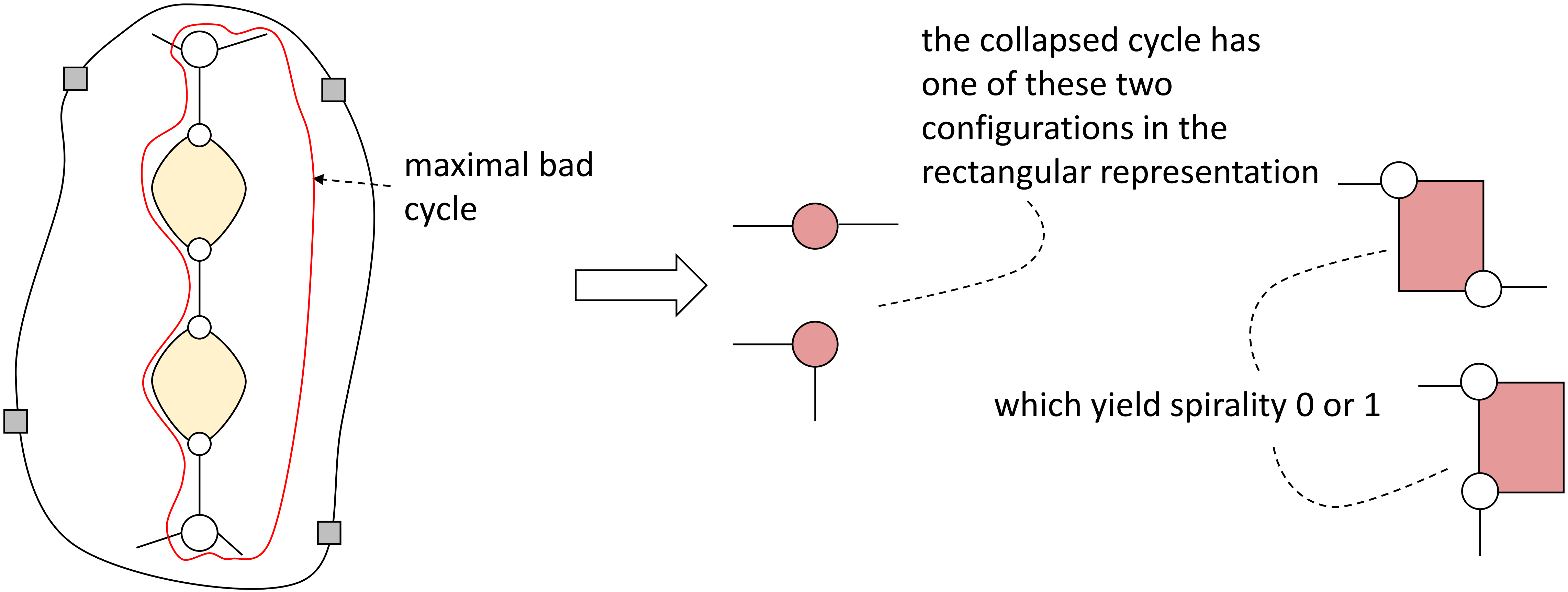
Case 1. the S-component is not inside a maximal bad cycle and all its edges are internal

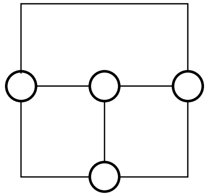


Key-Lemma: O3

Proof of O3 (<u>inner</u> S-components have spirality at most 4)

Case 2. the S-component is inside a maximal bad cycle that traverses the component





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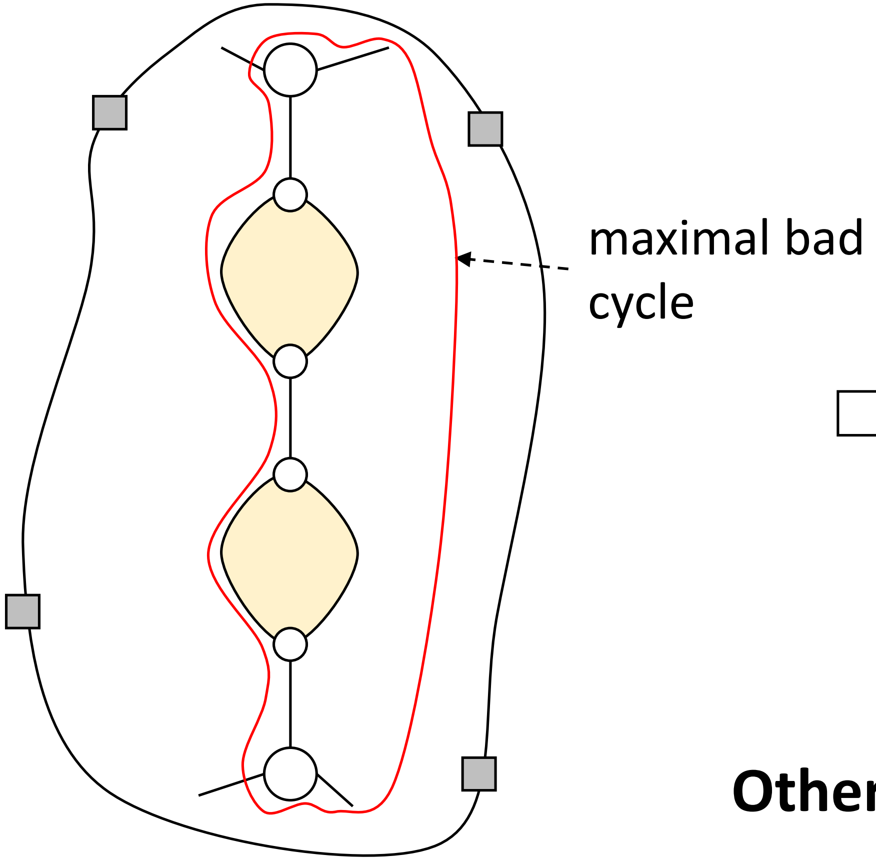
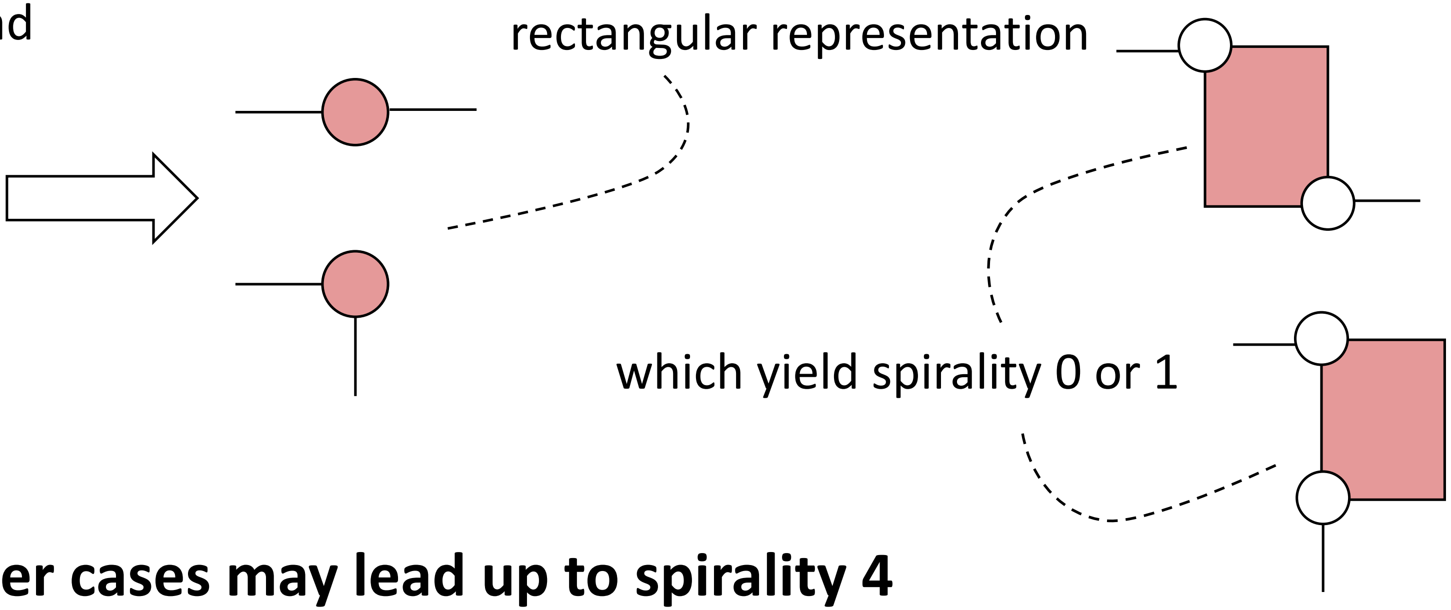
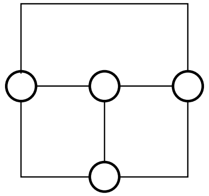


