Flips in Plane Graphs
Old Problems, New Results

Oswin Aichholzer

Graz University of Technology

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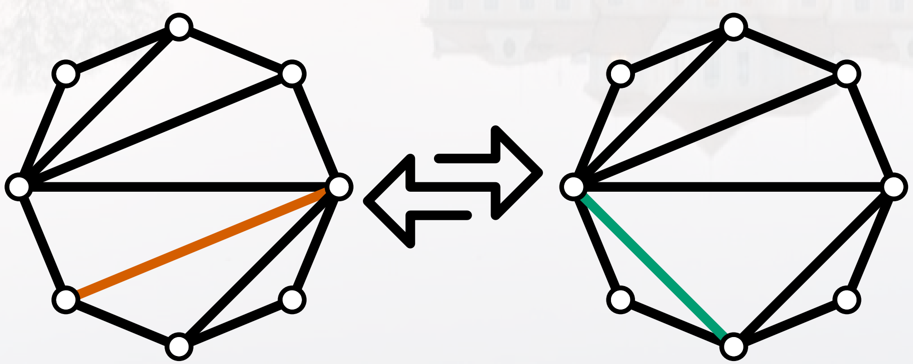

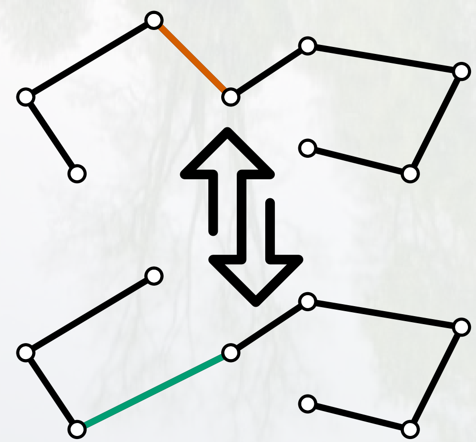

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Reconfiguration: process of changing a structure (configuration) into another - either through continuous motion or through **discrete changes, so-called flips**

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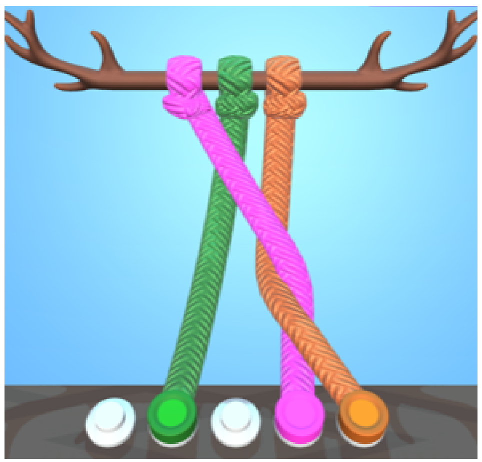

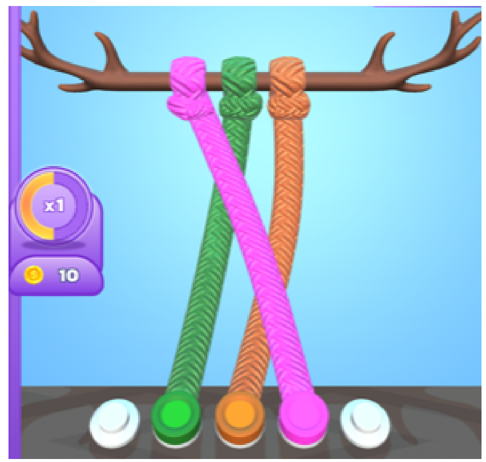

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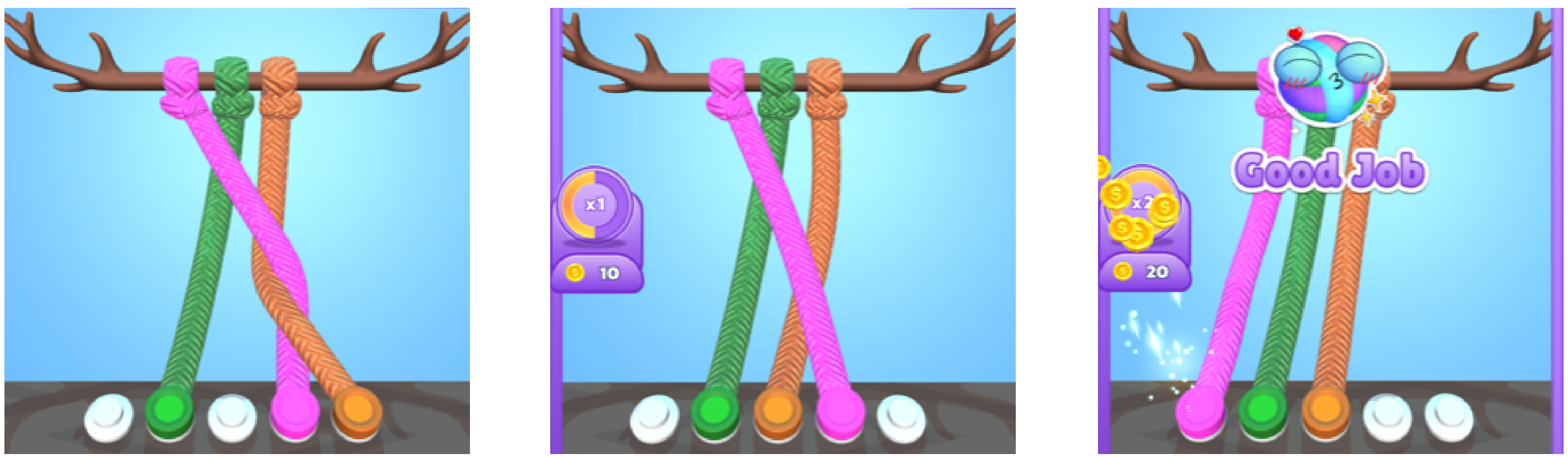

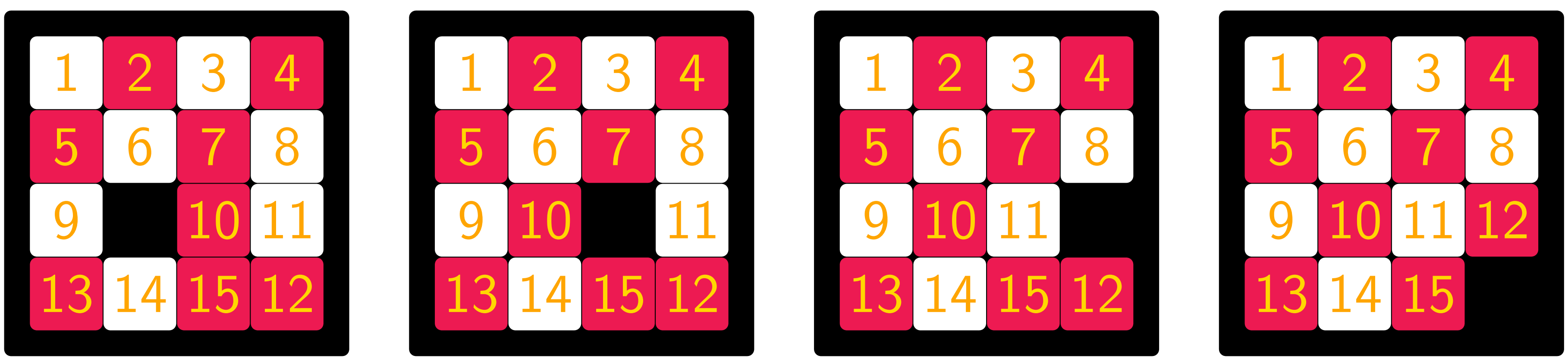

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  plane drawings of straight line graphs, labeled vertices
- Reconfiguration step:
  flip = exchange of (a bounded number of) edges

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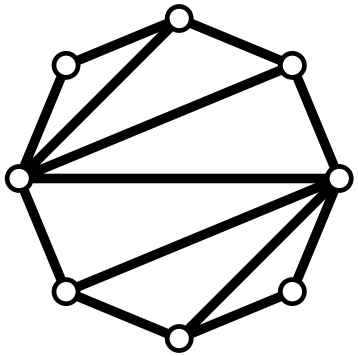

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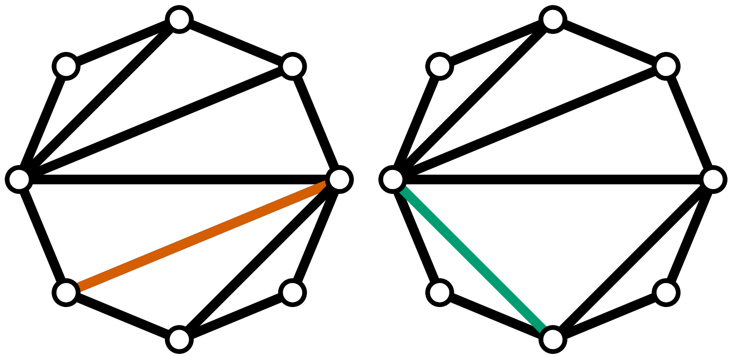

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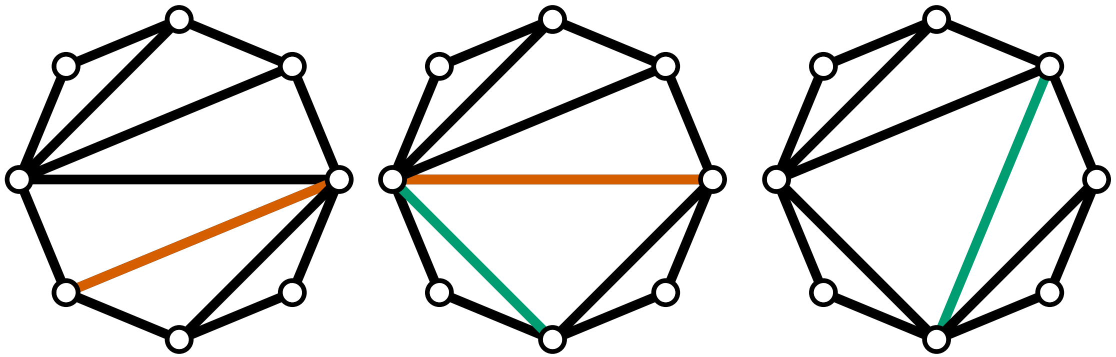

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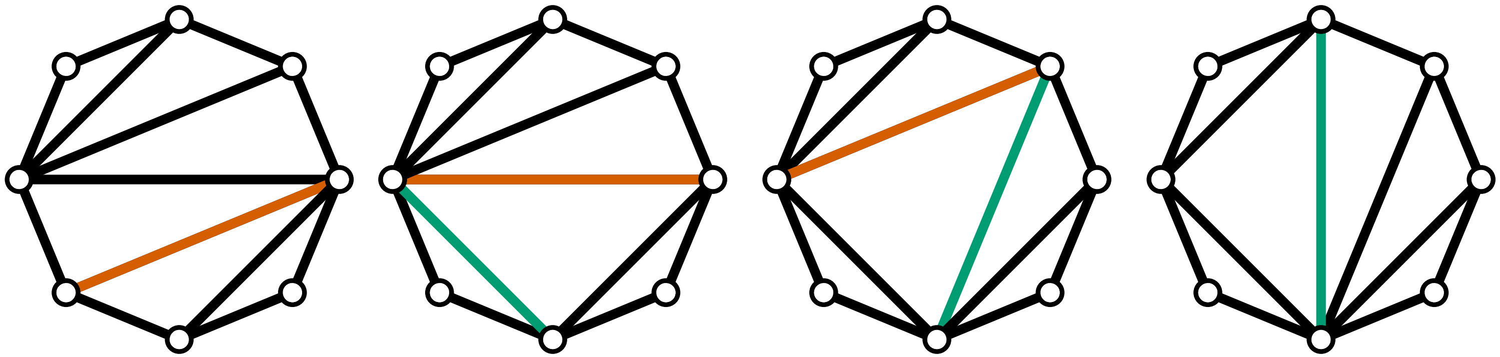

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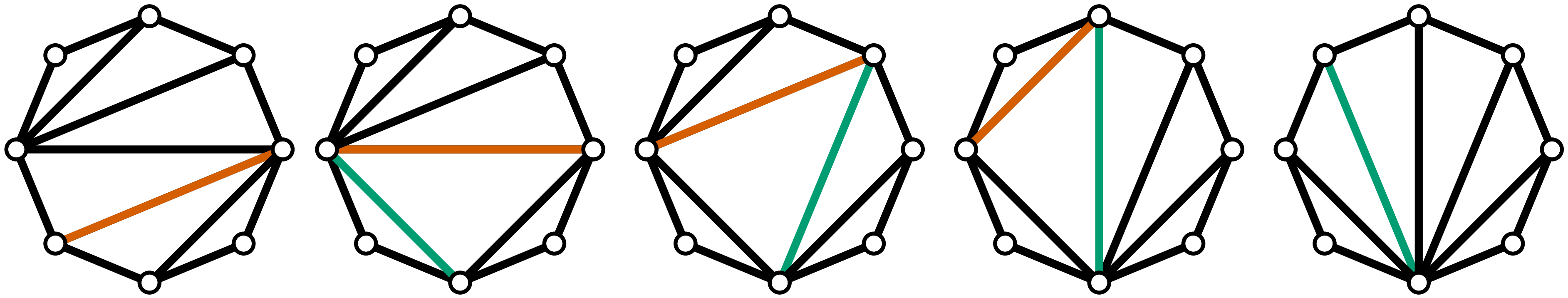

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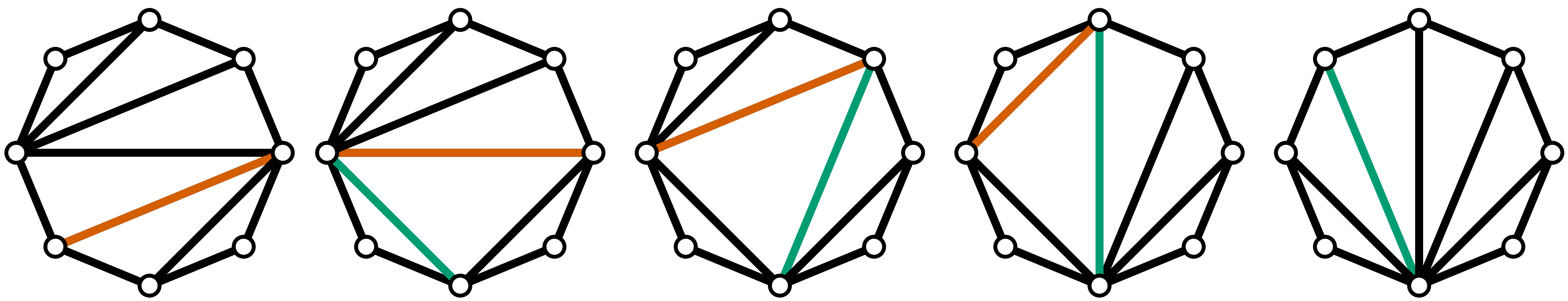

We always assume points to be in general position, i.e., no three points are on a common line

Reconfiguration

Three central questions for a set of configurations:

- Can we transform from any configuration to any other?
- How long does it take in the worst case?
- How long for a specific pair of configurations?

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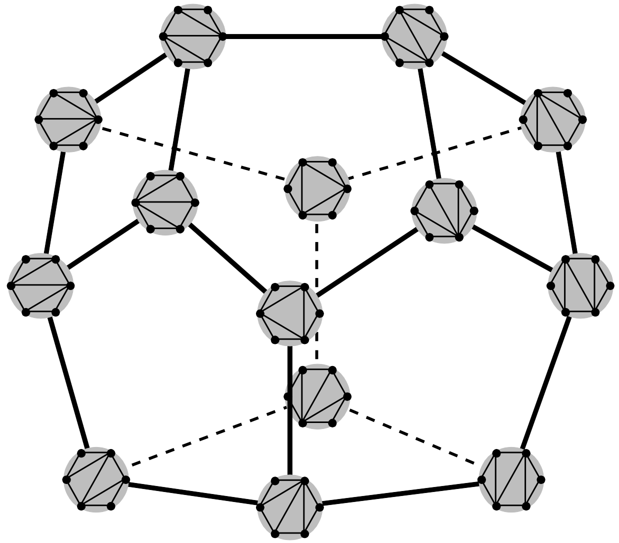
- vertex: each configuration (e.g. triangulation)
- edge: reconfiguration step (e.g. edge flip)



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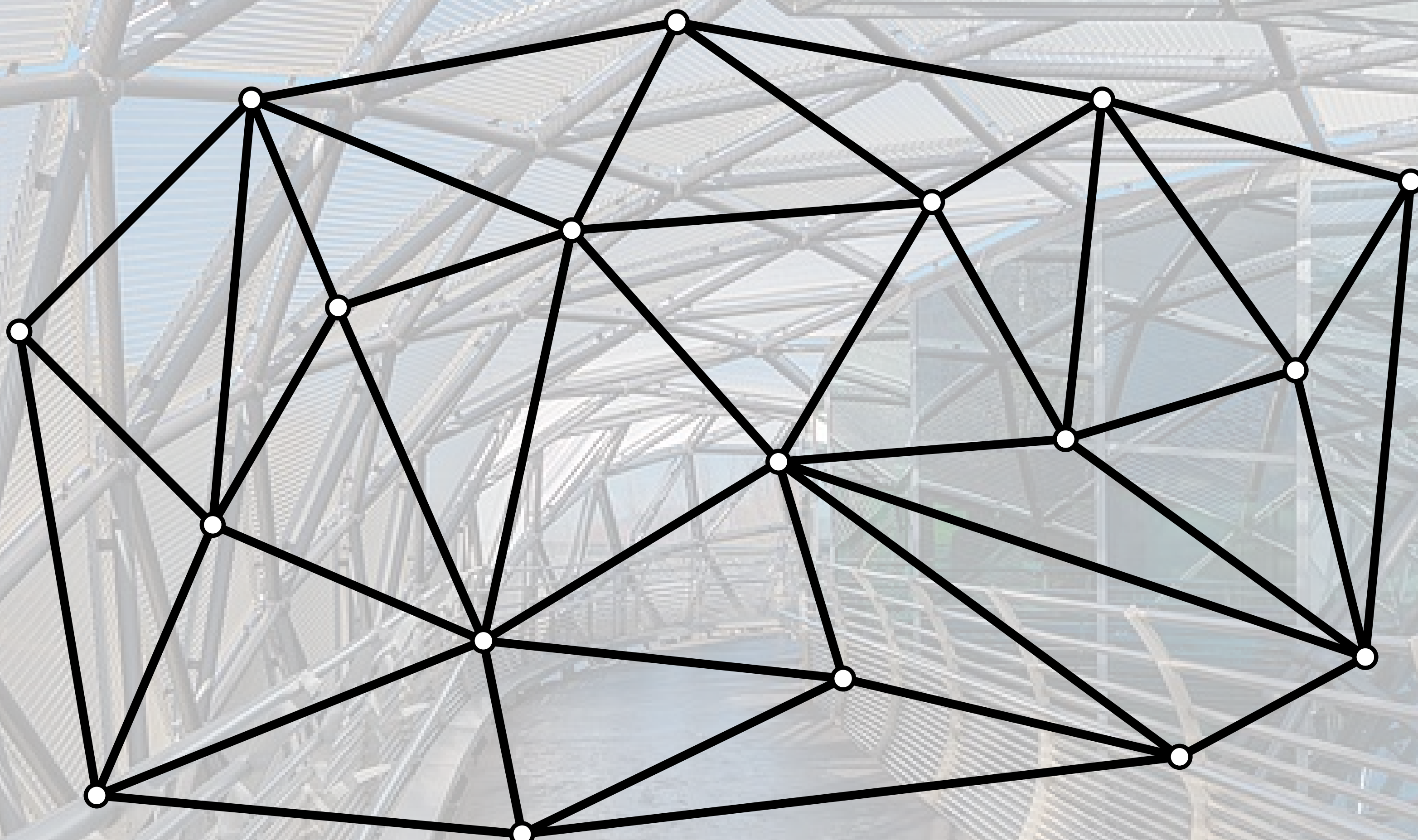
Reconfiguration graph:

- vertex: each configuration (e.g. triangulation)
- edge: reconfiguration step (e.g. edge flip)

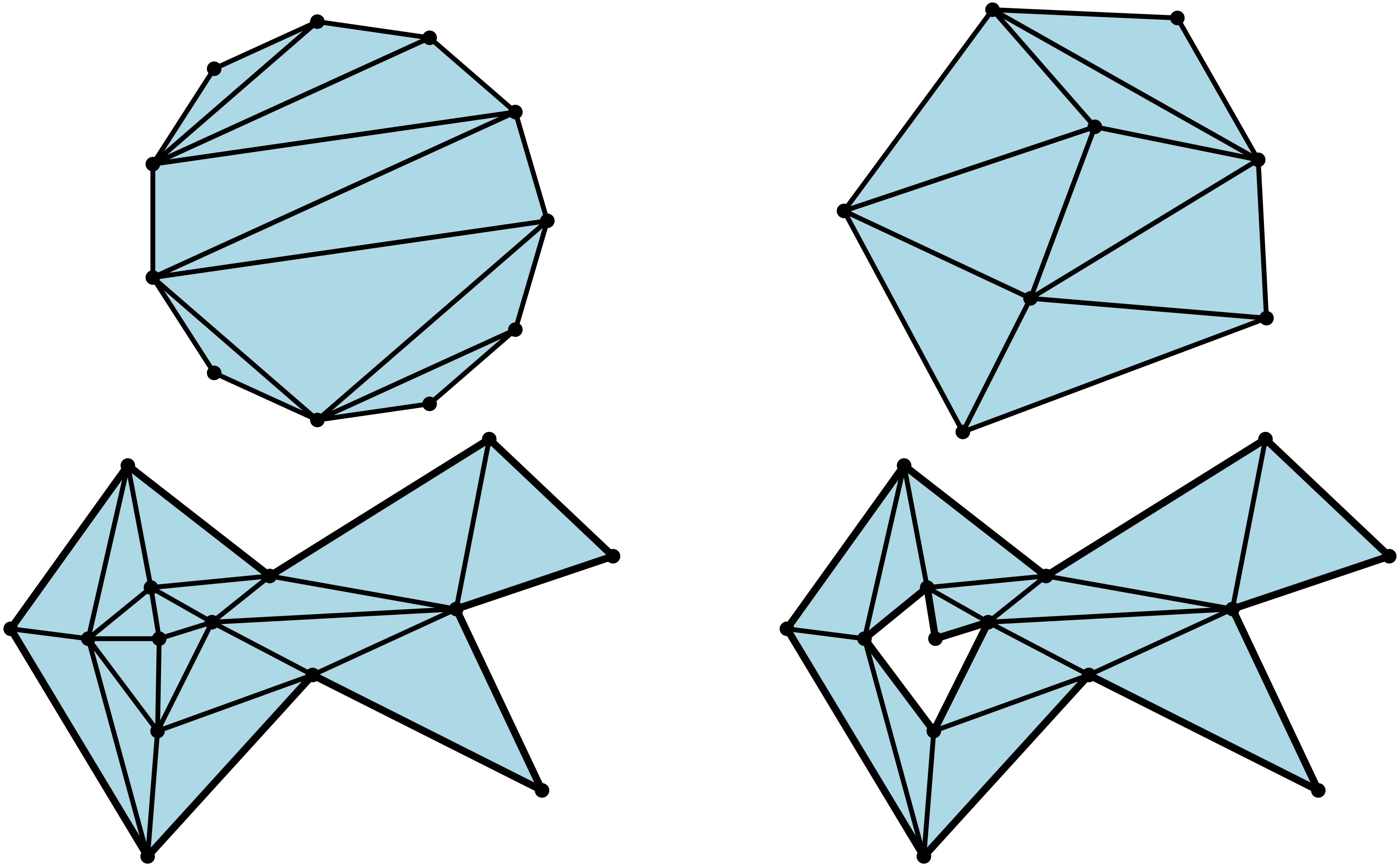
- Is the flip graph connected?
- What is the diameter (or radius) of the flip graph?
- What is the complexity of finding the shortest flip sequence between two given elements of the flip graph?

Triangulations



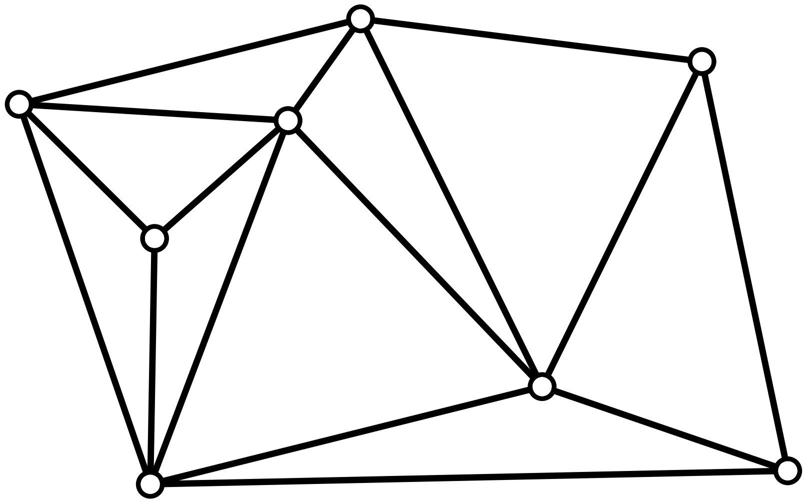
Sometimes also called "near-triangulation": straight-line embedding in the plane, where the outerface needs not to be a triangle

Triangulations



Triangulations – Edge Flip

Edge flip in triangulations:



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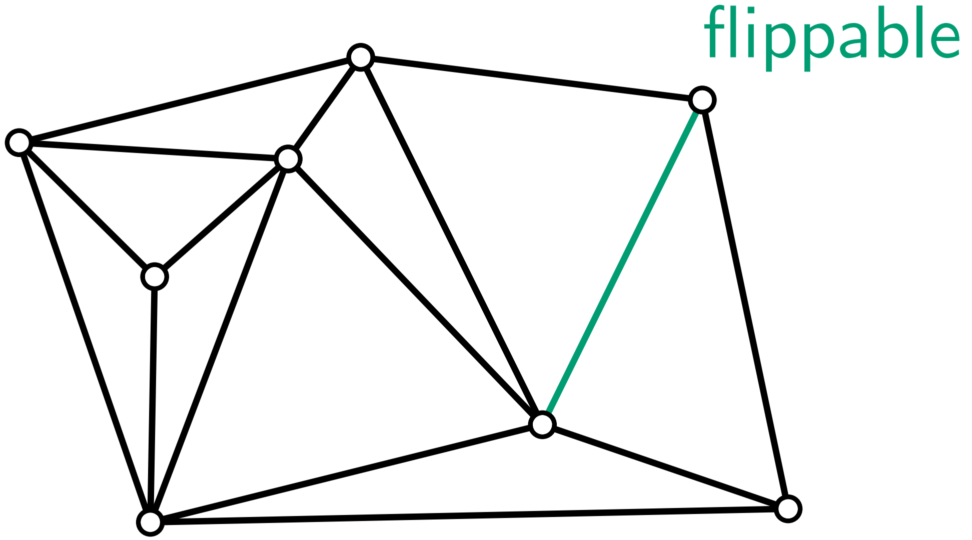

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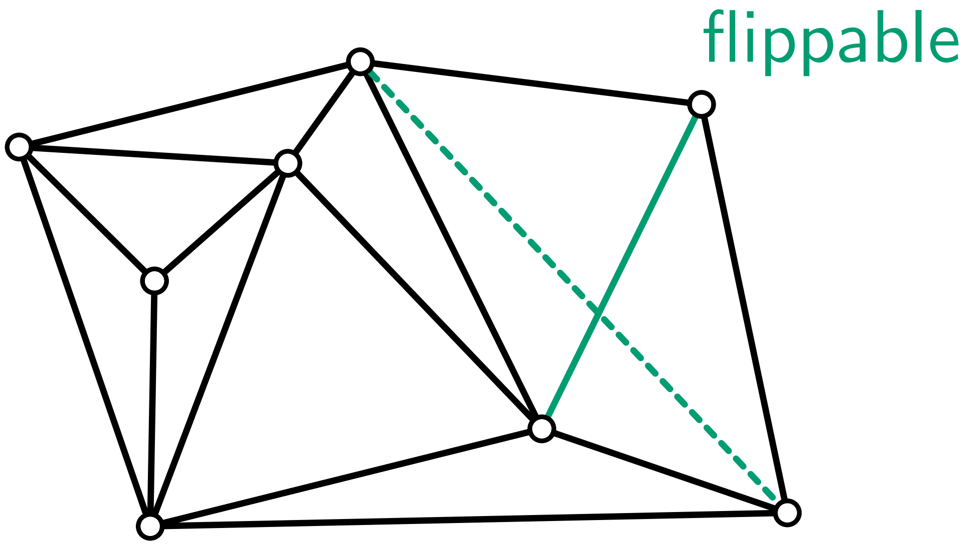

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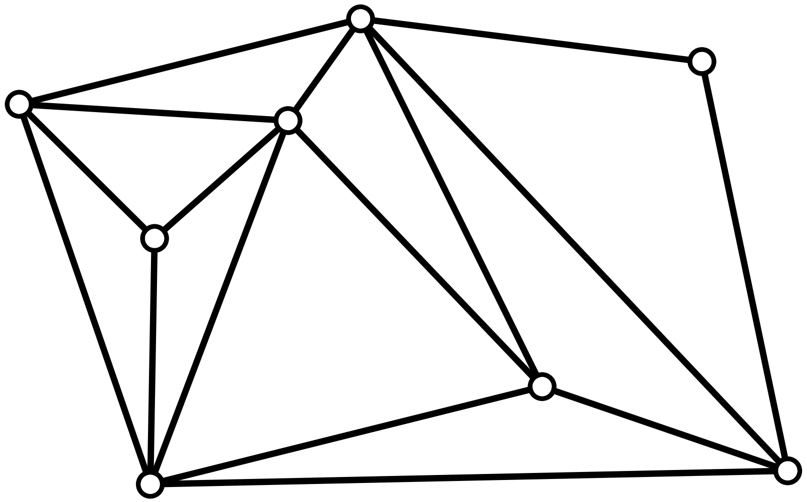

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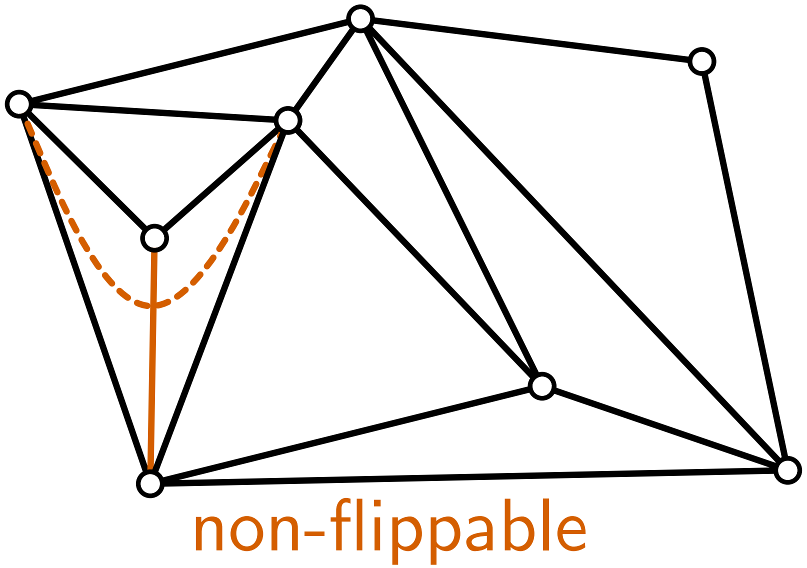

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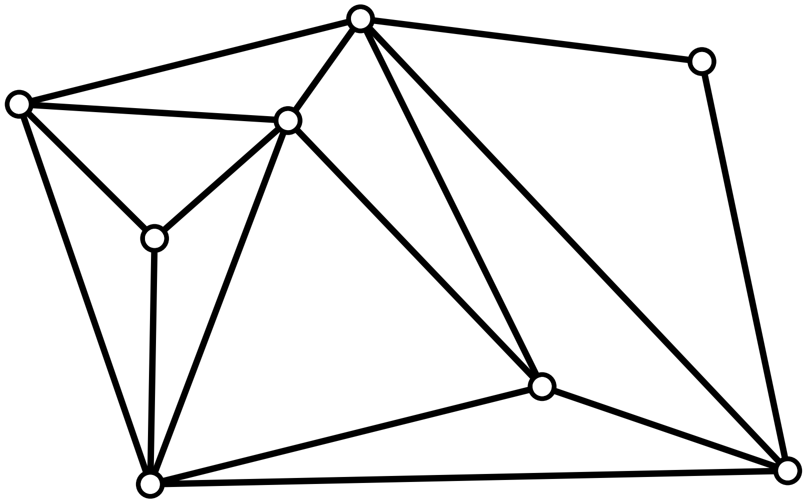