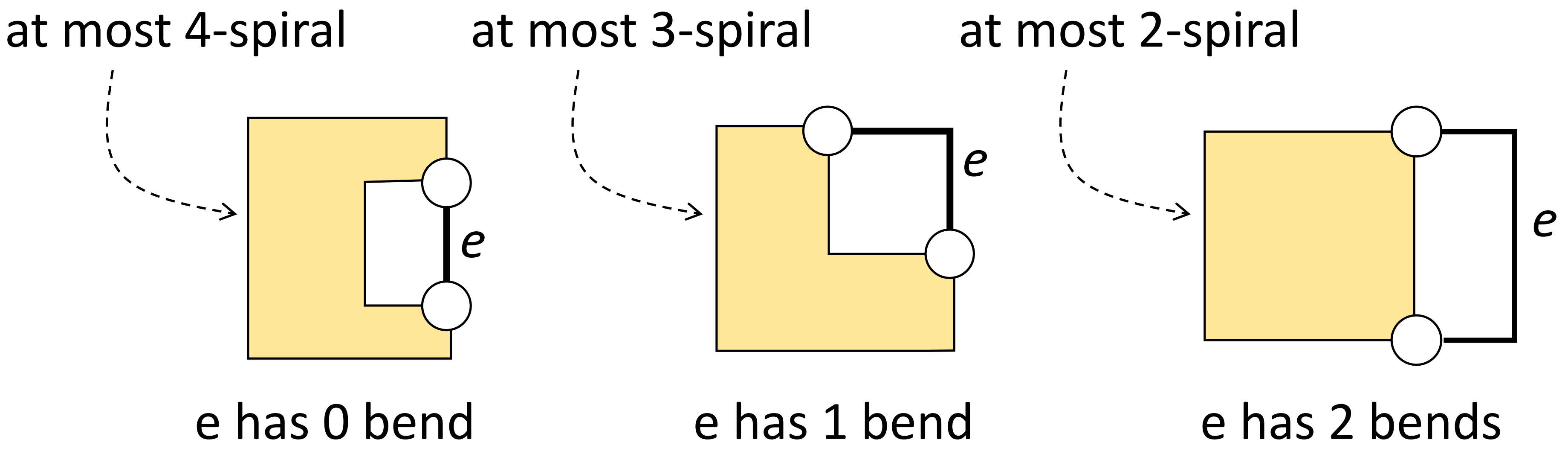


Other cases may lead up to spirality 4

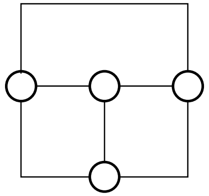


Key-Lemma: O3

Proof of O3 (a root child S-component has spirality at most 4)



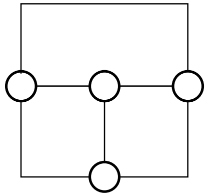
Higher values of spirality may only increase the number of bends



Algorithm

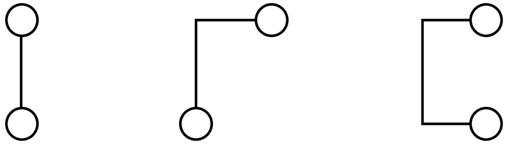
- **input**: biconnected planar 3-graph G with a reference edge *e*
- **output**: bend-min representation H of G with *e* on the external face

1. construct the SPQR-tree T of G with respect to *e*
2. visit the nodes μ of T **bottom-up**:
   - **μ inner node** ⇒ store in μ a candidate set of bend-min representations of G_{μ}- one for each distinct representative shape, thanks to the substitution theorem
   - **μ the root child** ⇒ construct H by suitably merging *e* with the candidate representations stored at the children of μ; consider {0, 1, 2} bends for *e*, thanks to O1 of the key-lemma

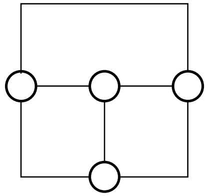


Candidate sets for the tree nodes

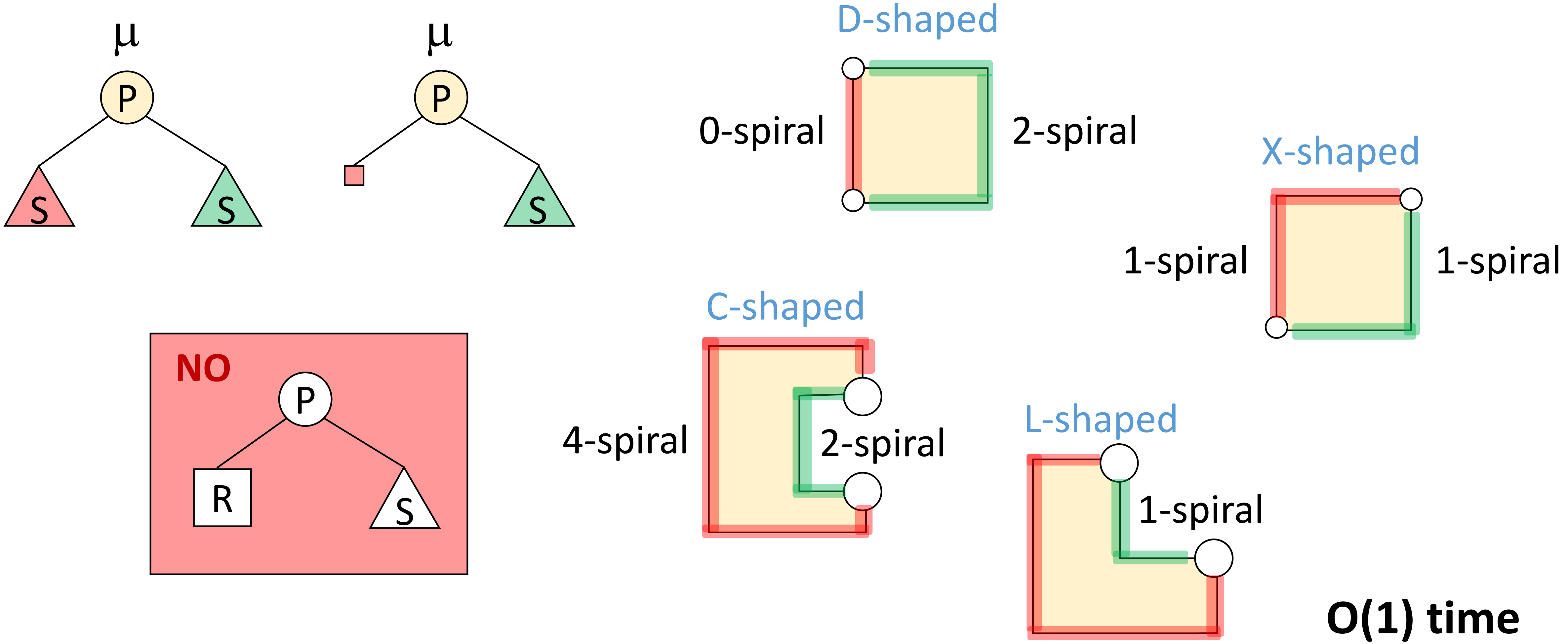
- **Q-node:** a representation for each number of bends in {0, 1, 2}
  - thanks to O1 of the key-lemma



- **P/R-node:** the cheapest D- and X-shaped representations for the inner nodes and the cheapest D-, C-, and L-shaped representations for the root child
  - thanks to O2 of the key-lemma
- **S-node:** the cheapest representation for each value of spirality in {0, 1, 2, 3, 4}
  - thanks to O3 of the key-lemma



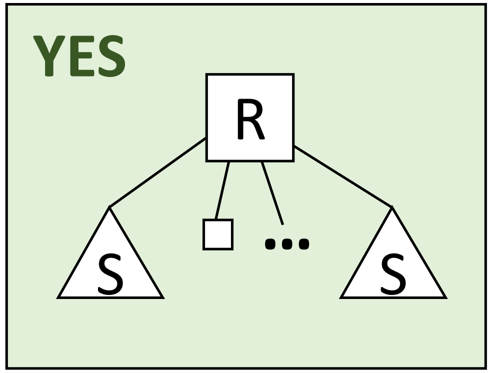
Candidate set of a P-node

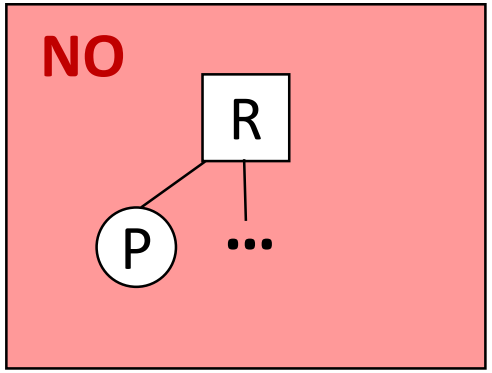


O(1) time

Candidate set of an R-node

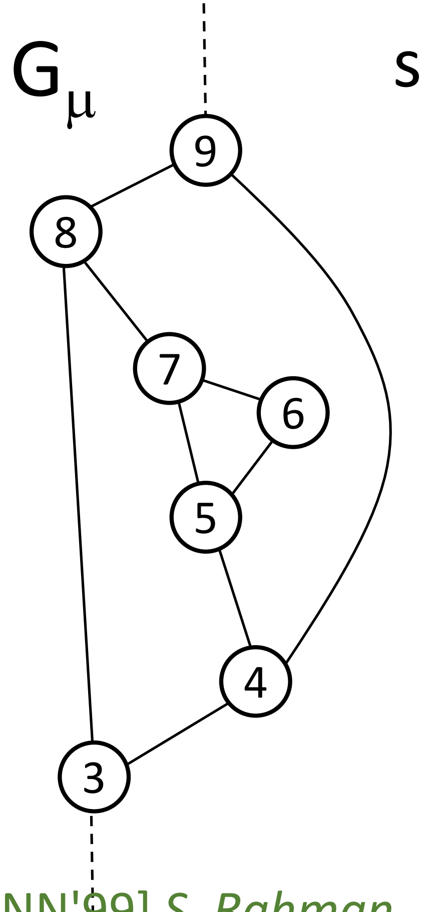
Each child of an R-node is either a Q- or an S-node



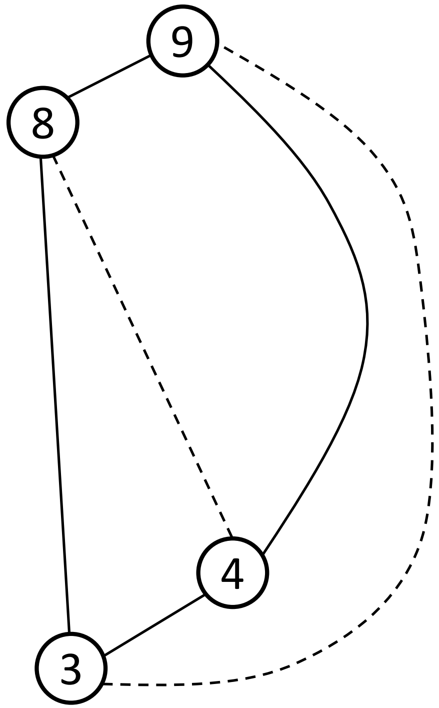


Candidate set of an R-node (sketch)