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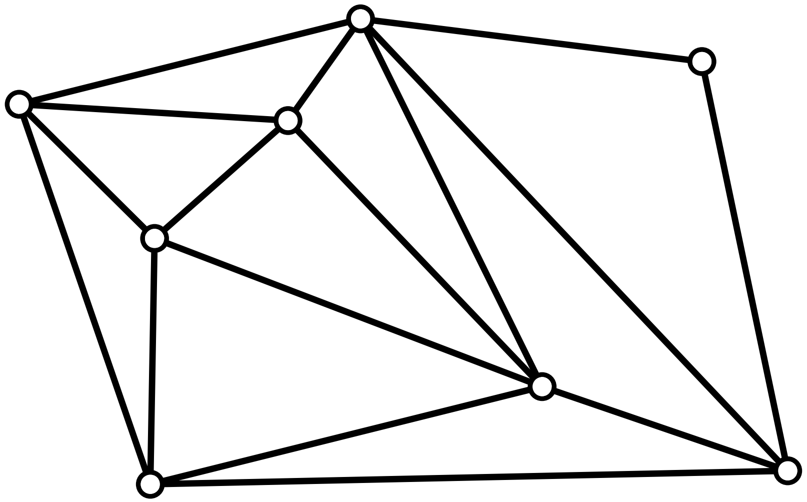

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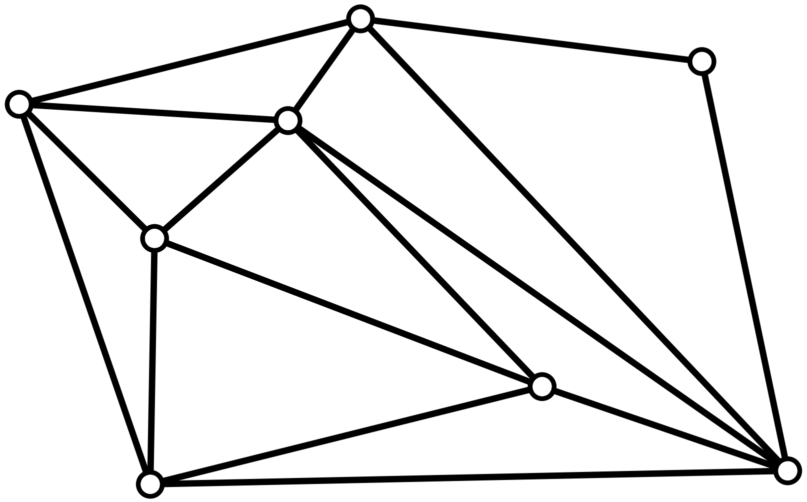
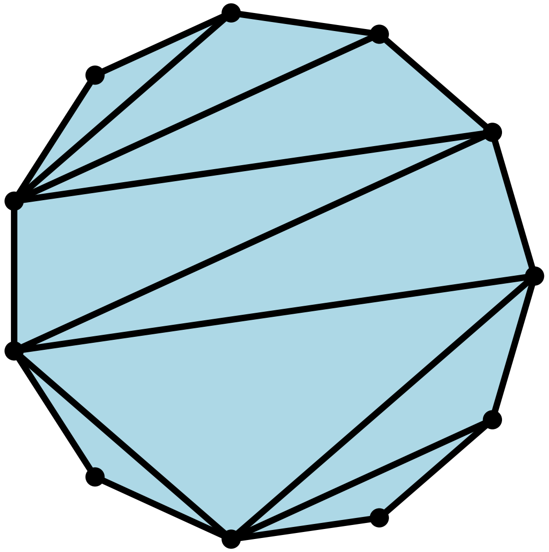
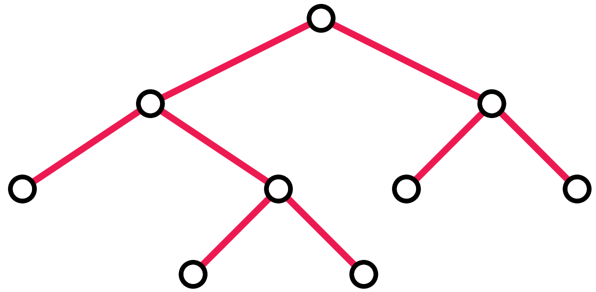


Triangulations – Convex Position

The convex case: bijection with binary trees.





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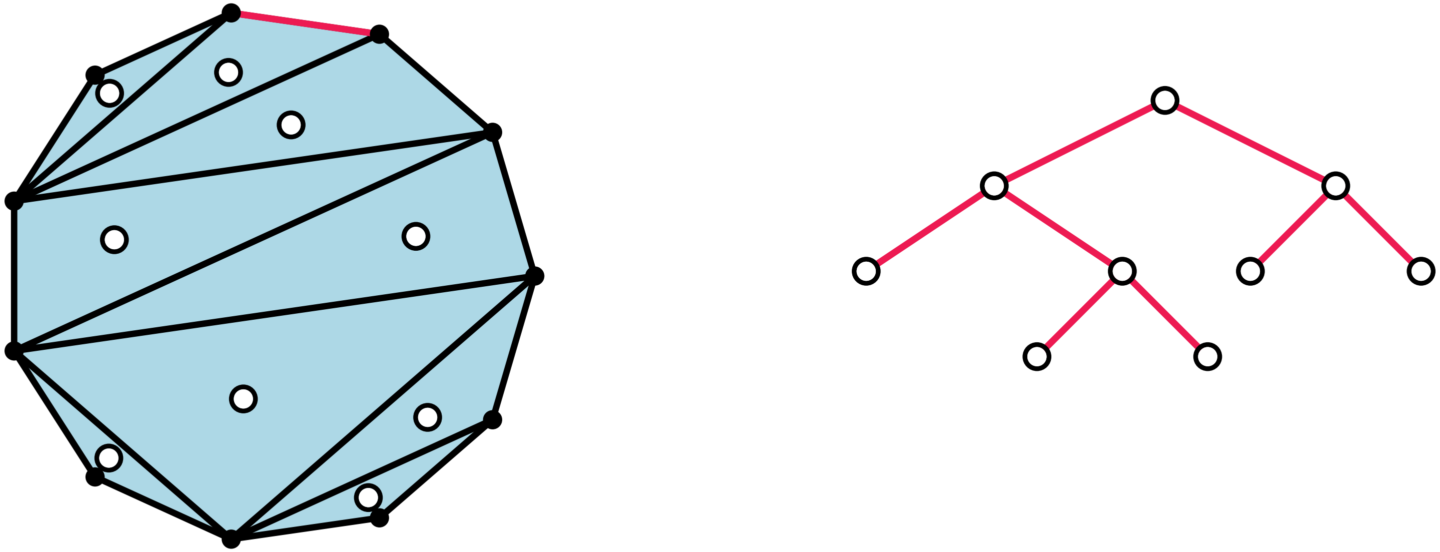

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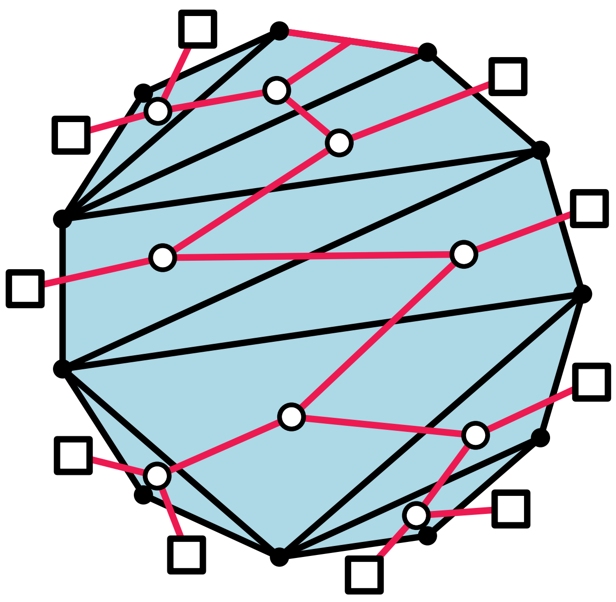

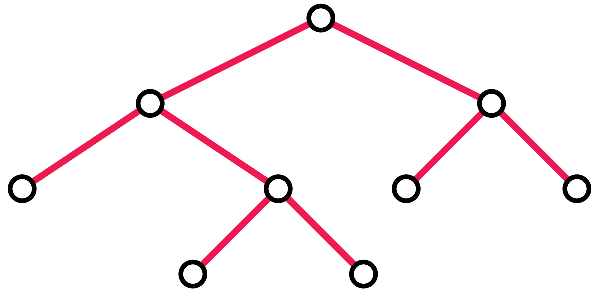

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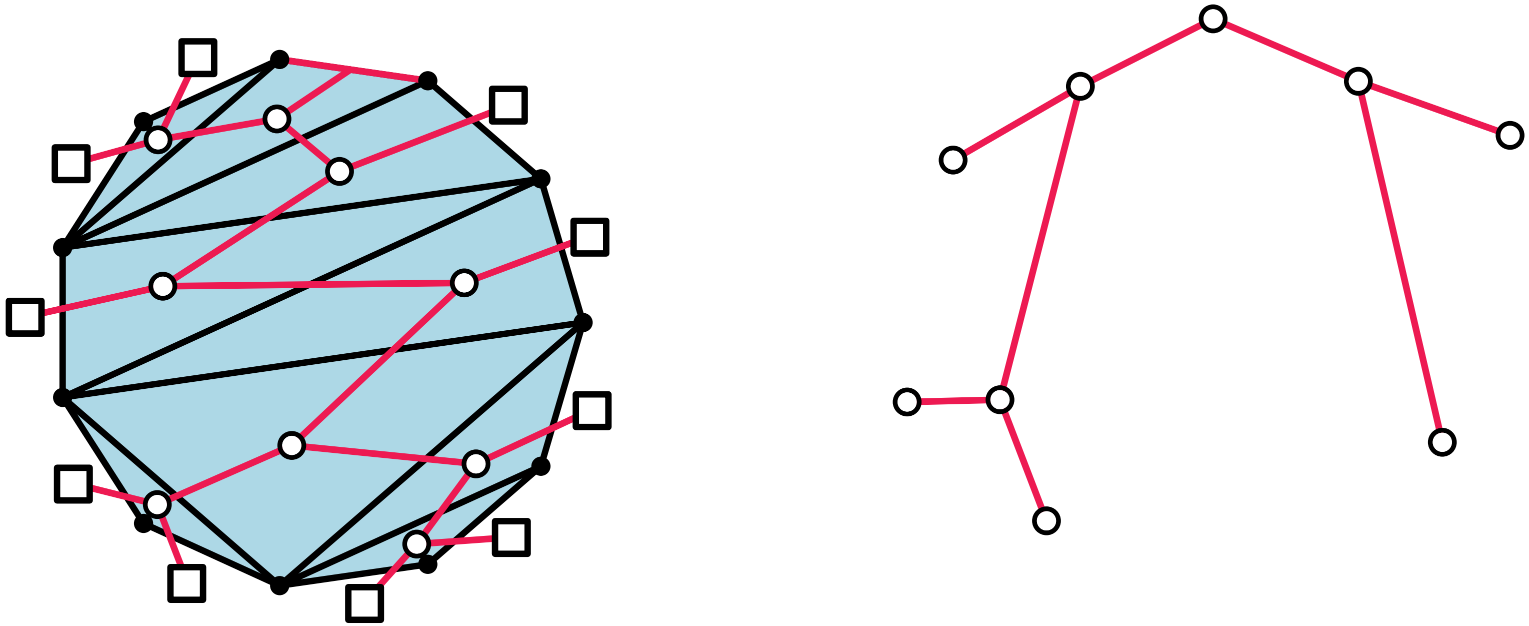

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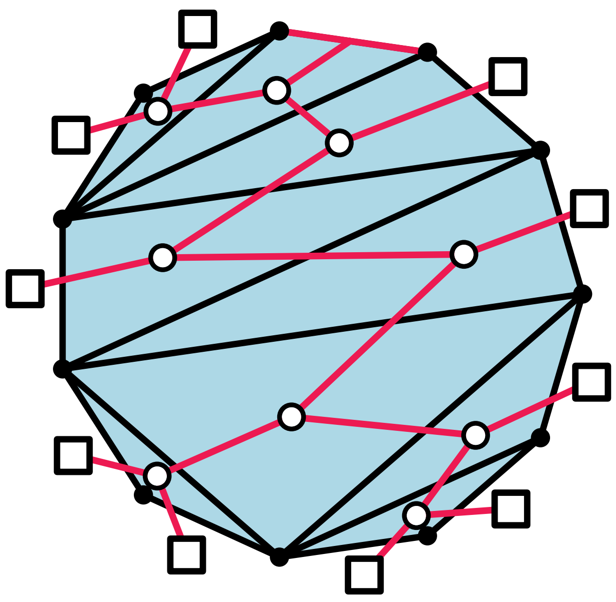

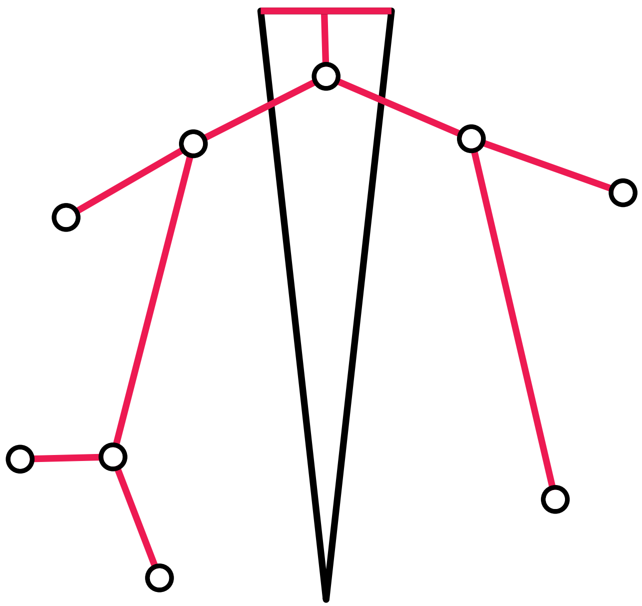

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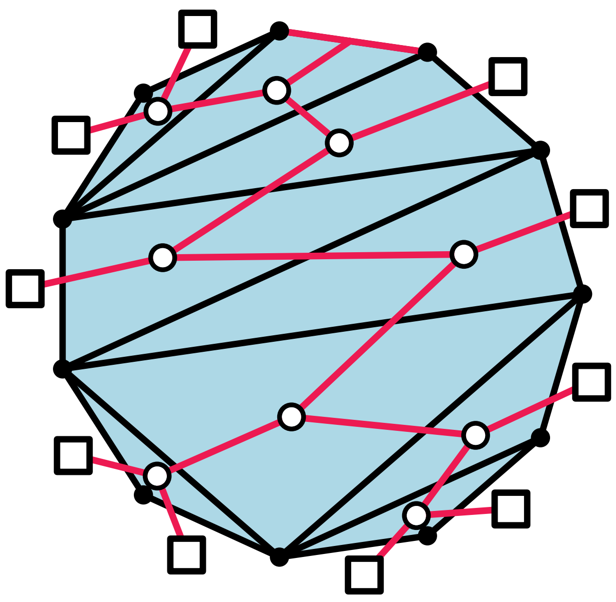