G_μ



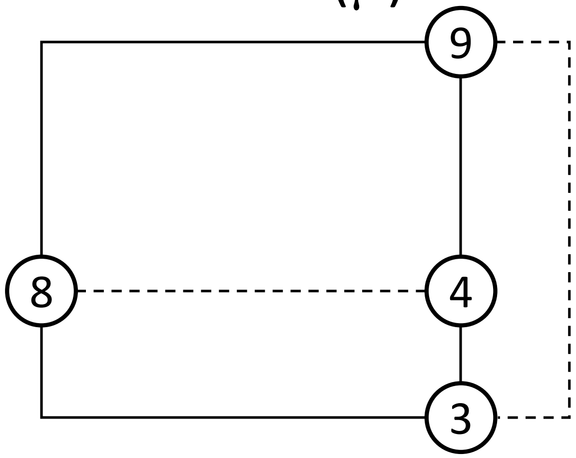
skel(μ)



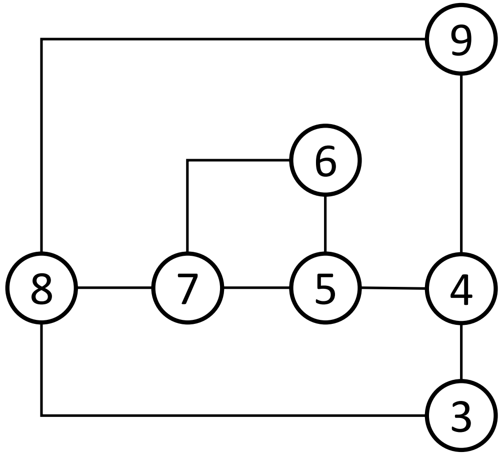
O(n)-time variant of [RNN'99]

bend-min 3-connected cubic (with constraints)

constrained bend-min repr. of skel(μ)

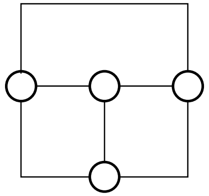


bend-min D-shaped

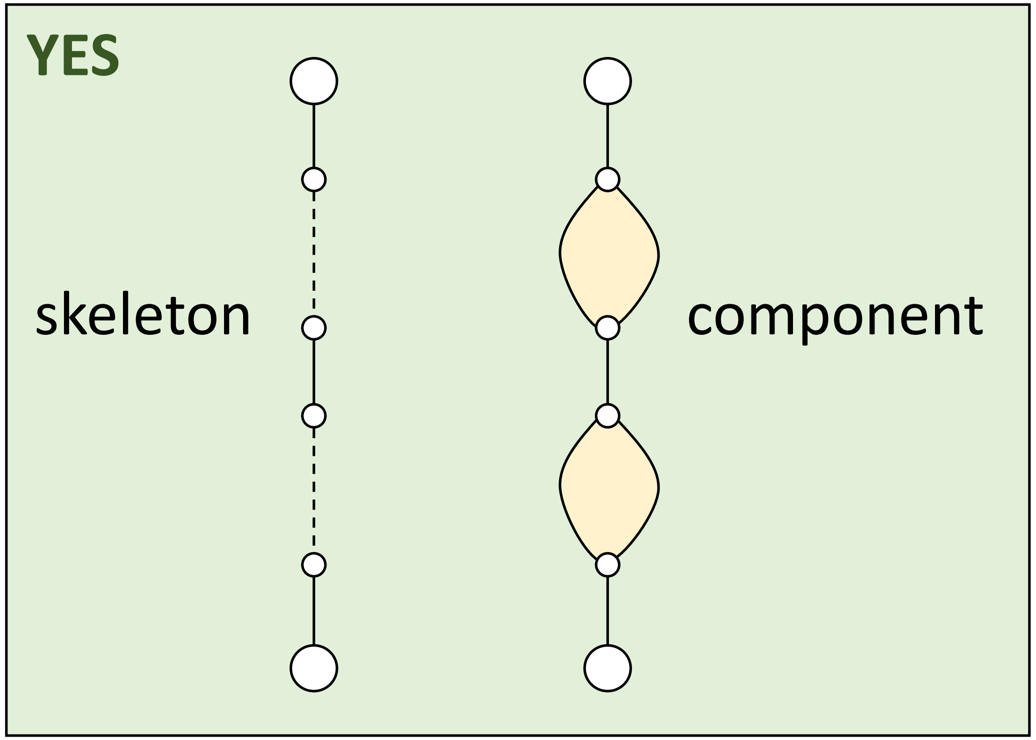


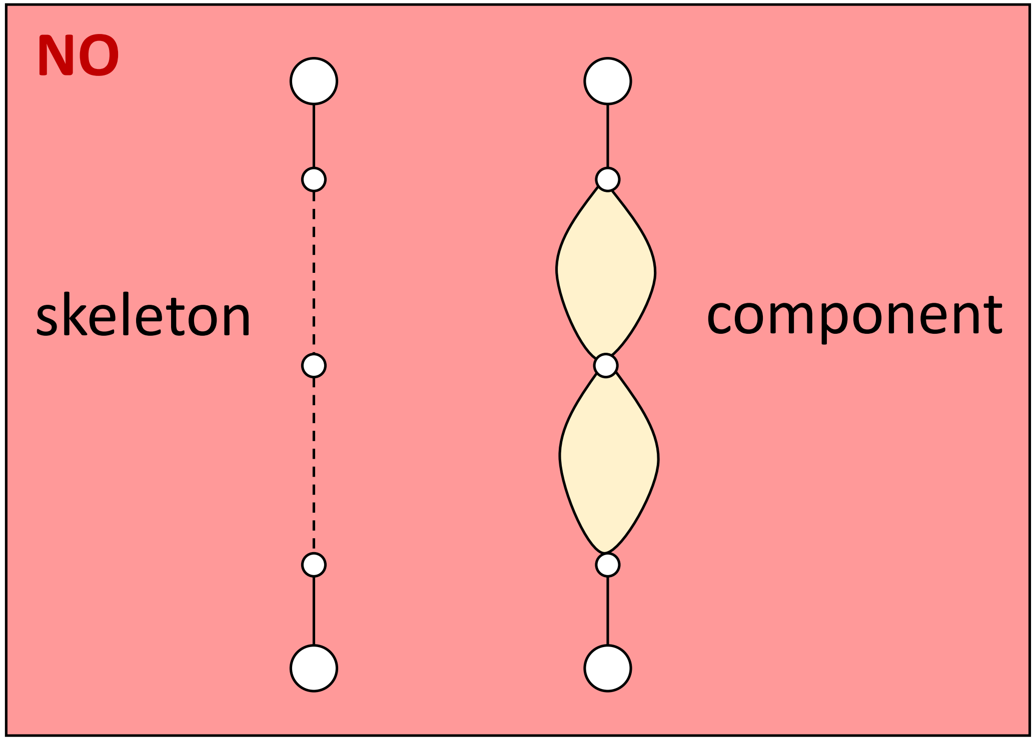
O(n_μ) time

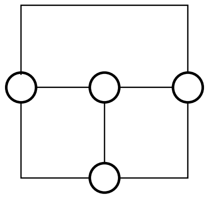
[RNN'99] *S. Rahman, S.-I. Nakano, T. Nishizeki*: A Linear Algorithm for Bend-Optimal Orthogonal Drawings of Triconnected Cubic Plane Graphs. J. Graph Algorithms Appl. 3(4): 31-62 (1999)



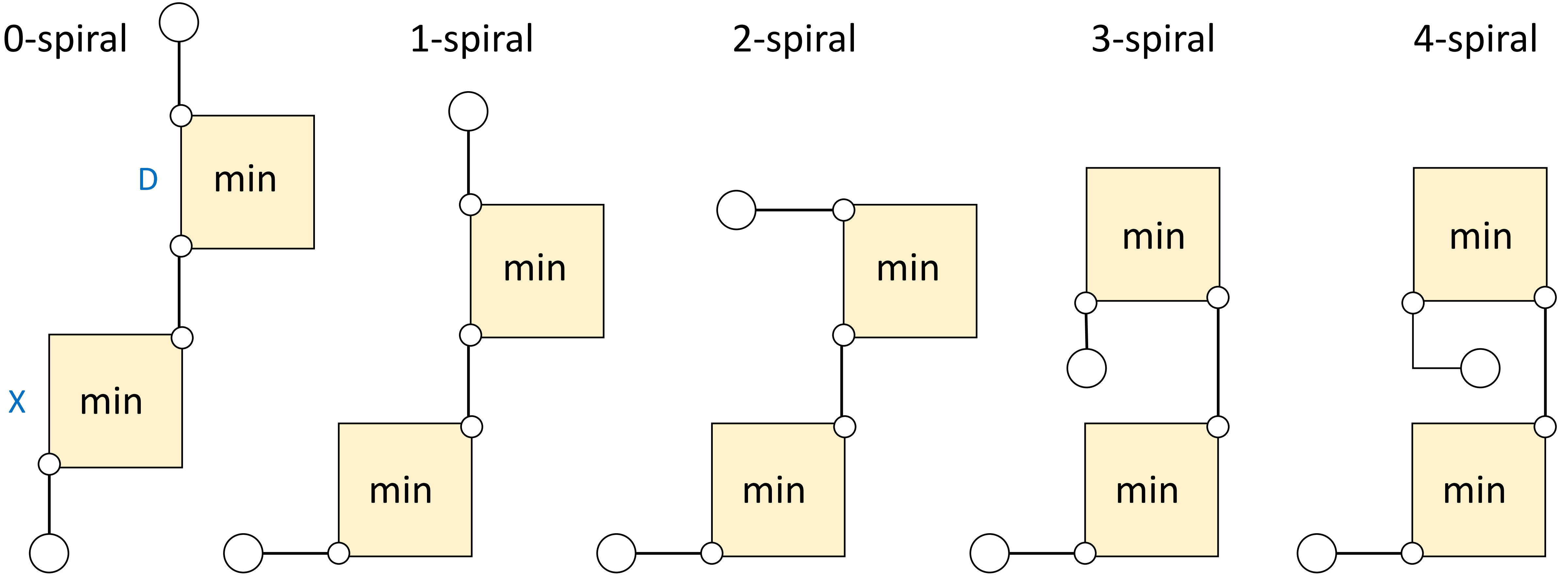
Candidate set of an S-node





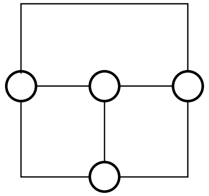


Candidate set of an S-node



#(extra bends) = max{0, spirality – (#D-shaped + #Q-nodes – 1)}

$O(n_\mu)$ time

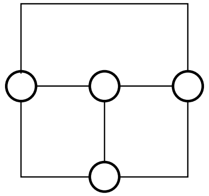


Question

- Is there a subquadratic-time algorithm to compute a bend-minimum orthogonal drawing of a planar 3-graph?

$O(n^2)$

Can we do better?



Question

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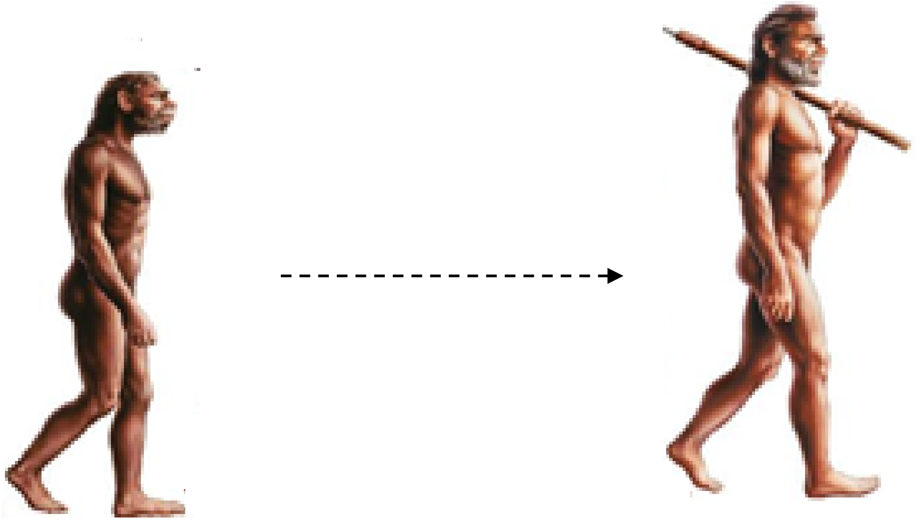
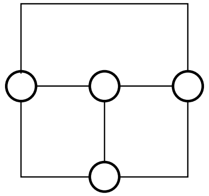


$O(n^2)$

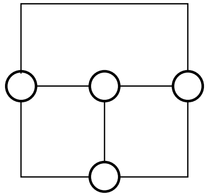
O(n)-time algorithm

Ingredients:

- new data structure for the rigid components
- labeling procedure for the candidate sets
- reusability principle for the SPQR-tree nodes



Bend-minimum orthogonal drawings of planar 4-graphs

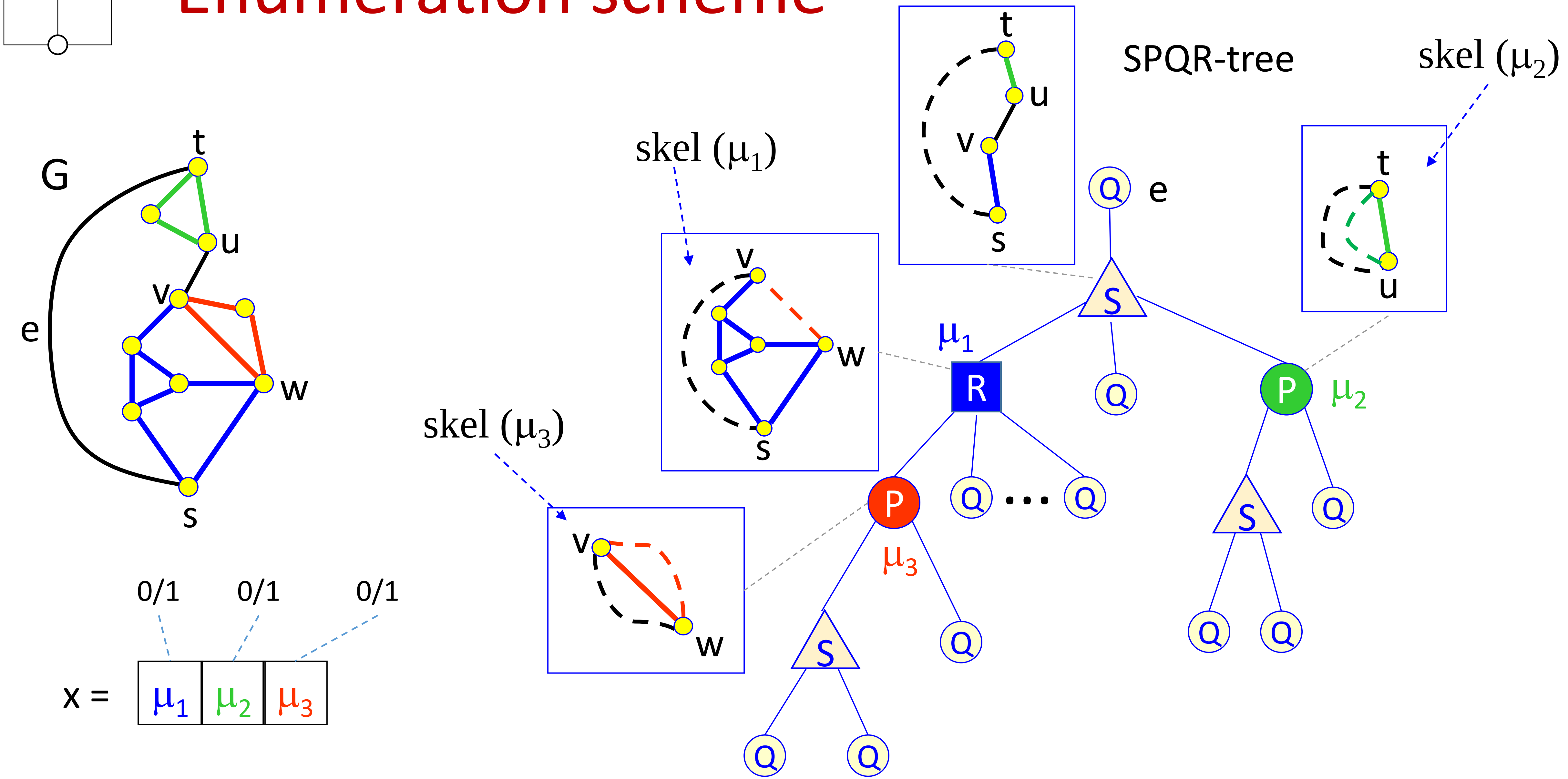


Bend-min of planar 4-graphs

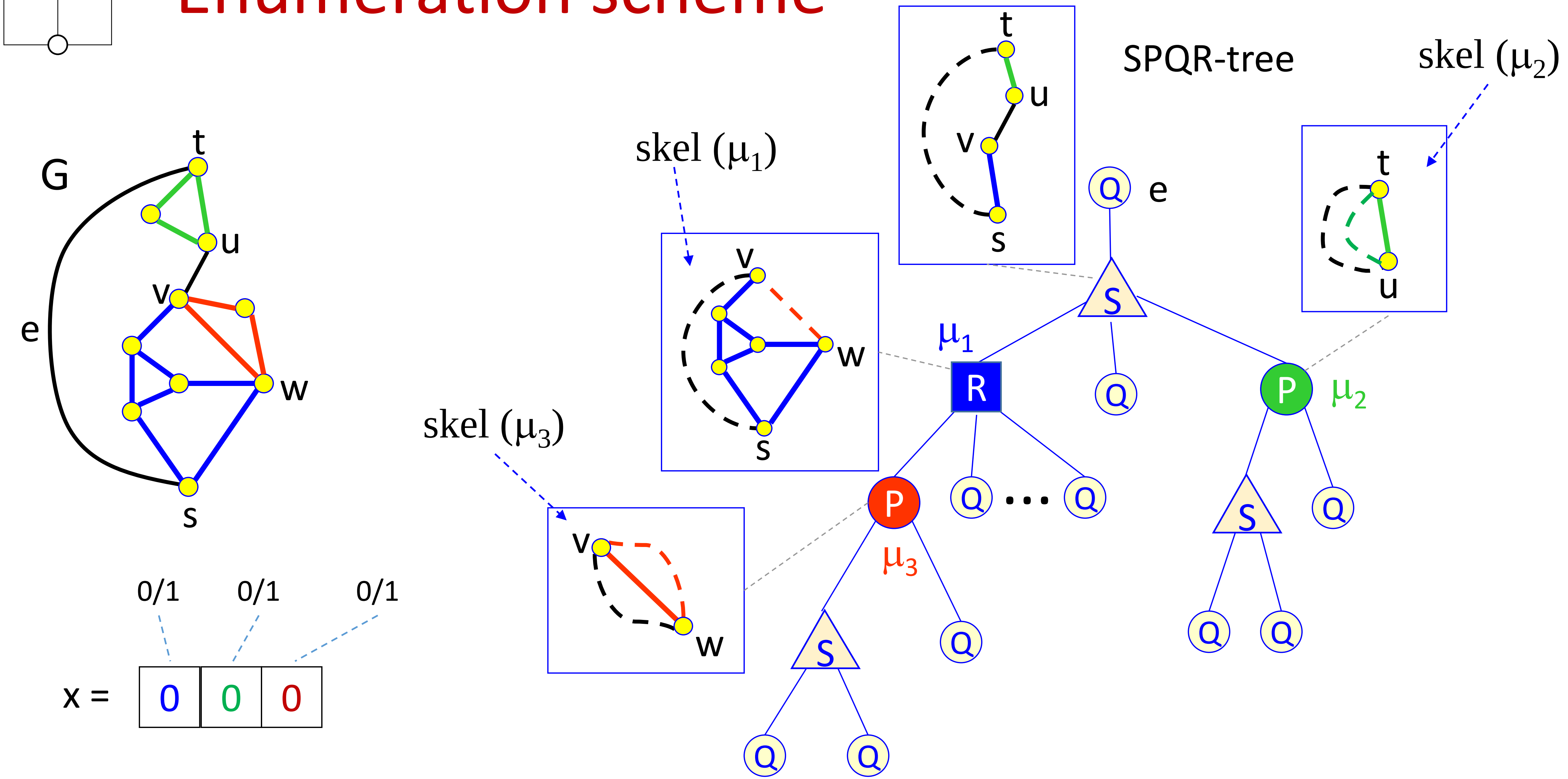
- Branch-and-bound algorithm for a biconnected graph G
  - *P. Bertolazzi, G. Di Battista, W. Didimo: Computing Orthogonal Drawings with the Minimum Number of Bends. IEEE Trans. Computers 49(8): 826-840 (2000)*