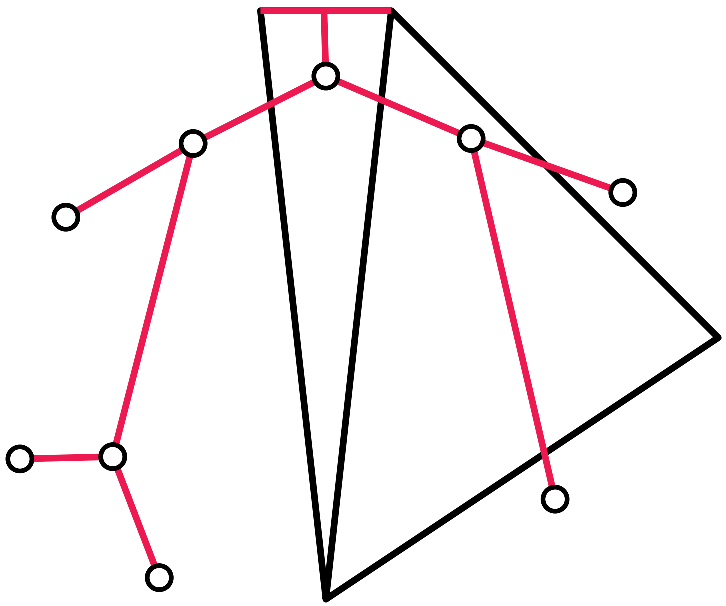

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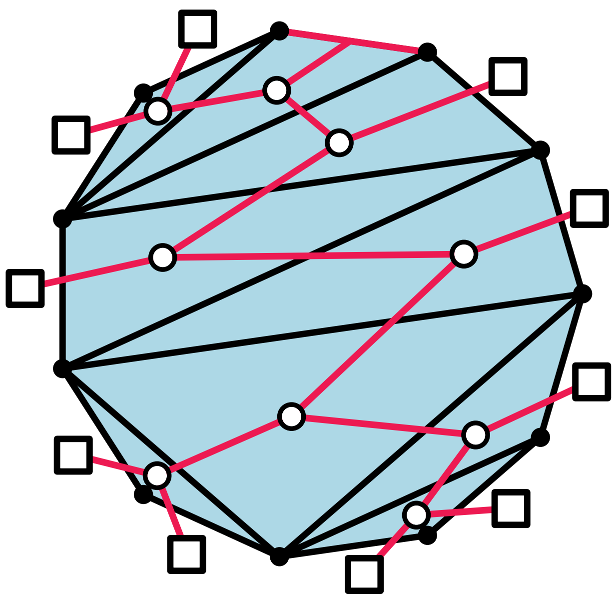

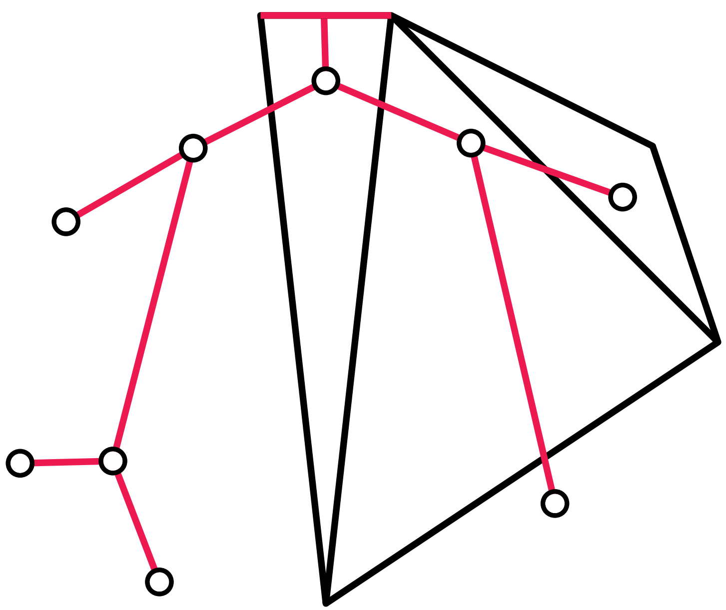

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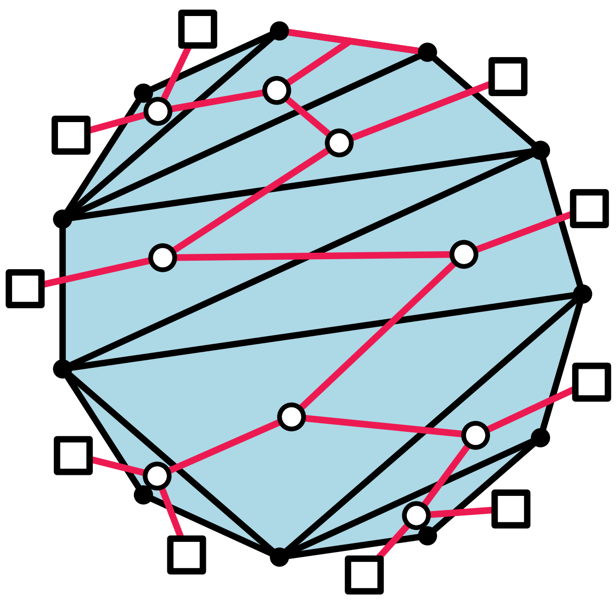

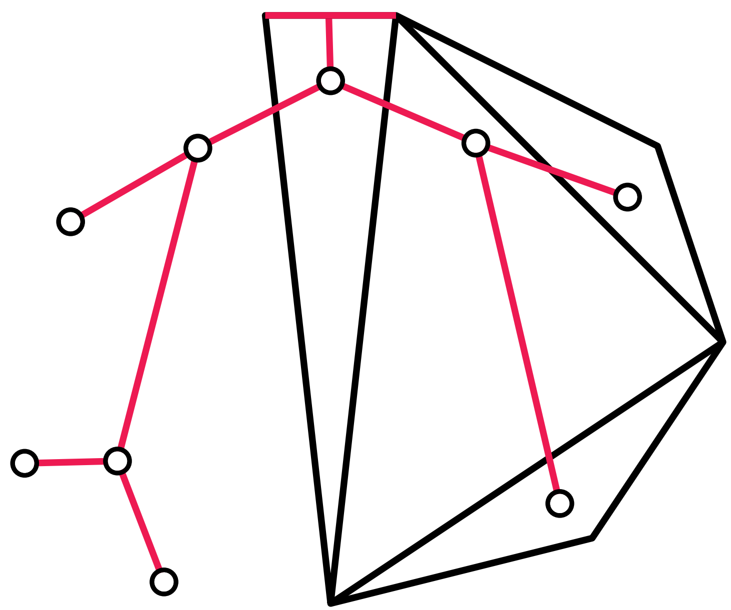

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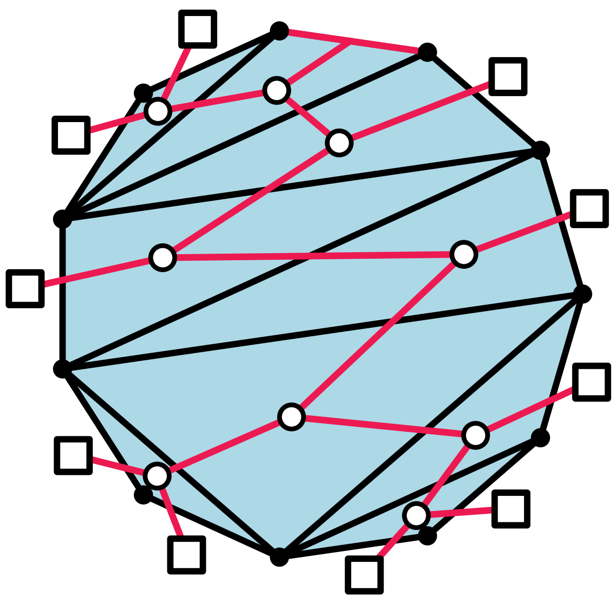

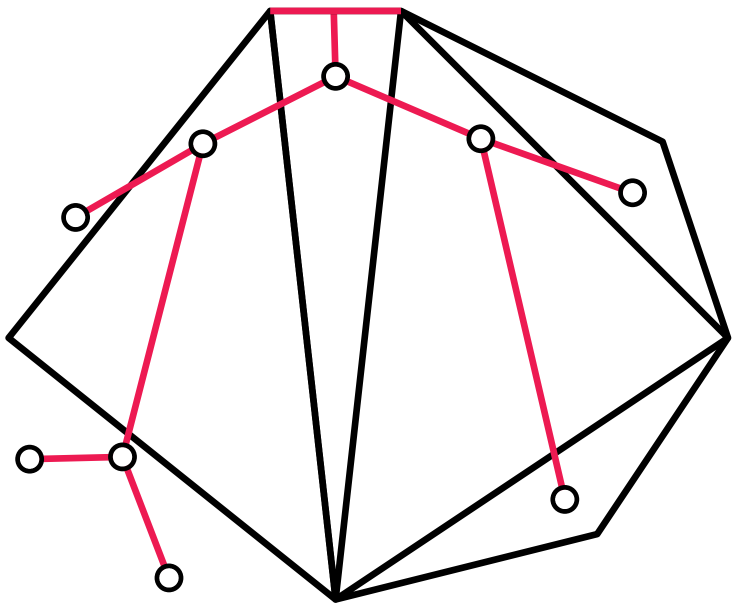

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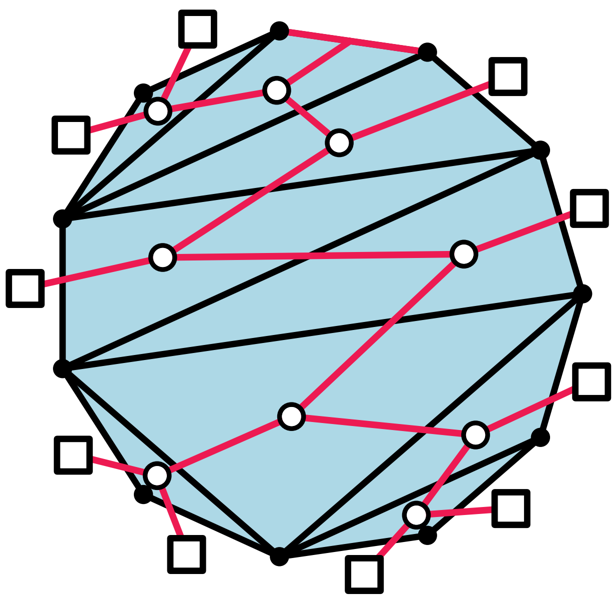

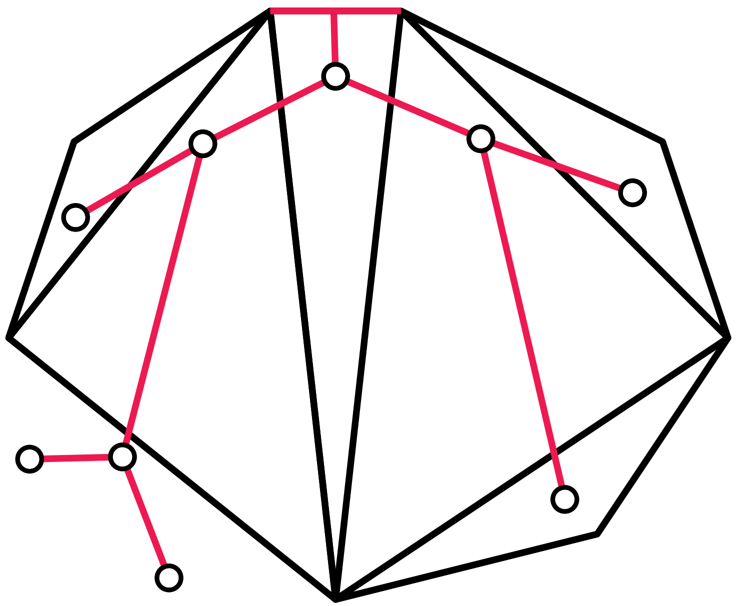

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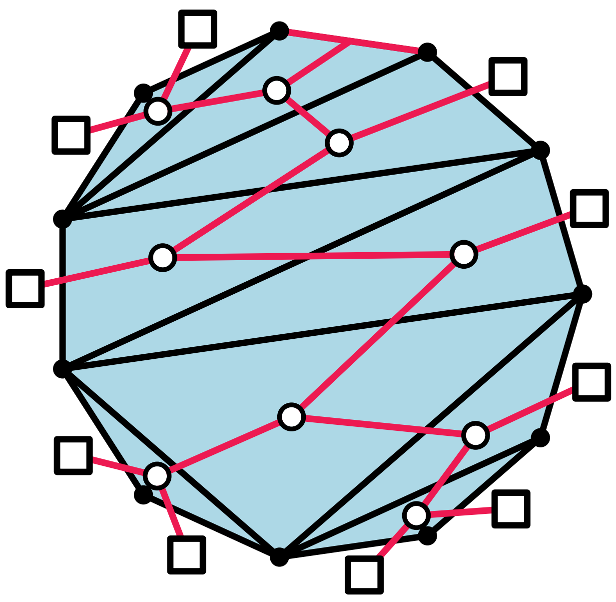

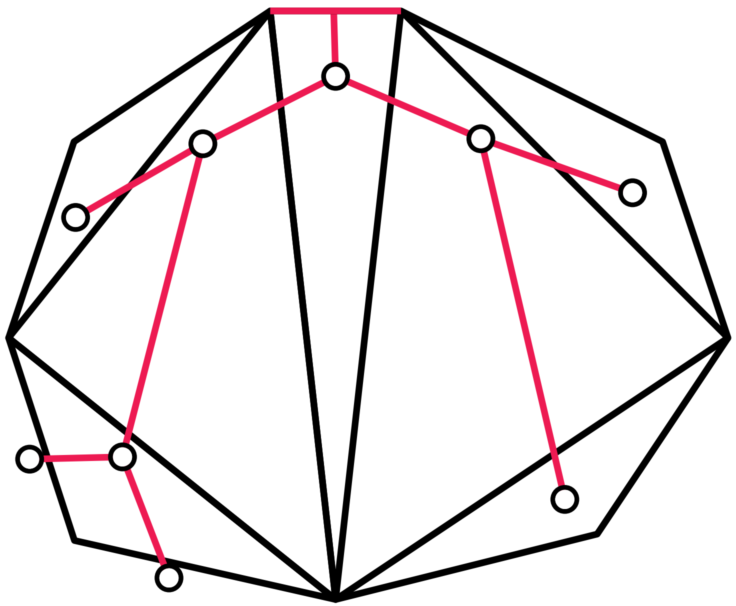

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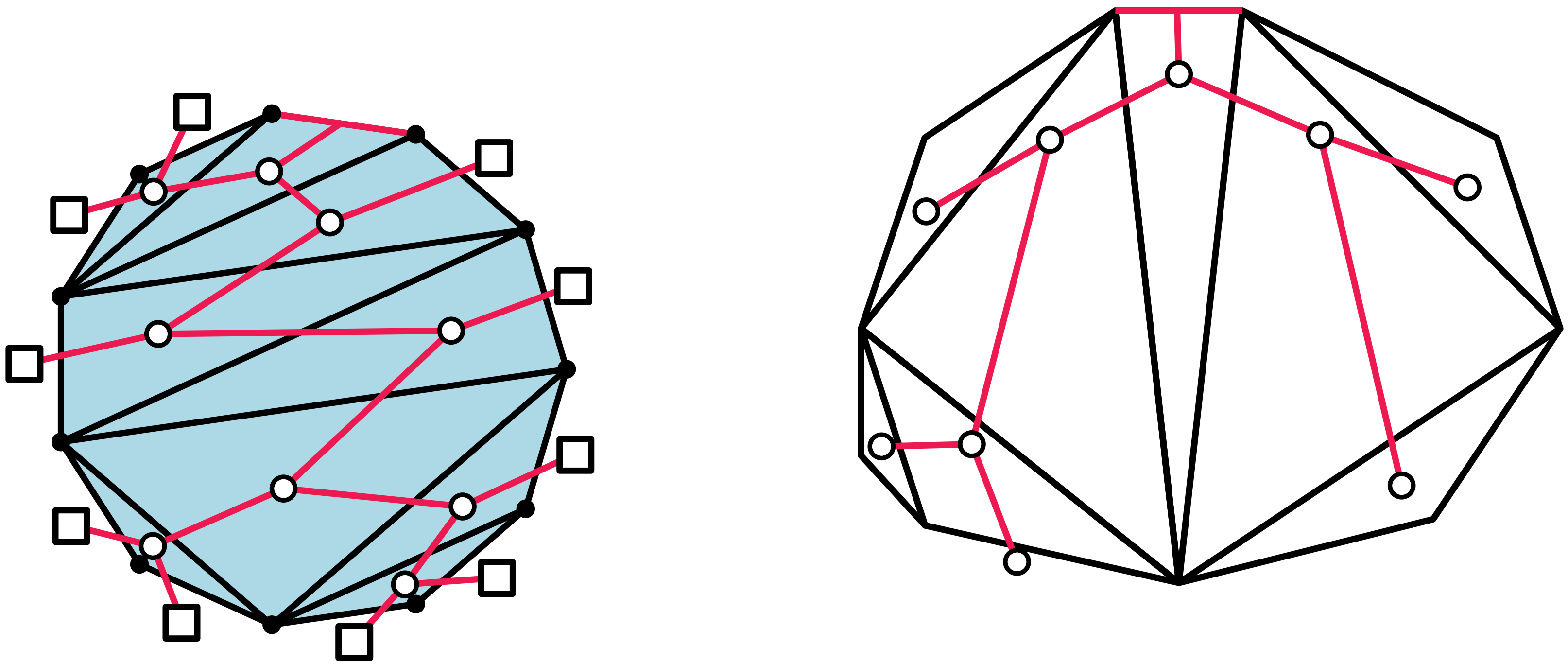

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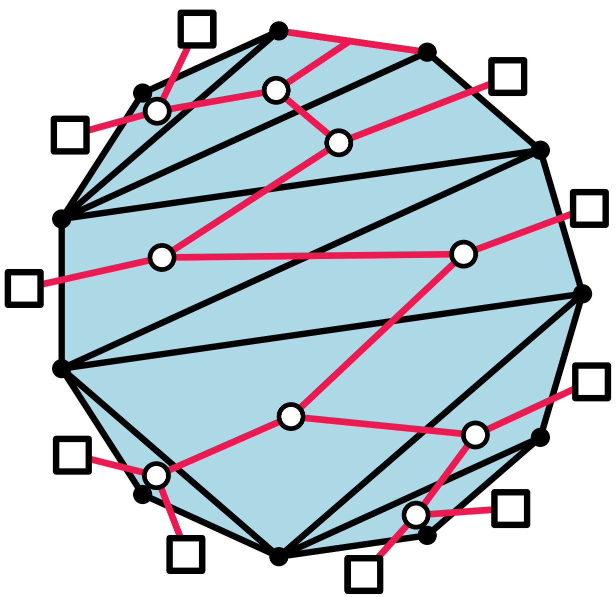

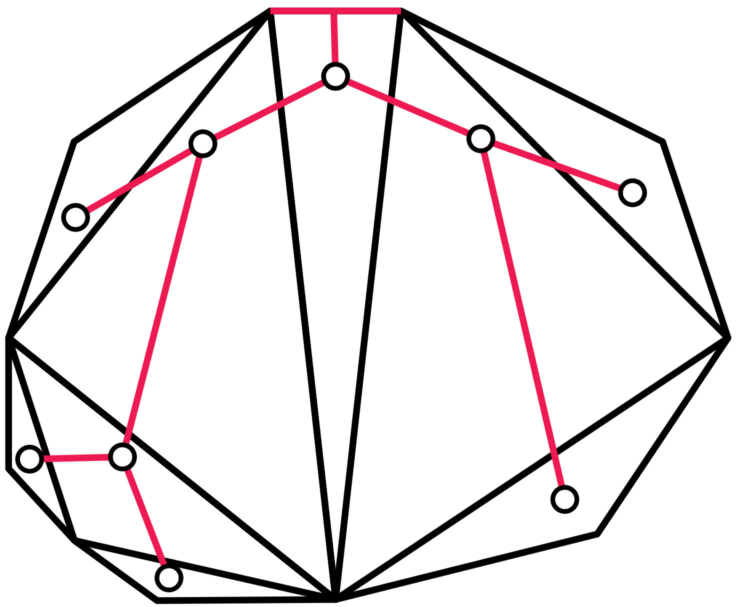

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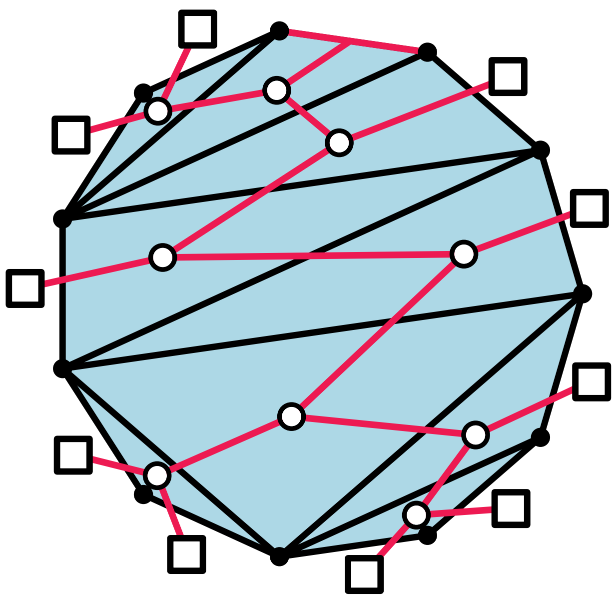
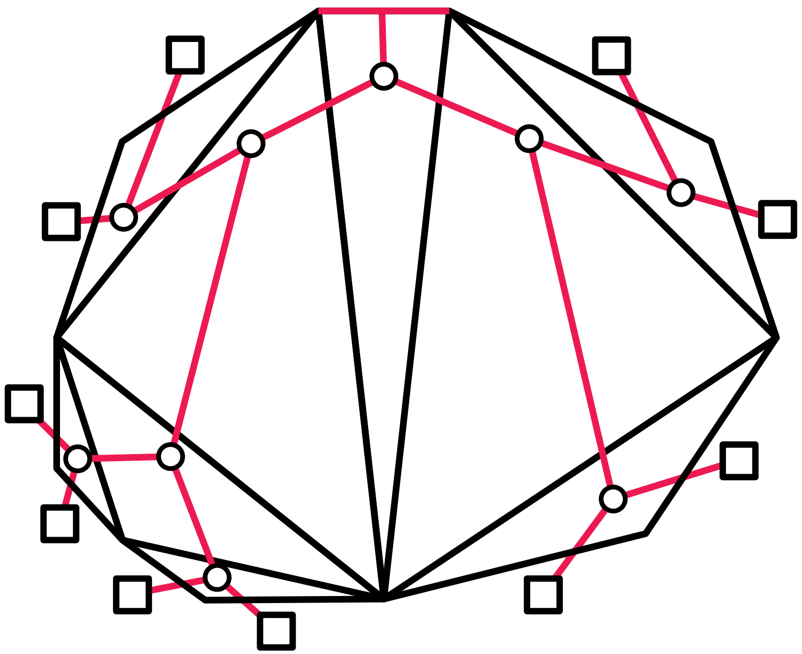
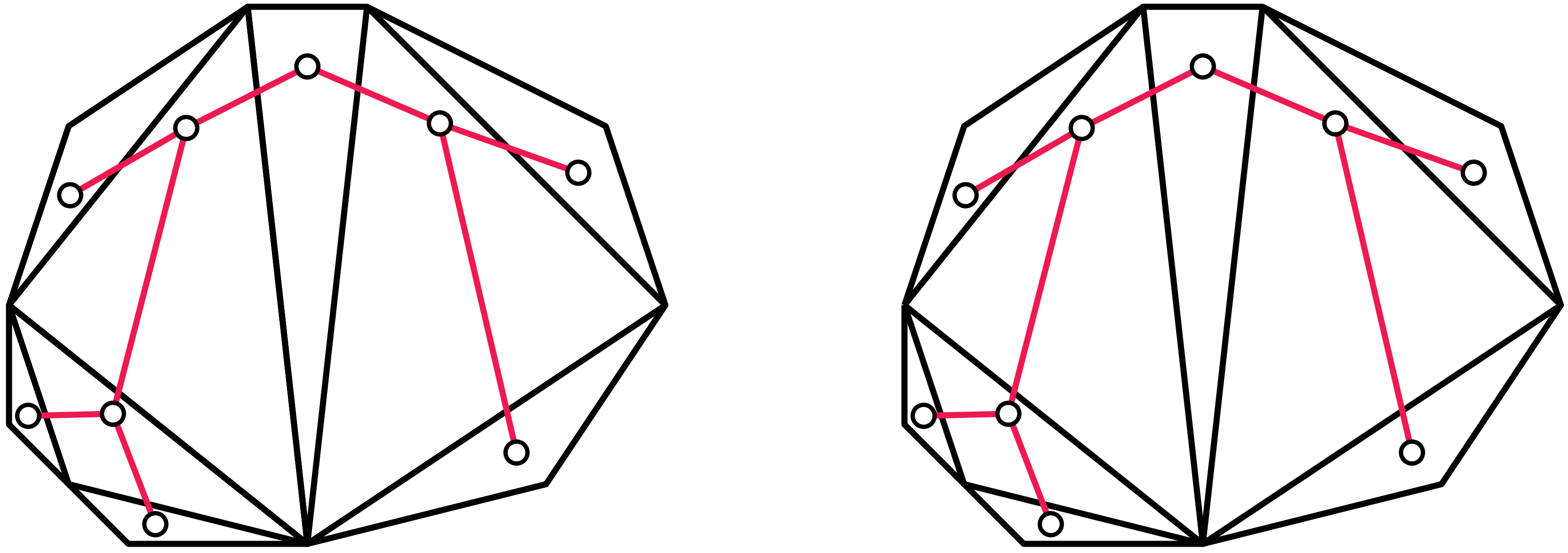




Edge Flip – Rotation in Binary Tree



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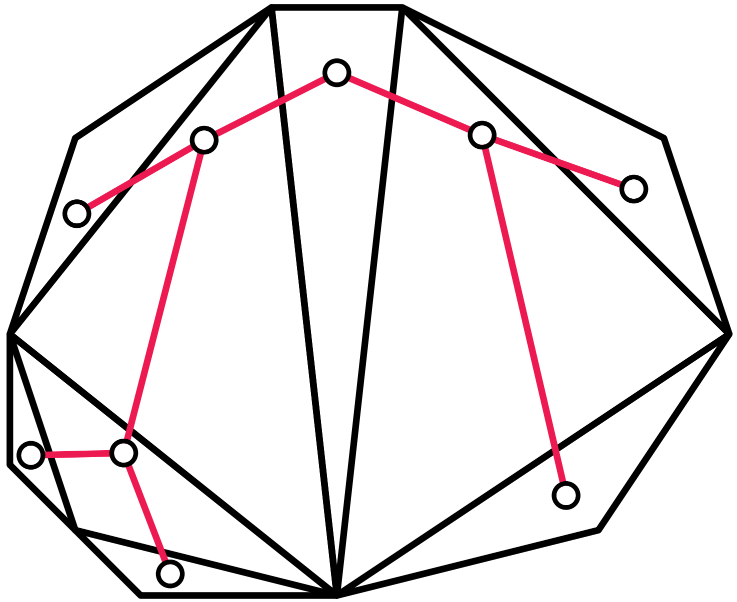

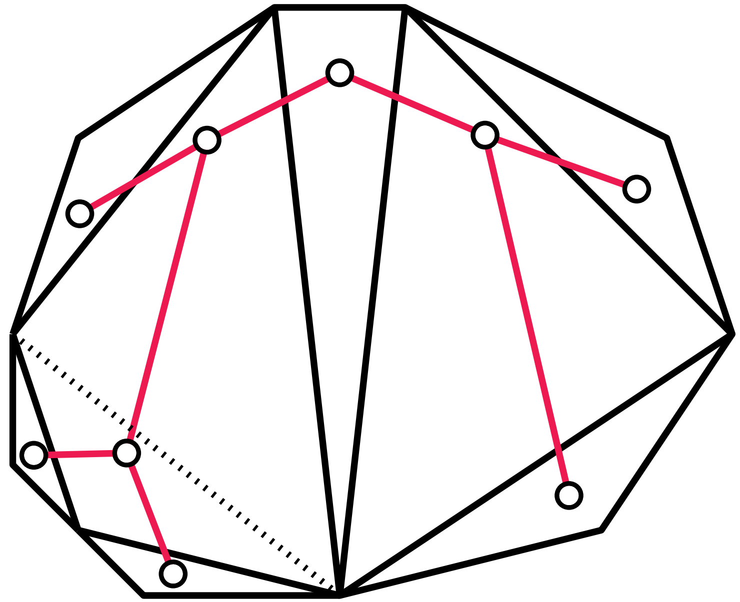

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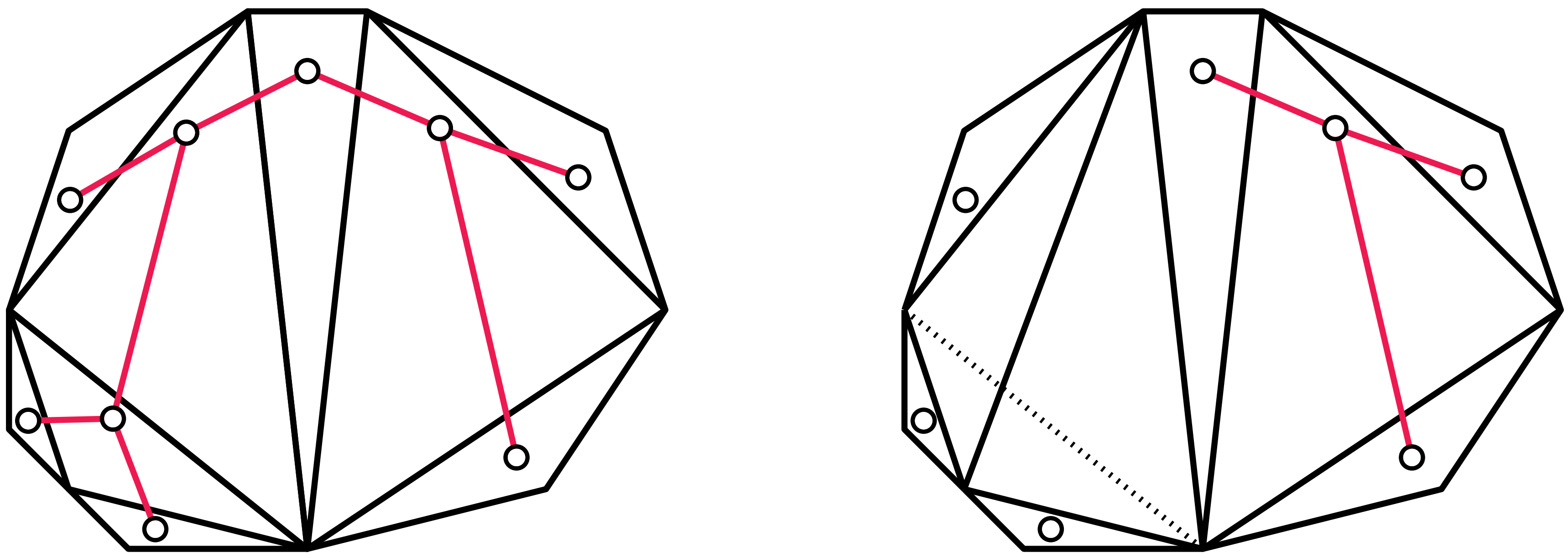

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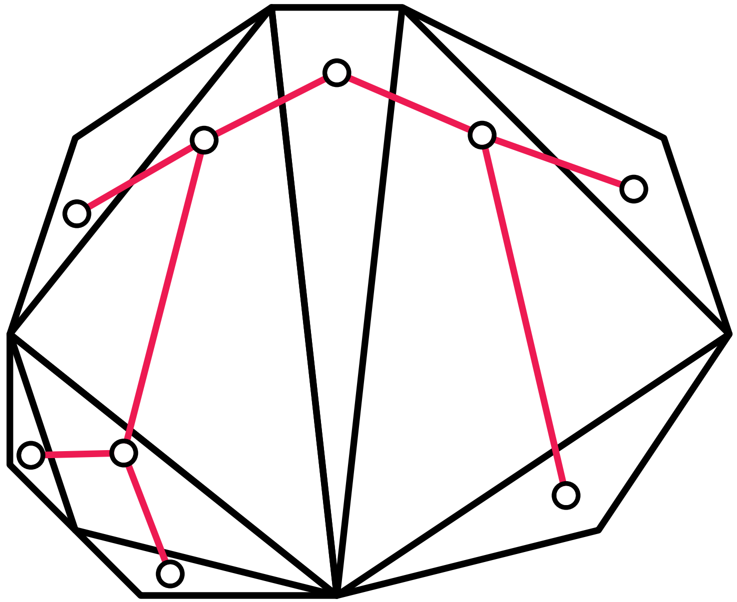