- Ingredients:
  - enumeration scheme for the planar embeddings of G
  - effective lower bounds on the number of bends
  - simple upper bounds on the number of bends

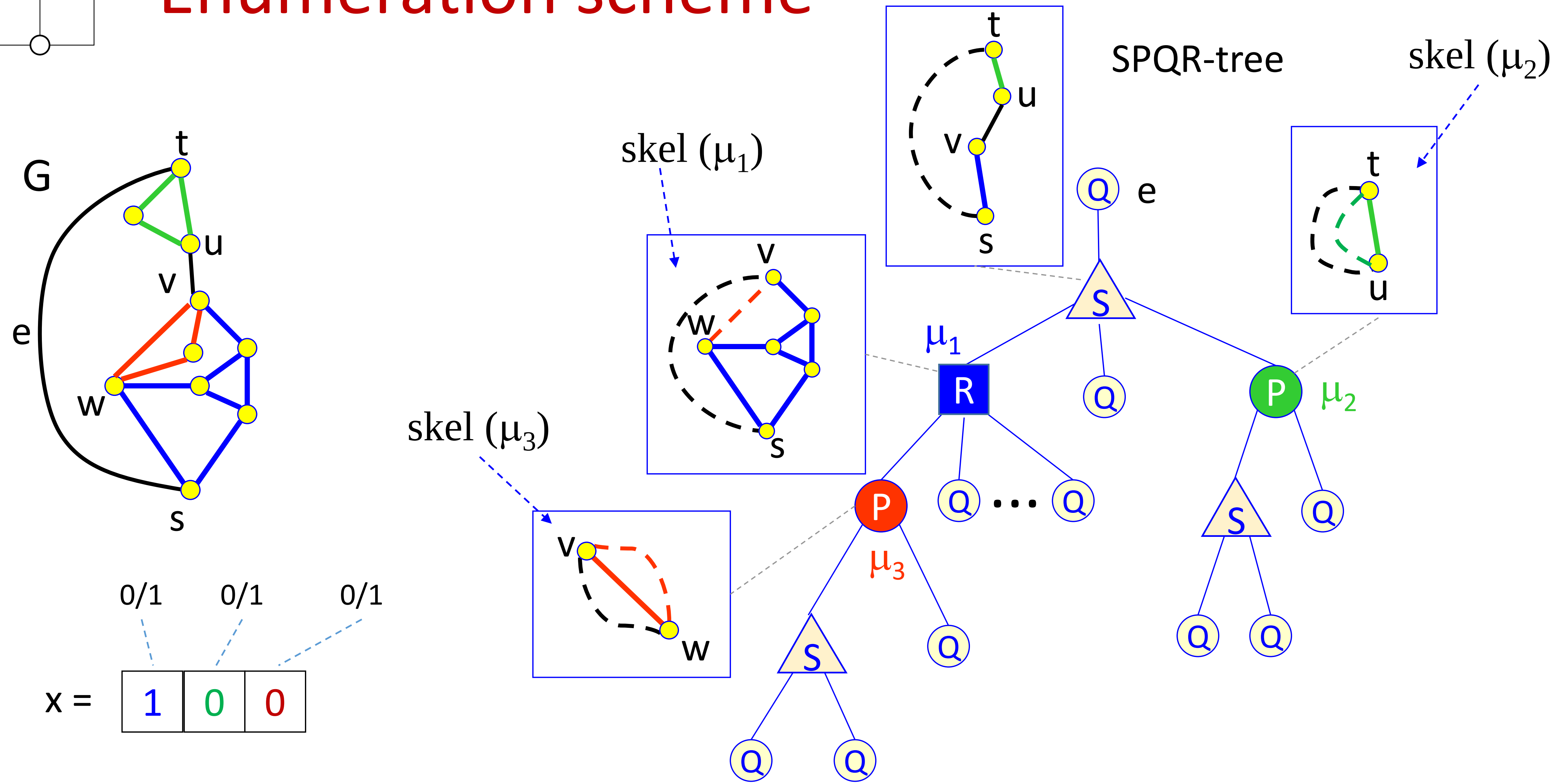
Enumeration scheme



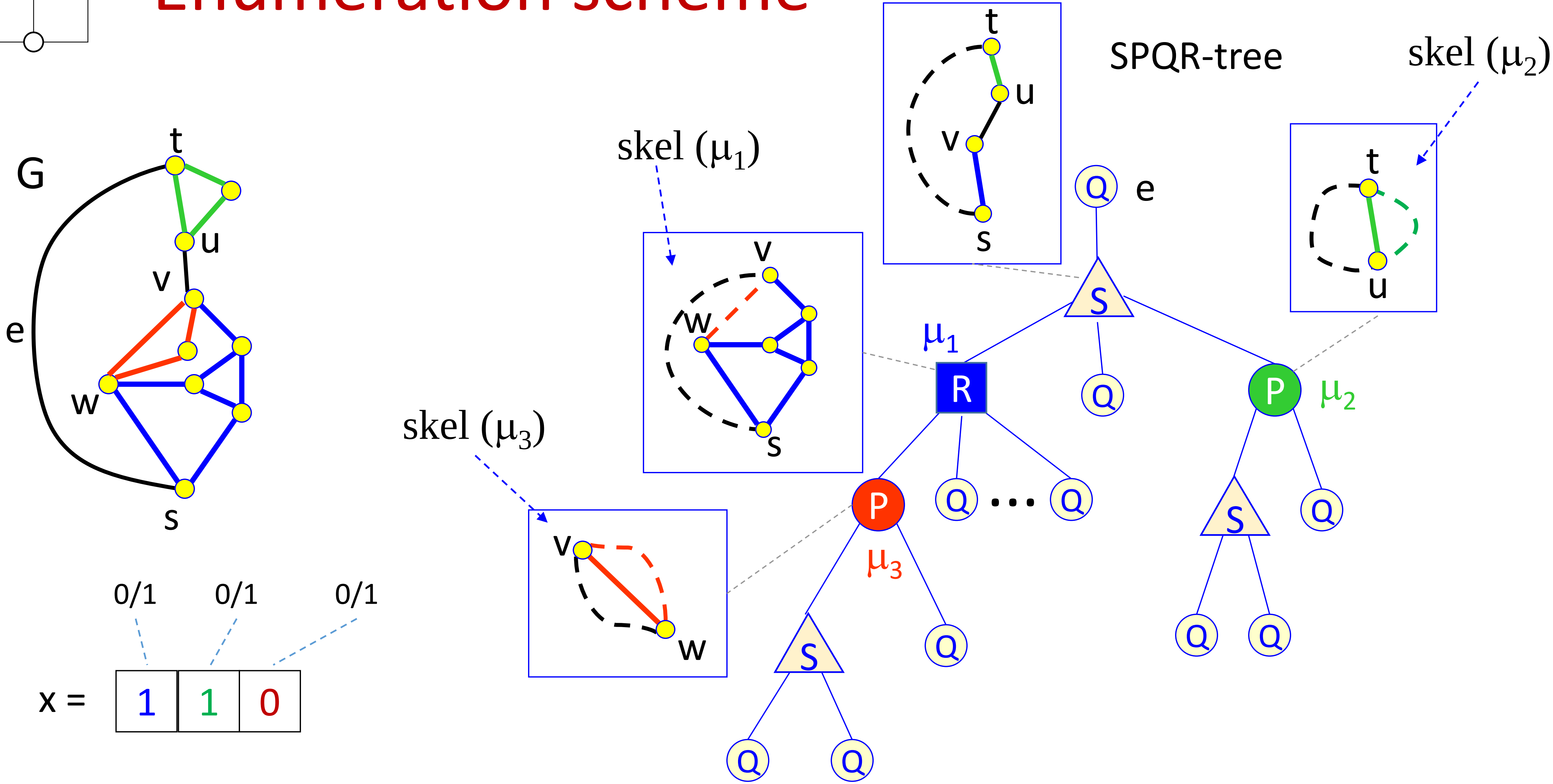
Enumeration scheme



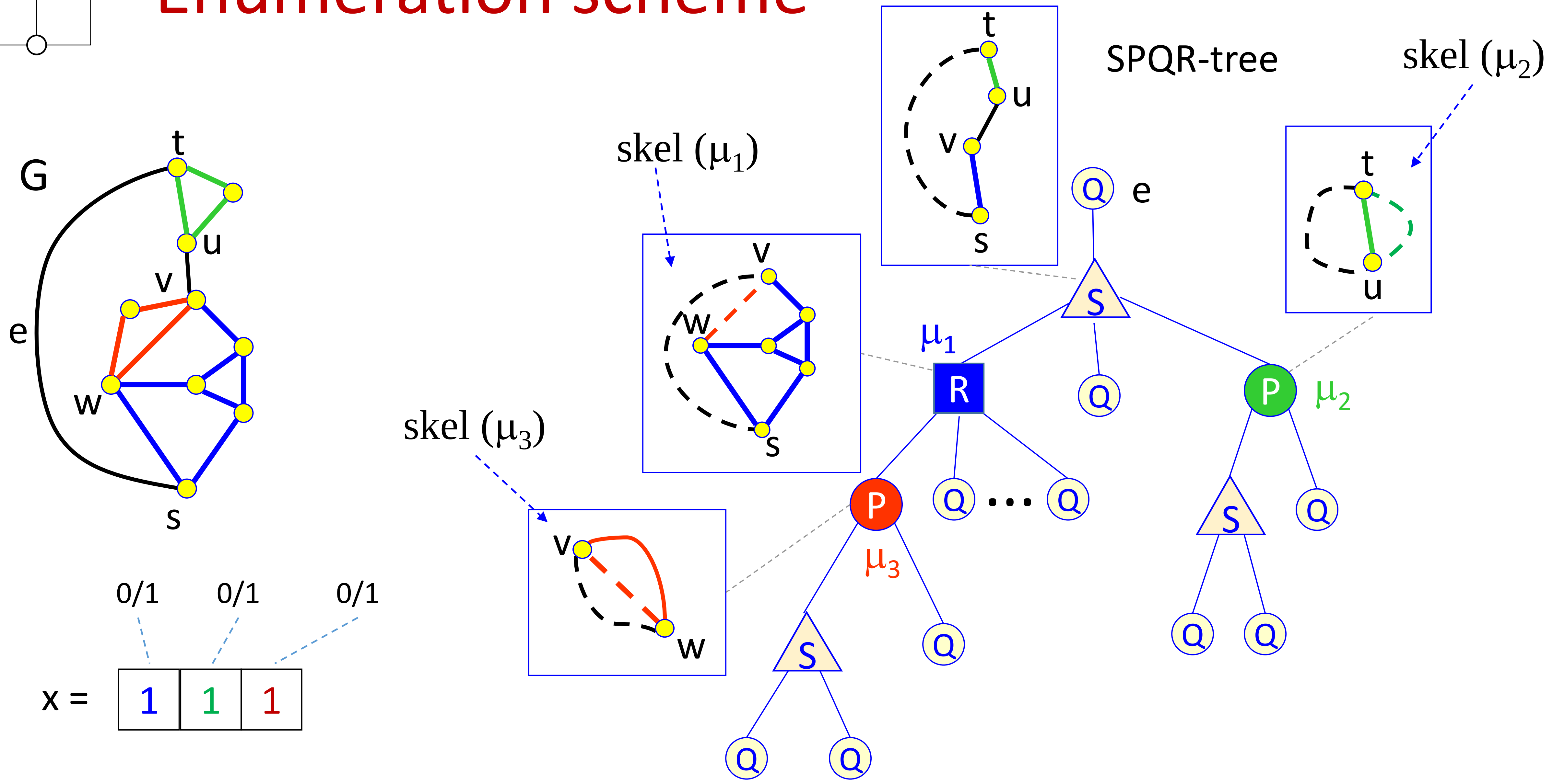
Enumeration scheme



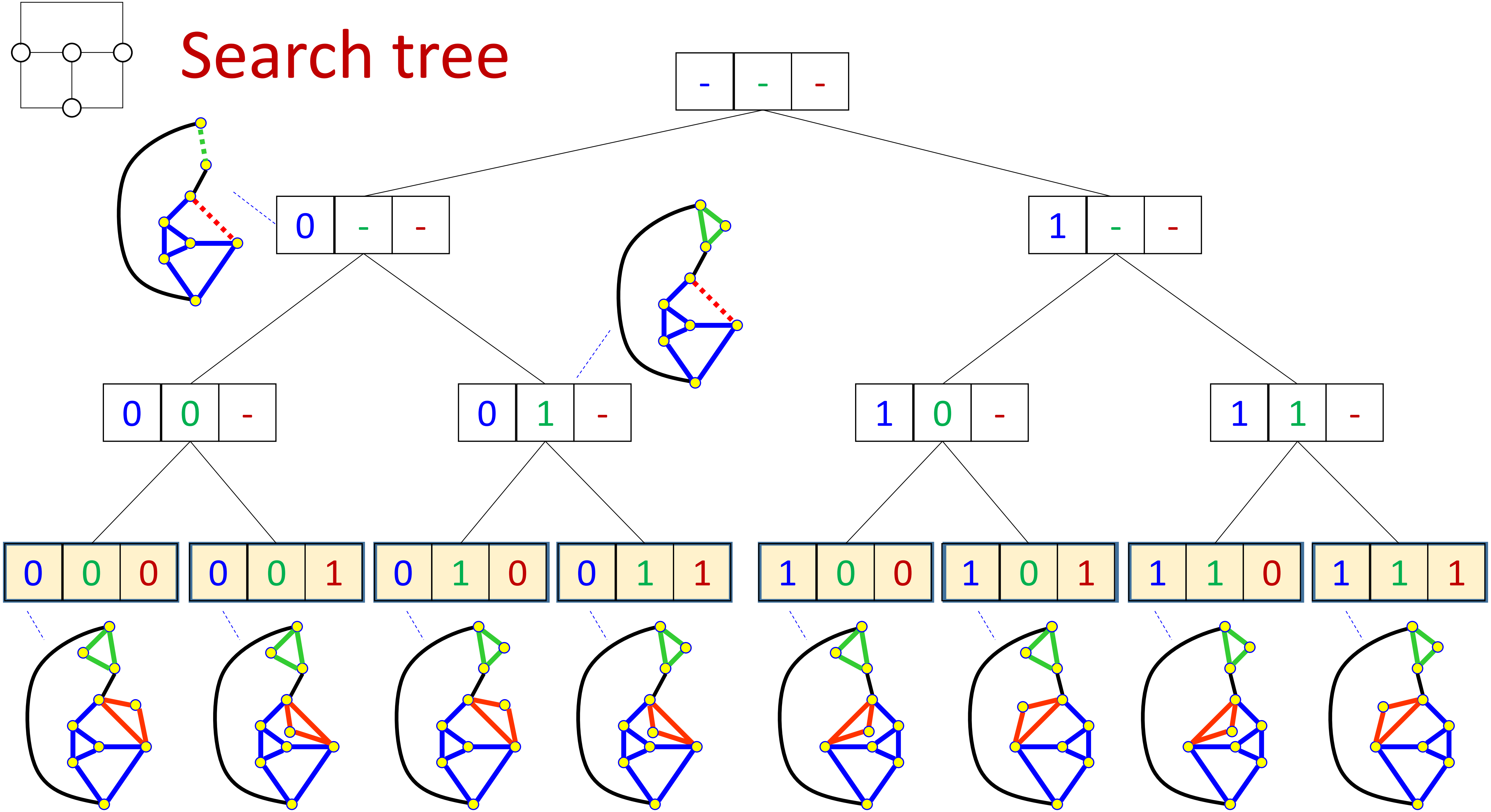
Enumeration scheme



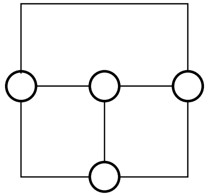
Enumeration scheme

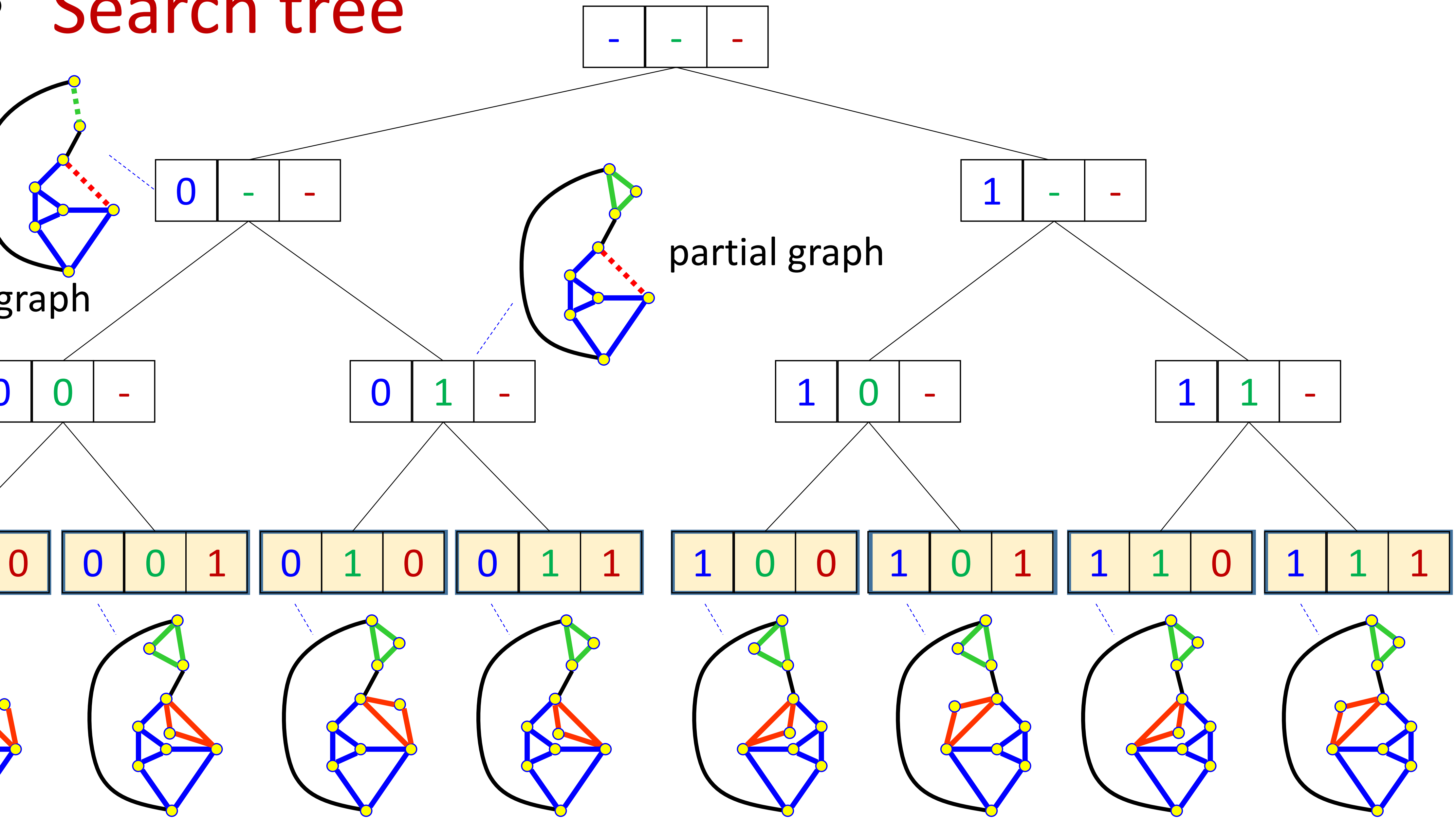


Search tree

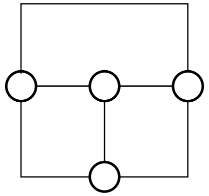


Search tree





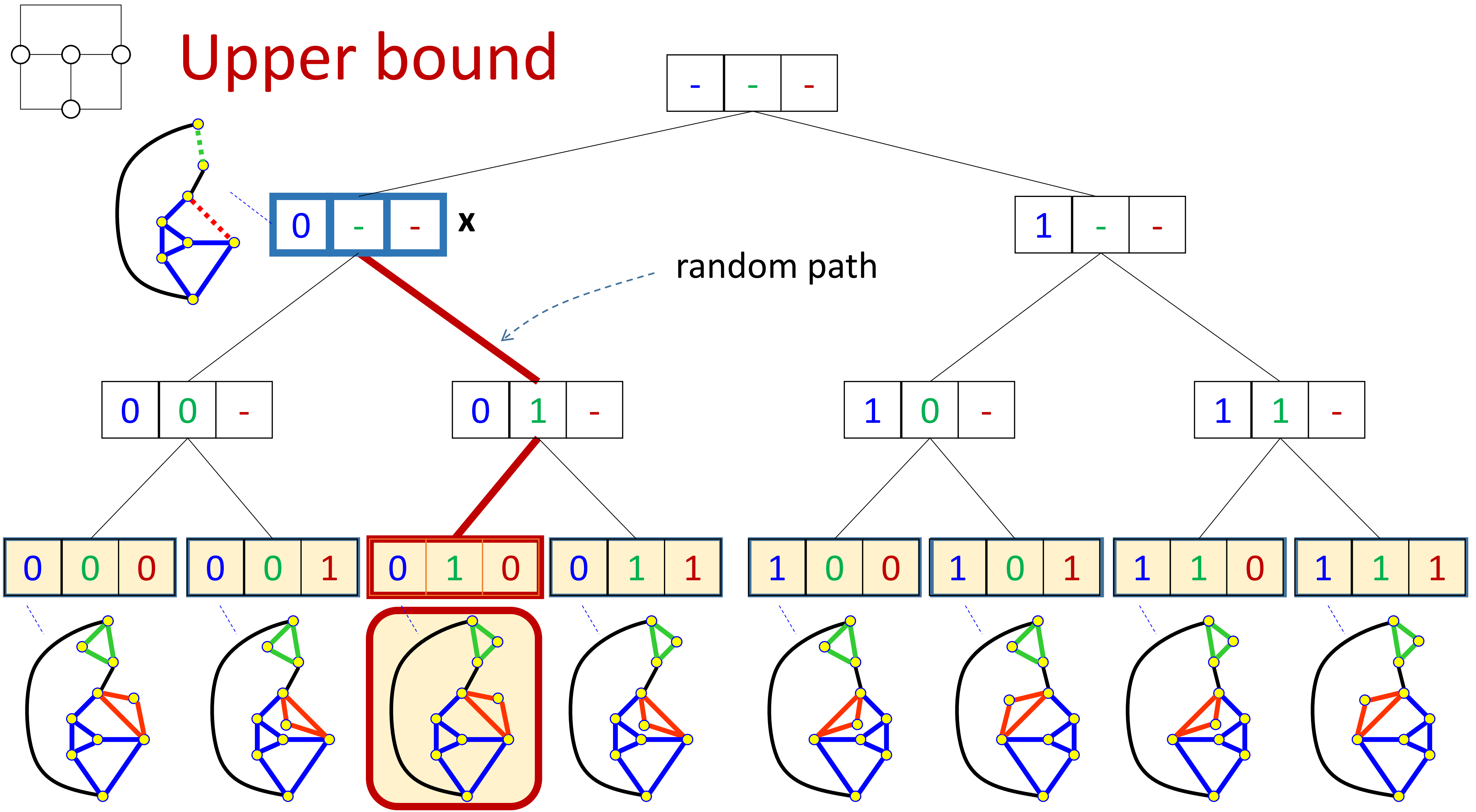
- - -
0 - -
1 - -
partial graph
partial graph
0 0 -
0 1 -
1 0 -
1 1 -
partial graph
0 0 0
0 0 1
0 1 0
0 1 1
1 0 0
1 0 1
1 1 0
1 1 1



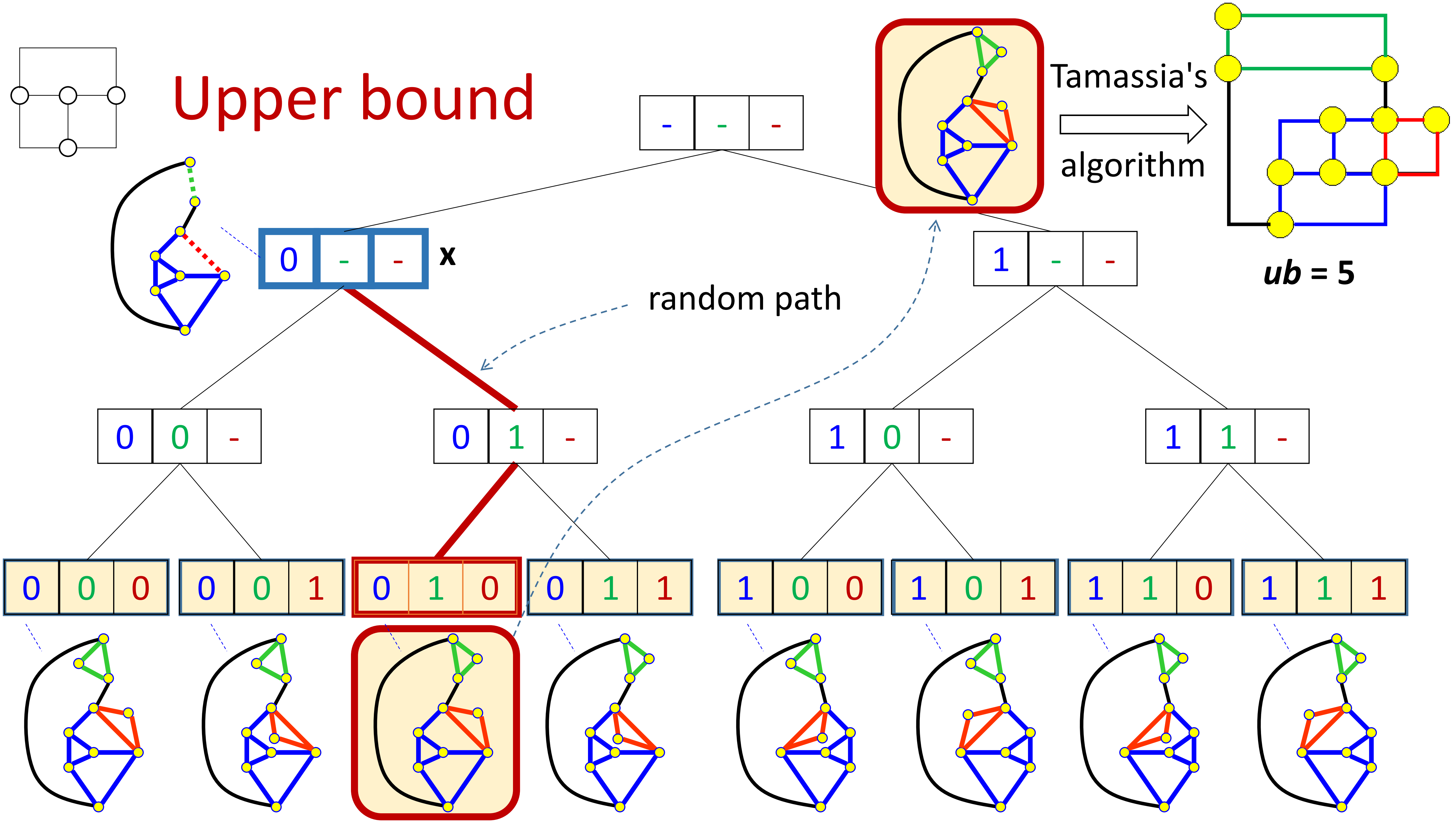
Branch-and-Bound algorithm

- mb $\leftarrow +\infty$ // minimum number of bends known so far
- visit the search tree from the root (use a BFS or DFS)
- when a node x is visited:
  - compute an upper bound *ub* on the number of bends of an orthogonal representation with embedding in the subtree rooted at x
    - **If** (ub < mb) **then** mb $\leftarrow$ ub
  - compute a lower bound *lb* on the number of bends of an orthogonal representation with embedding in the subtree rooted at x
    - **If** (lb > mb) **then** cut x and its subtree
- **return** mb