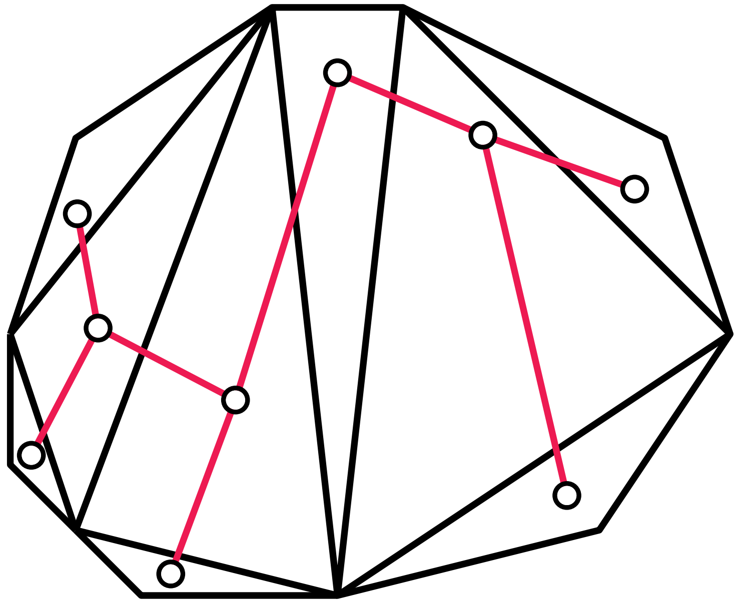

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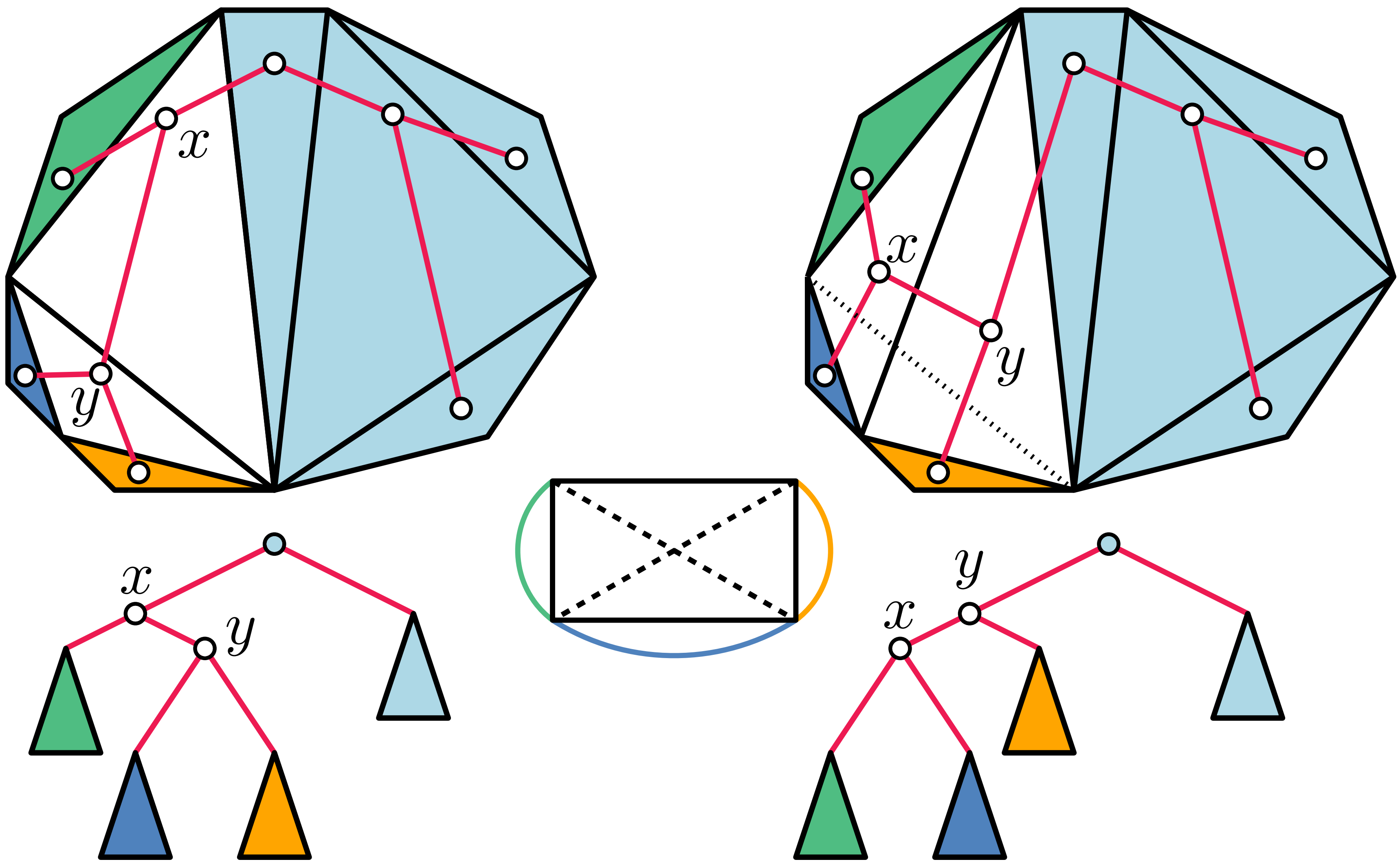

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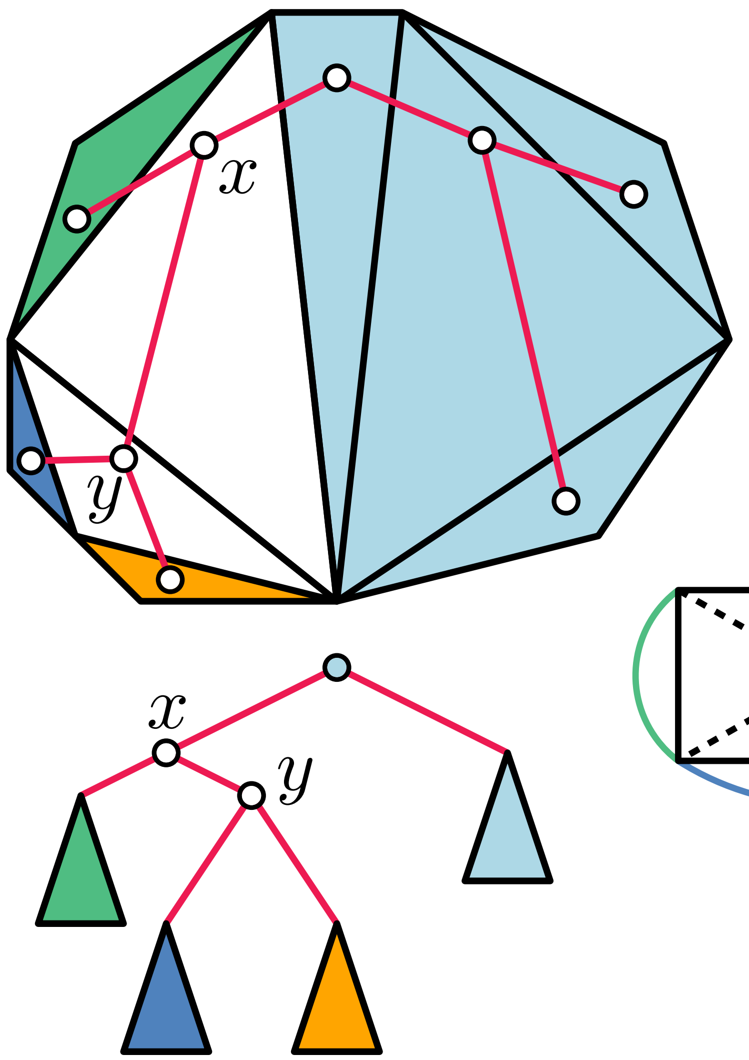


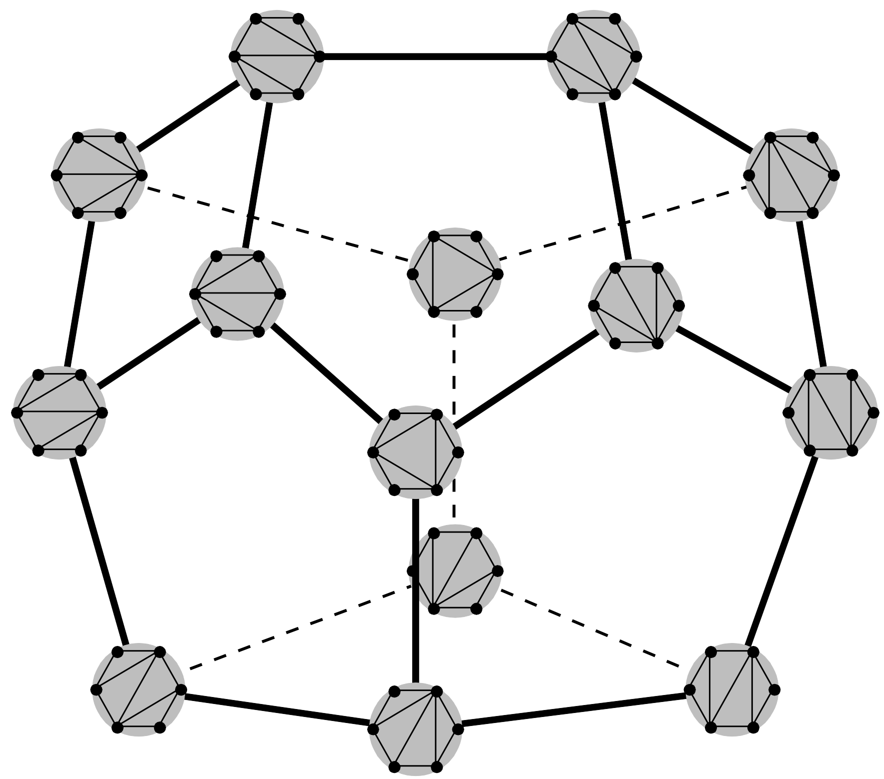
What is the shortest rotation sequence between two trees?

Triangulation Flip Graph

- Vertex: Each triangulation of the given set of points
- Edge: two triangulations can be transformed into each other by one edge flip.

For points in convex position:

- connected
- C_{n-2} elements
- an $(n-3)$-dimensional polytope, called "associahedron"



Triangulations

Classic result for Flips in Triangulations:

- In the combinatorial setting all n-triangulations are connected [Wagner 1936, diameter $O(n^2)$] with diamter $O(n)$ [Komuro 1997]
- For general point sets [Lawson 1972] showed that the flip graph is connected with diameter $O(n^2)$. He uses a canonical triangulation, triangulated in x-sorted order.
- Later [Lawson 1977] used the Delaunay triangulation as canonical triangulation, again $O(n^2)$ Delaunay flips.

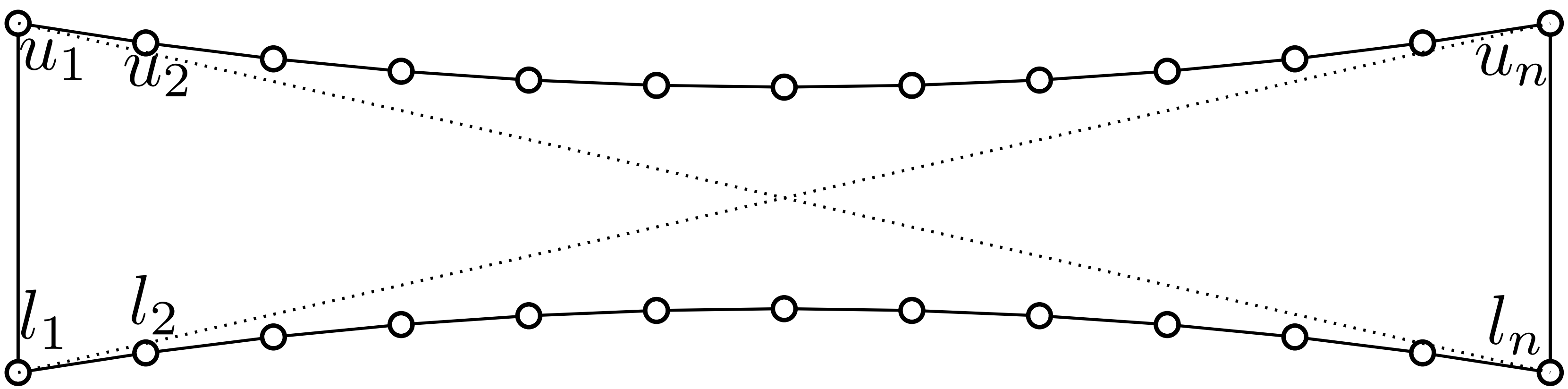
Triangulations

Classic result for Flips in Triangulations (cont'):

- Without canonical triangulation: The flipdistance can be bounded by the number of crossings (there is always a flip reducing the number of crossings) [Hanke, Ottmann, Schuierer 1996]
- Computing lower bounds for the flipdistance between two given triangulations can be computed in polynomial time [Eppstein 2007]
- Diameter for sets with k onion layers: $O(nk)$
- Diameter for n-gons with k reflex vertices: $O(n+k^2)$; both results [Hurtado, Noy, Urrutia, 1995]
- Points in convex position: Diameter $2n-10$ [Sleator, Tarjan, Thurston 1988]; tight for $n>12$ [Pournin 2014]

Triangulation Flip – Lower Bound

Lower bound $\Omega(n^2)$ [Hurtado, Noy, Urrutia 1999]



- Double chain with $2n$ points
- The drawn edges are *unavoidable* for any triangulation, that is, they can not be crossed by any other edge
- Only the inner of the polygon is relevant

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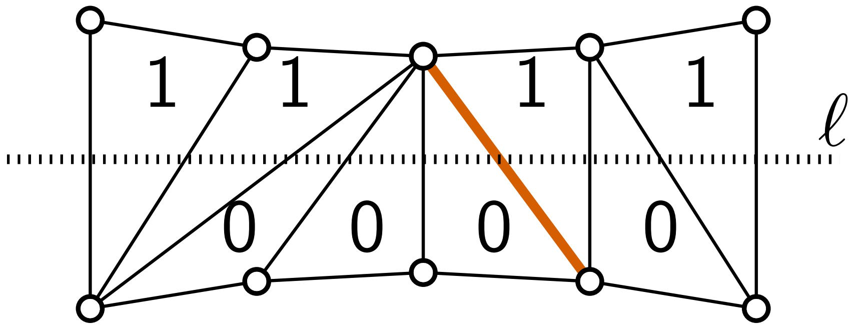

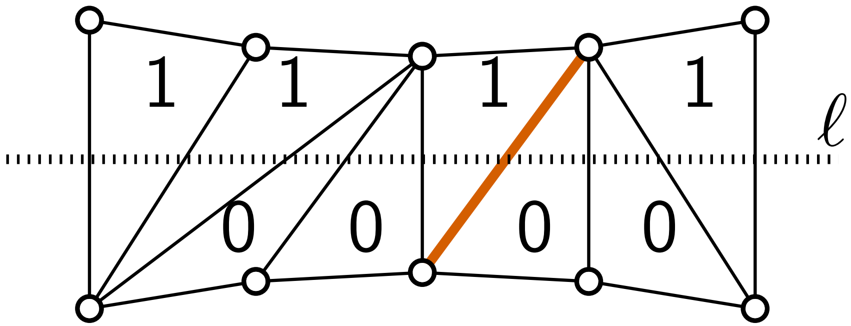

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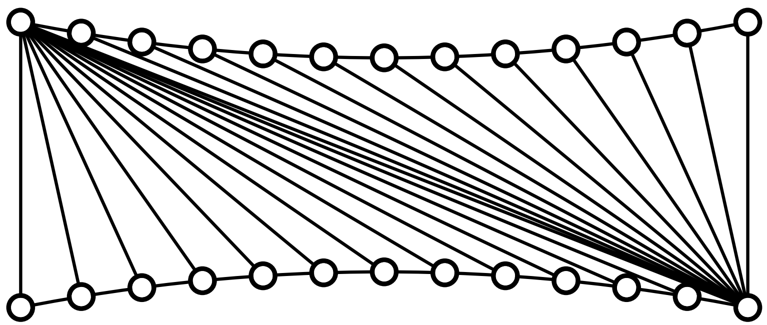
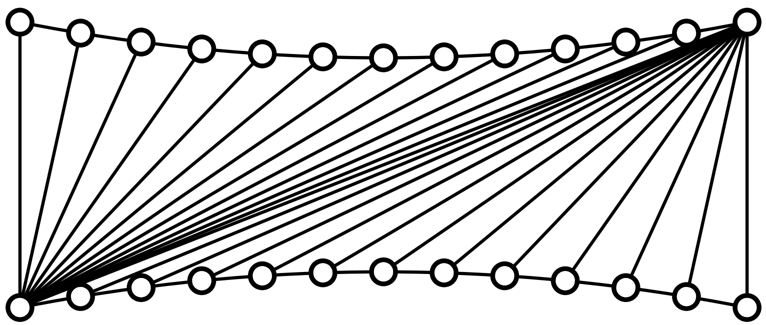
- We have a sequence of $n-1$ ones and $n-1$ zeros
- A flip is possible between a 1 triangle and a 0 triangle
- The two adjacent numbers are switched

Triangulation Flip – Lower Bound

Lower bound $\Omega(n^2)$ [Hurtado, Noy, Urrutia 1999]





00000 ...11111   11111 ...00000

- "All zeros have to be moved to the right"
- There are $n-1$ zeros and $n-1$ ones
- Therefore we need at least $(n-1)^2$ flips
- The flip graph has (at least) quadratic diameter

Triangulations

So we have tight upper bounds $O(n)$ for convex sets, and $O(n^2)$ for general sets, but what is the **complexity** of computing the **shortest flip sequence** between two given triangulations?

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- Computing the flip distance between two triangulations of a **planar point set** is
  - NP-complete [Pilz 2012] and [Lubiw, Pathak 2012]
  - APX-hard [Pilz, 2014] (reduction from VERTEX COVER), i.e., no polynomial-time algorithm to approximate the flip distance by $1+\varepsilon$, $\varepsilon \geq 0.36$ exists
  - fixed-parameter tractable for flip distance k: $O^*(k \cdot 32^k)$ [Feng, Li, Meng, Wang 2021], for convex sets $O^*(3.82^k)$ [Li, Xia 2025]

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- Computing the flip distance between two triangulations of a **planar point set** is
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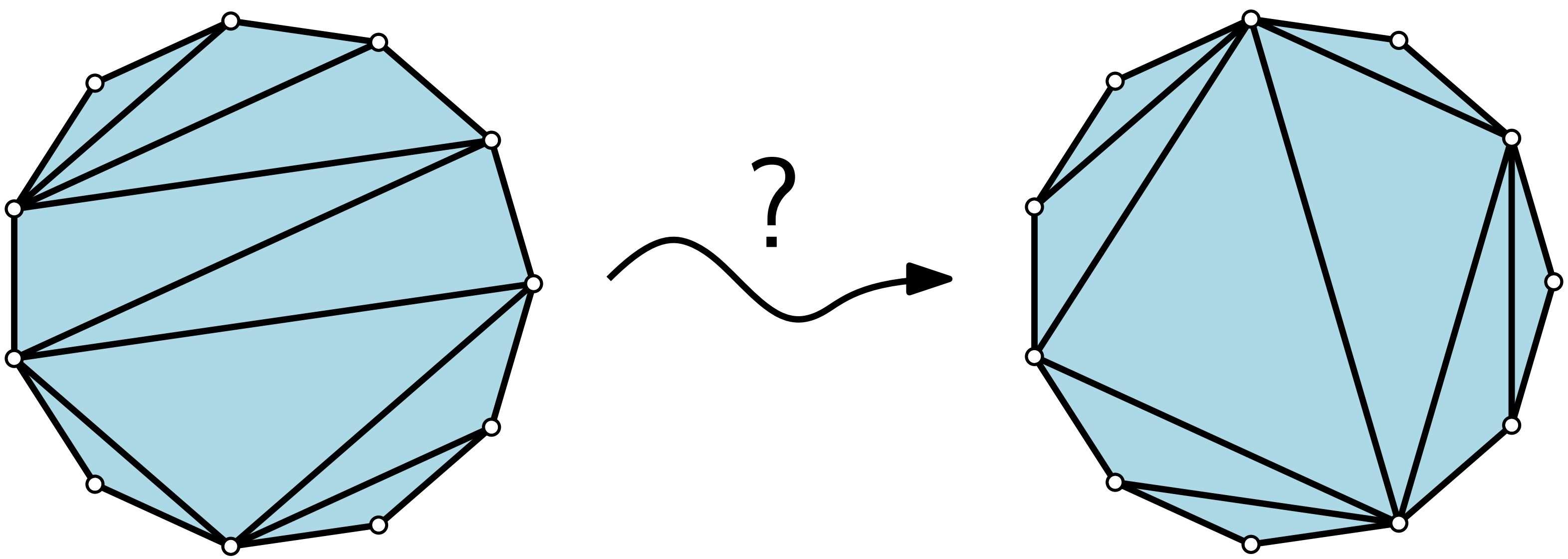
- Computing the flip distance between two triangulations of a **simple polygon** is NP-complete [A., Mulzer, Pilz, 2015] reduction from $\textsc{Rectilinear Steiner Arborescence}$

Triangulations

Open Problem 1: What is the complexity of the flip distance of two triangulations of a set of points in convex position (the rotation distance between two binary trees)?



Open whether it is in P or NP-hard, but $\leq 2n - 10$ flips

Happy Edge Property

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The Happy Edge Property does not hold for triangulations of general point sets or simple polygons [Hernando, Hurtado, Noy 2002] ...
$\rightarrow$ this was the key to construct NP-hardness gadgets
... but for triangulations of a set of points in convex position [Sleator, Tarjan, Thurston 1988]
$\rightarrow$ so maybe **Open Problem 1** is in P?

Crossing-free Spanning Trees

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A reconfiguration step (flip) for a crossing-free spanning tree removes one edge, and inserts another, such that the resulting drawing is again a crossing-free spanning tree

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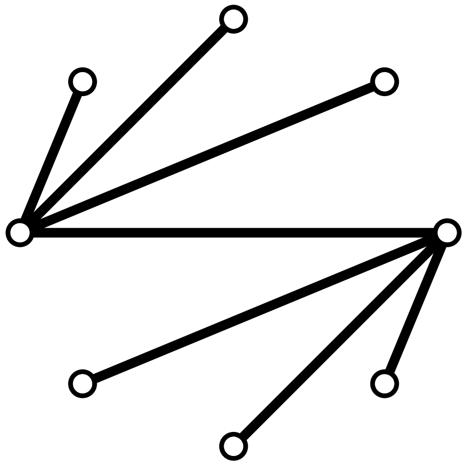

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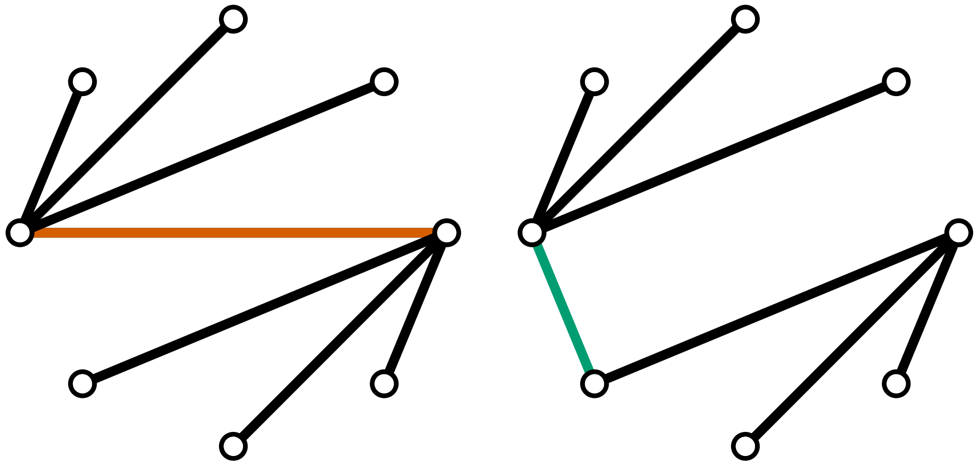

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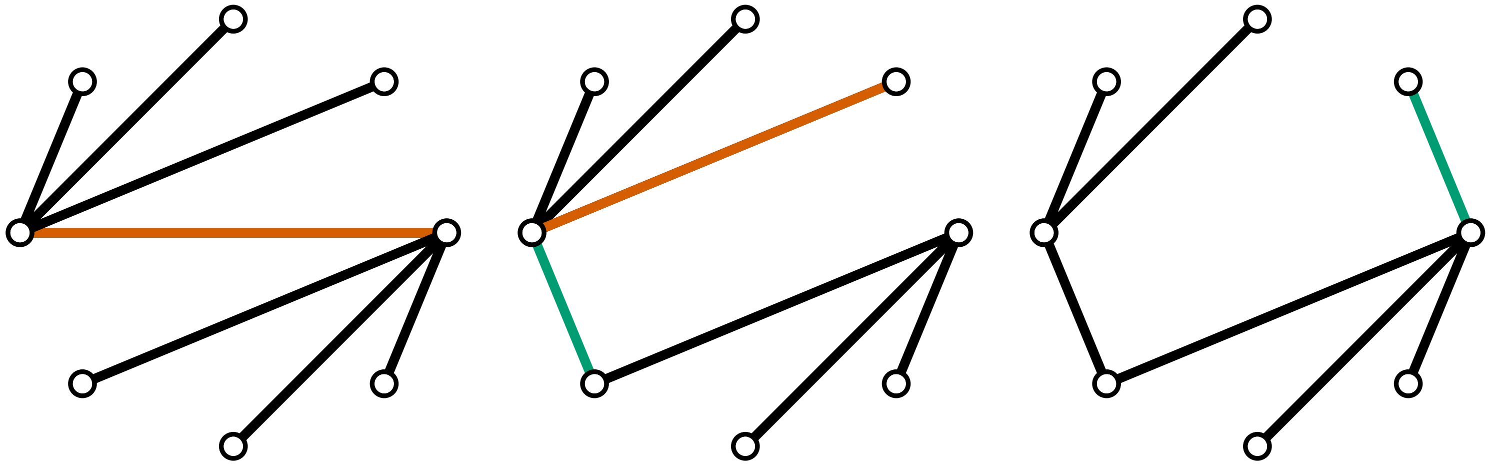

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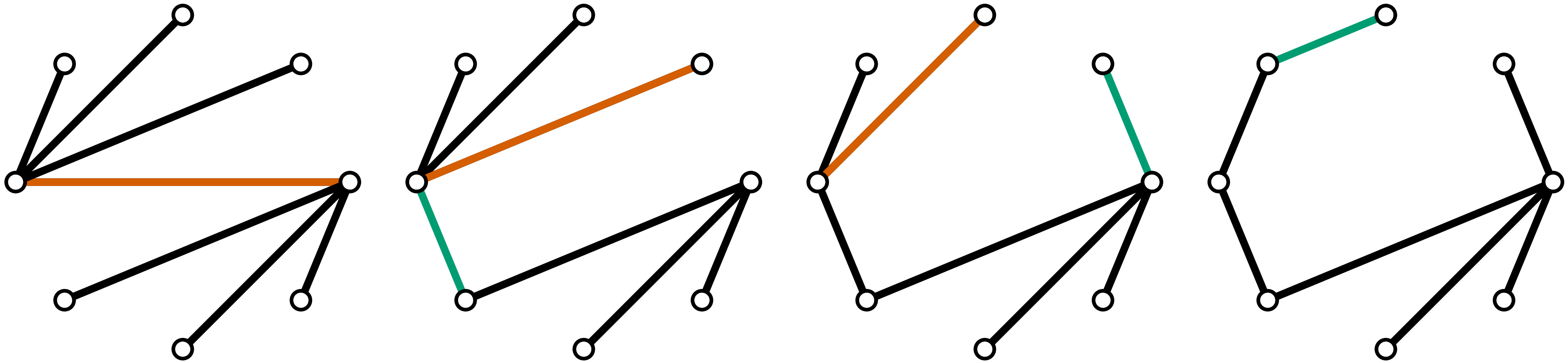

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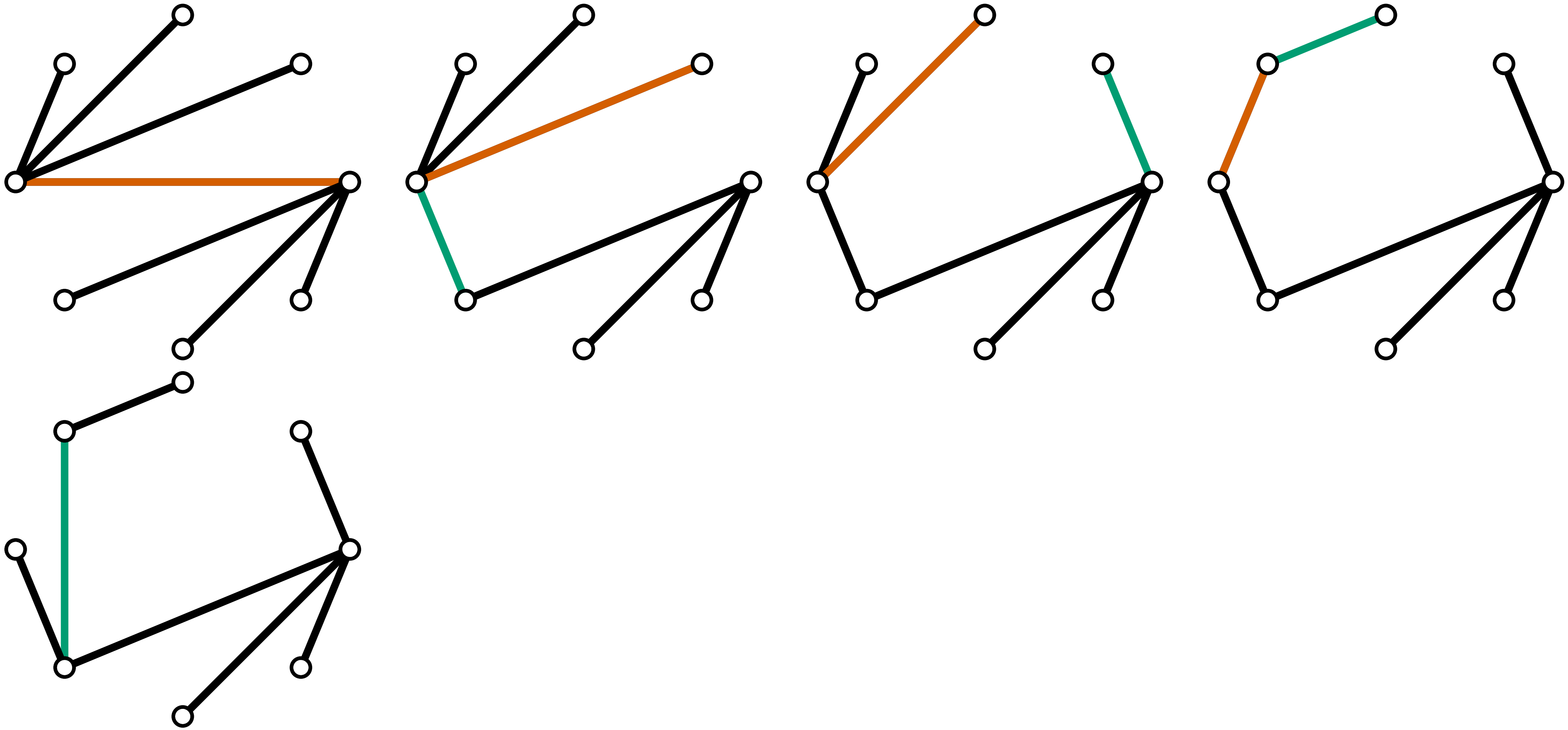