Upper bound



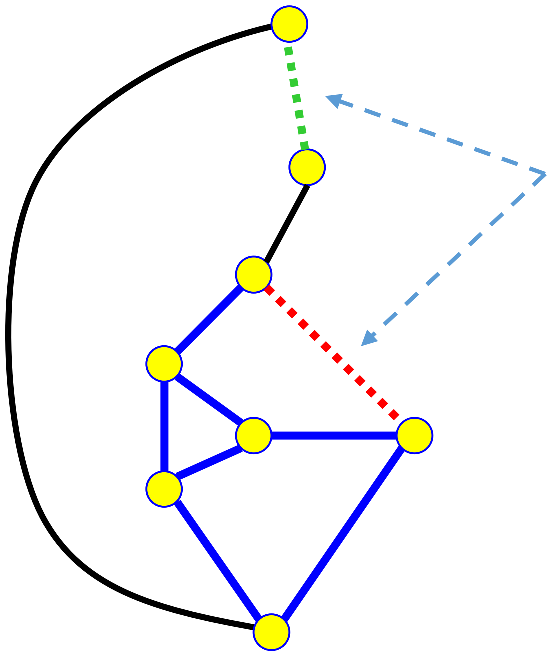
Upper bound



- - -
0 - -
x
1 - -
random path
Tamassia's
algorithm
ub = 5
0 0 -
0 1 -
1 0 -
1 1 -
0 0 0
0 0 1
0 1 0
0 1 1
1 0 0
1 0 1
1 1 0
1 1 1

Lower bound: Notation

partial graph



EV = set of virtual edges

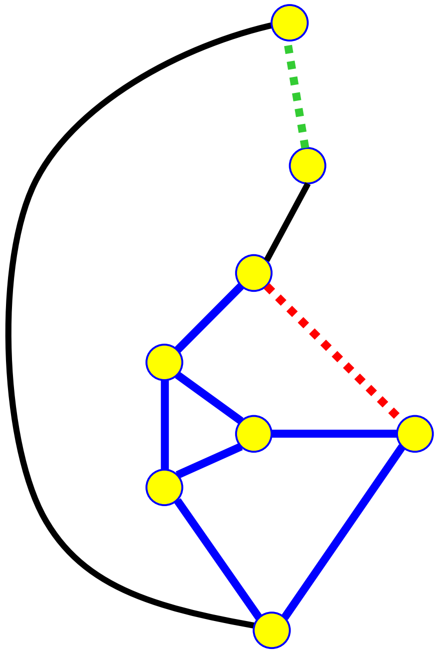
ER = set of real edges

$\mathbf{b_{ER}(H)}$= # of bends of H along the real edges

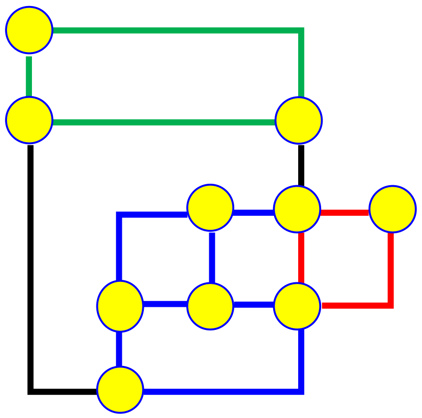
Lower bound: Preliminary lemma

- H' = representation of G' with minimum bends on ER
- H = bend-min representation of G that preserves the embedding of G'

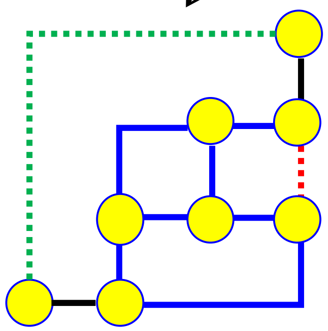
partial graph G'



$$\mathbf{b_{ER}(H') \leq b_{ER}(H)}$$



b_{ER} (H) = 3



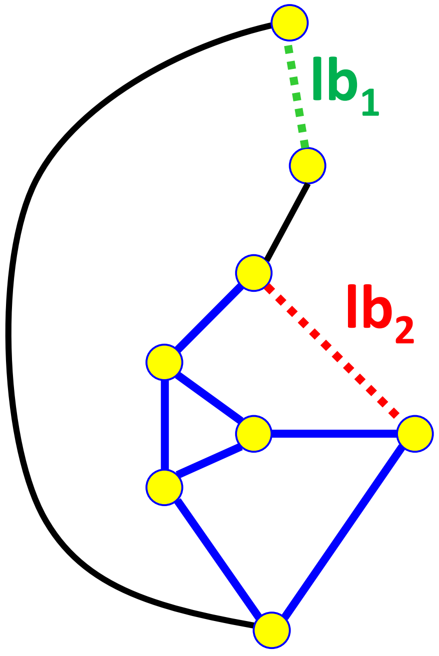
b_{ER} (H') = 2

b_{ER}(H') can be computed by imposing cost 0 for the bends on the virtual edges in Tamassia's flow network

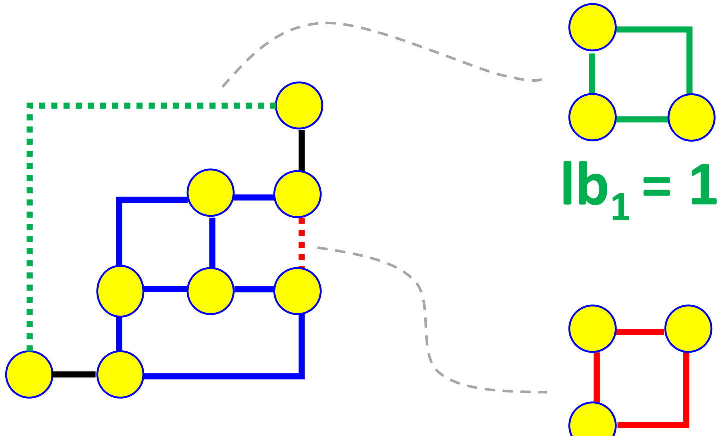
Lower bound: Recursive approach

- **lb_i**= lower bounds on the # of bends in the pertinent graph of a component G_i

partial graph G'



$$\boldsymbol{lb = b_{ER}(H') + \Sigma_i\, lb_i}$$



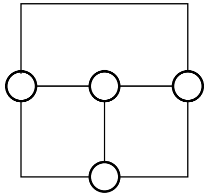
b_{ER} (H') = 2

$lb_1 = 1$

$lb_2 = 1$

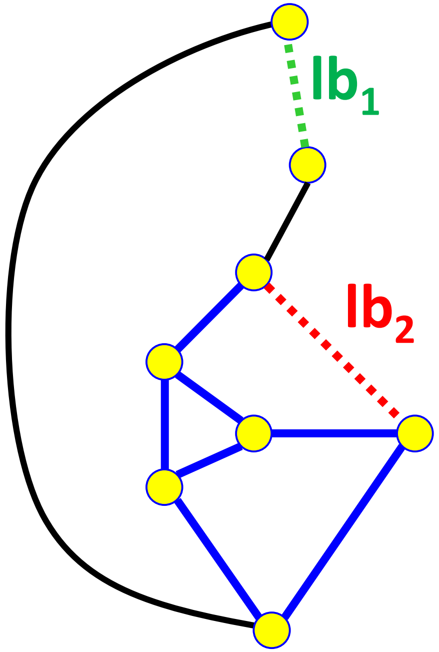
$lb = 4$

the set of **lb_i** can be computed through a bottom-up visit of the SPQR-tree in a pre-processing step



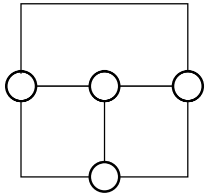
Lower bound: Further improvement

partial graph G'



- If some lb_i is zero, replace the corresponding virtual edge with a simple path π between the poles of G_i and regard the edges of π as real edges

$$\boldsymbol{lb = b_{ER}(H') + \Sigma_i\, lb_i}$$



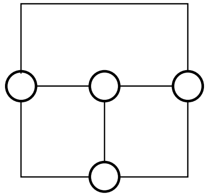
Some experimental data

density/vertices	10	20	30	40	50	60	70	80	90	100
1.1	6	10	10	25	25	10	10	4	13.33	0
1.2	37.5	32.38	27	26.33	41.3	38.67	32.1	17.32	33.28	31.76
1.4	20.82	22.31	19.99	19.92	22.35	28.99	24.88	16.59	20.36	14.2
1.6	19.75	15.05	20.76	12.16	13.14	12.4	15.92	11.87	14.61	12.65
1.8	13.04	11.05	10.46	10.08	8.15	9.94	4.07	4.77	4.21	

% avg. improvement on the number of bends w.r.p. to a bend-minimum orthogonal drawing in the fixed embedding setting

Additional reading

- *P. Mutzel, R. Weiskircher*: Bend Minimization in Planar Orthogonal Drawings Using Integer Programming. SIAM Journal on Optimization 17(3): 665-687 (2006)



Bend-min of planar 4-graphs: Open problem

- **Problem**: Let G be a biconnected 4-planar graph with a given combinatorial embedding. is there an $o(n^{2.5})$-time algorithm that computes a bend-minimum orthogonal drawing of G overall possible choices of the external faces? (the combinatorial embedding is preserved)