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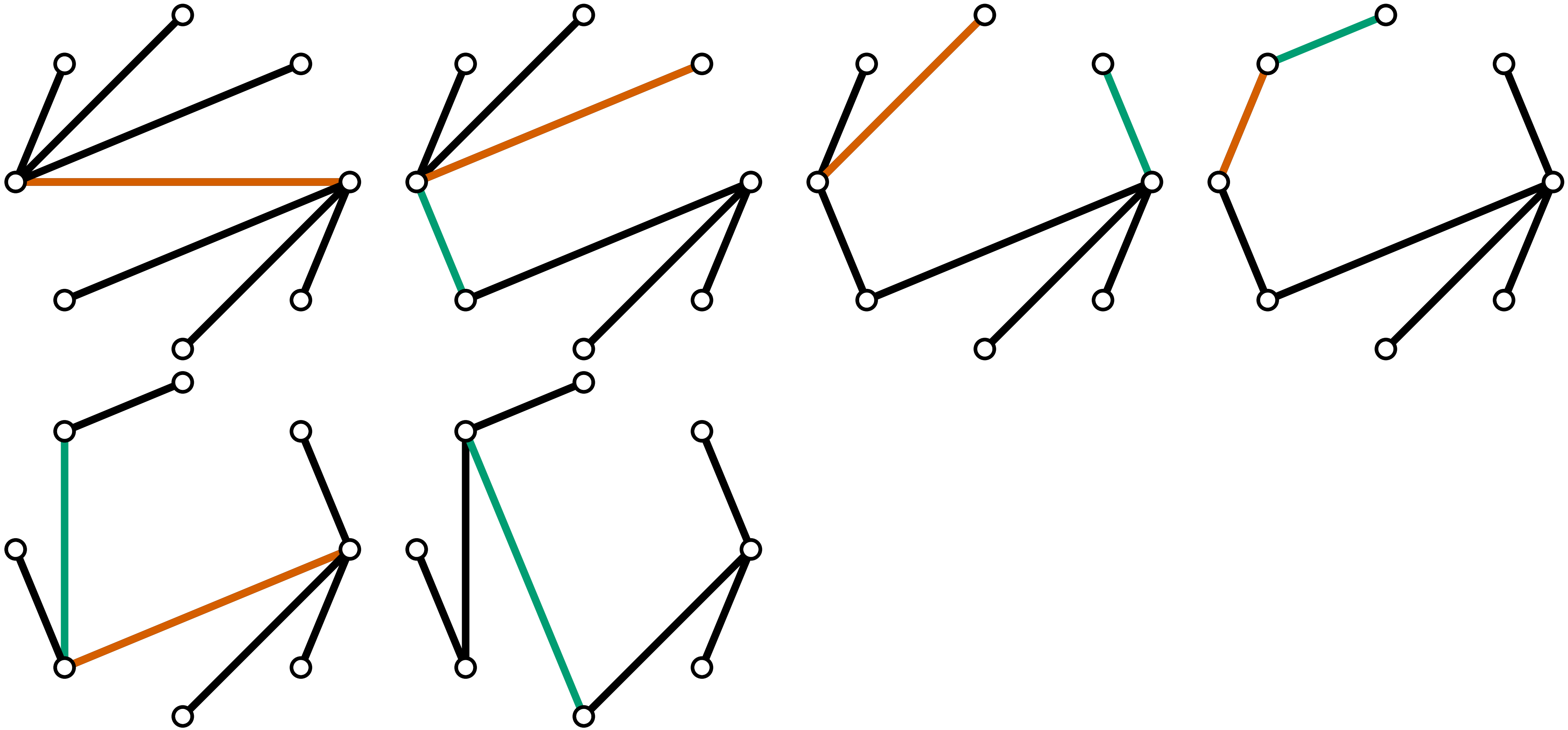

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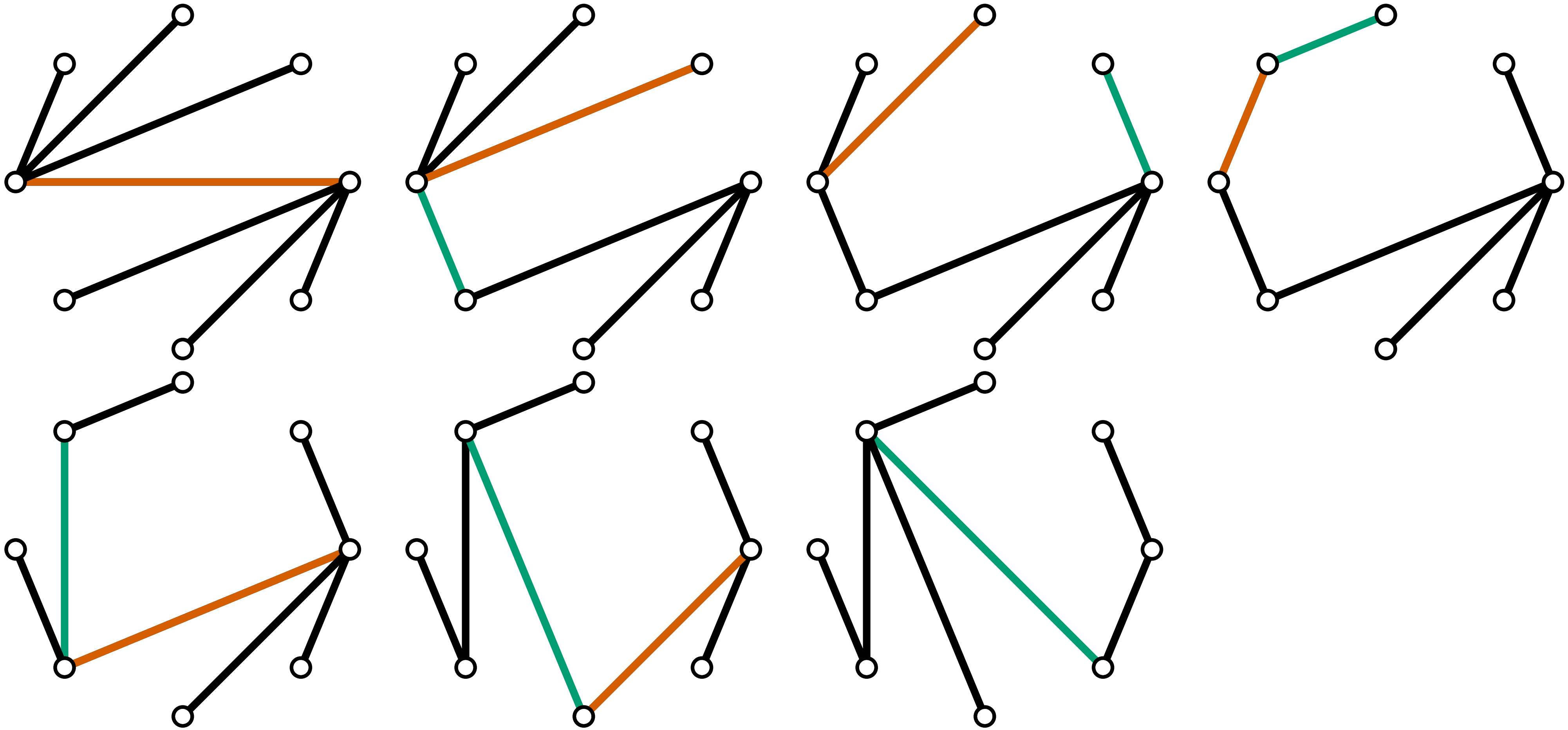

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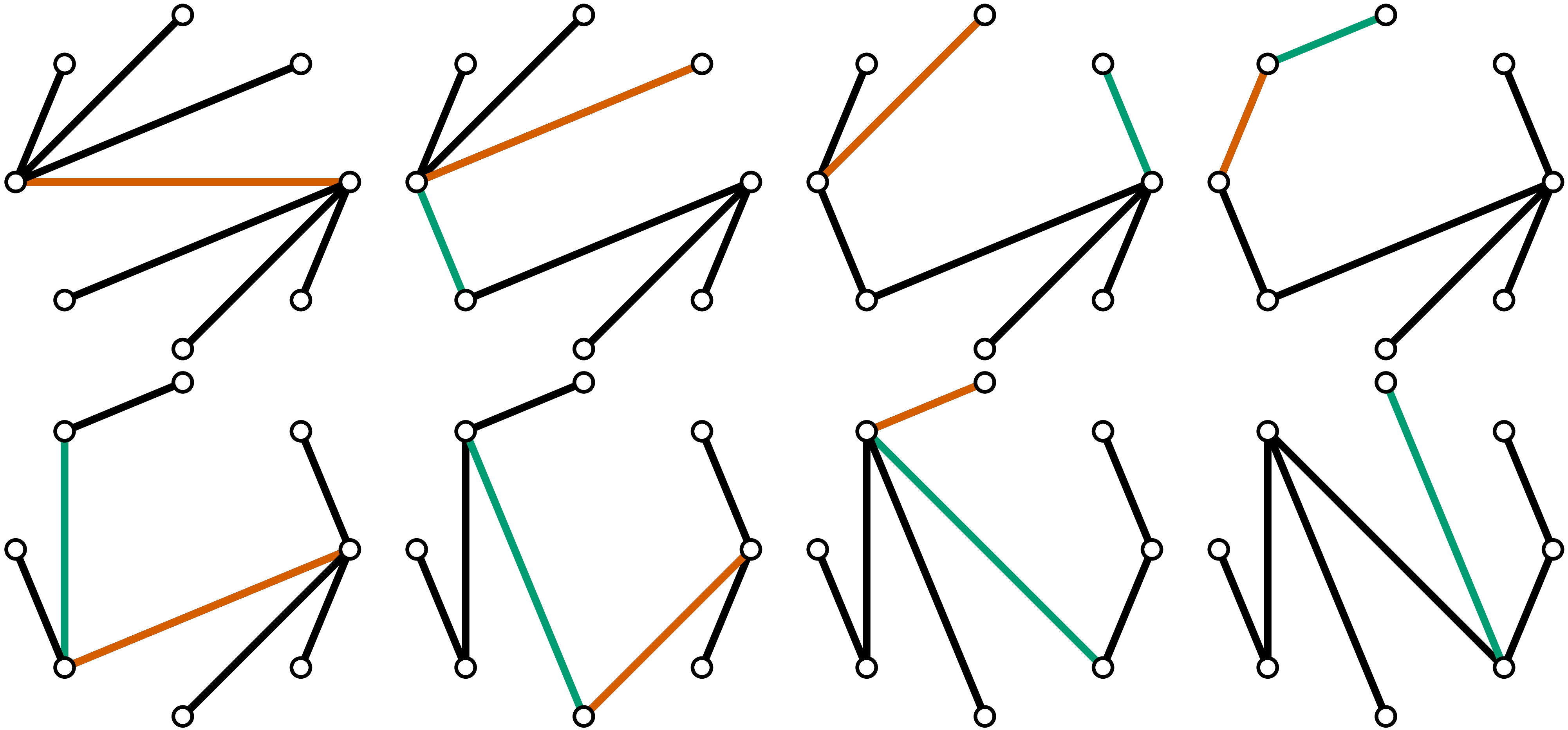

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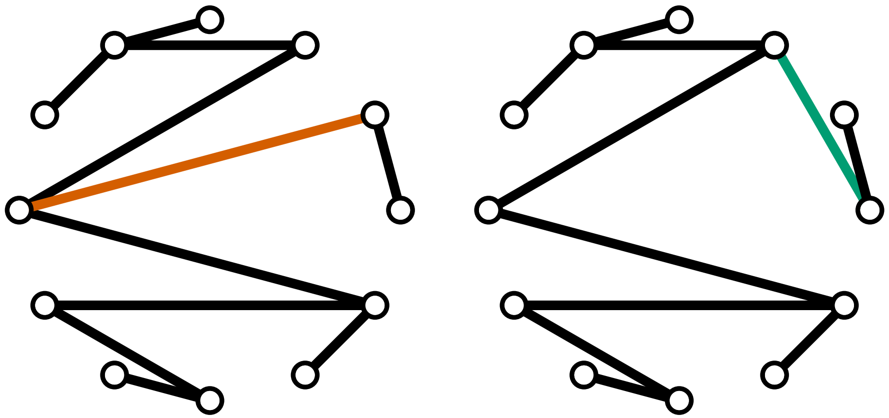
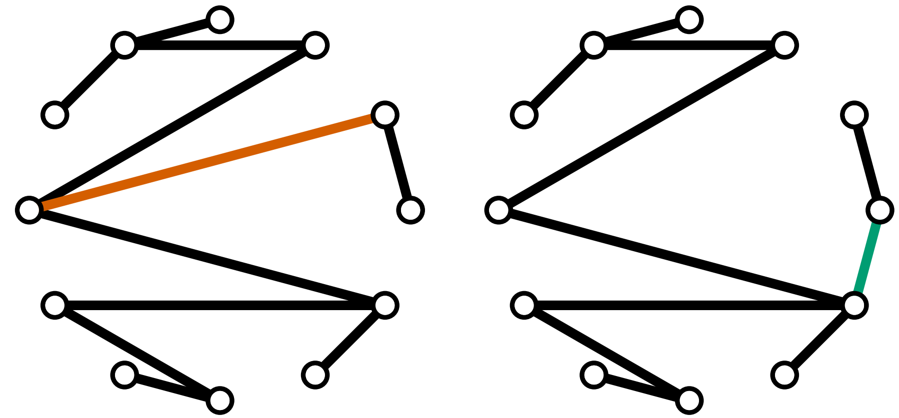
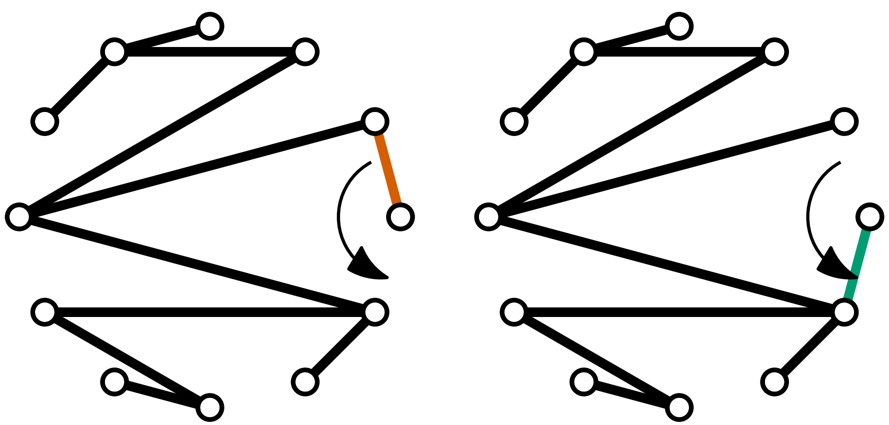


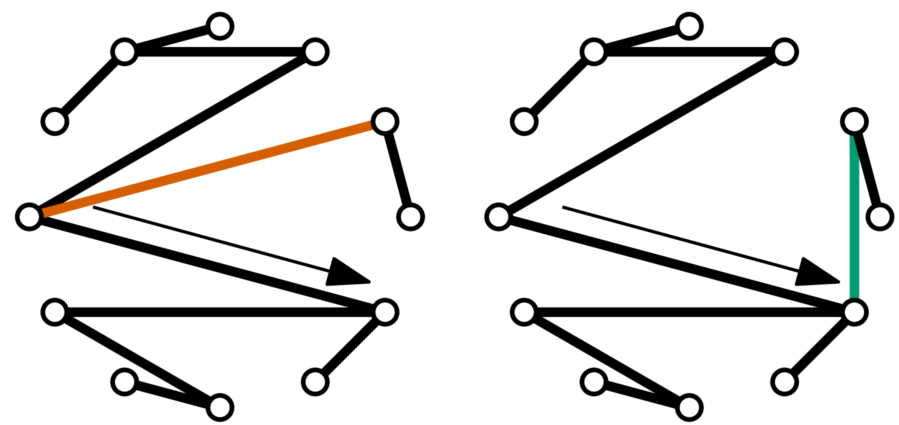
edge exchange



compatible edge exchange



edge rotation



edge slide

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Lower Bound:

Upper Bound:

Upper Bound: (Compatible Flips)

$1.5n$ [HHMMN, 1999]

$2n - 4$ [AF 1996]

$2n - log(n)$ [ABBDDKLLTU 2022]

$2n - \sqrt{n}$ [BGNP 2023]

$1.95n$ [BMPW 2024]

$\frac{14}{9}n$ $\frac{5}{3}(n-1)$ [BKUV 2025]

$\frac{5}{3}(n-1)$ New Result

Further details:

Friday April 11

Session 8.B (Chateau Restaurant)

12:15 - 12:30 *Oswin Aichholzer, Joseph Dorfer and Birgit Vogtenhuber*

Flipping Plane Spanning Trees Compatibly

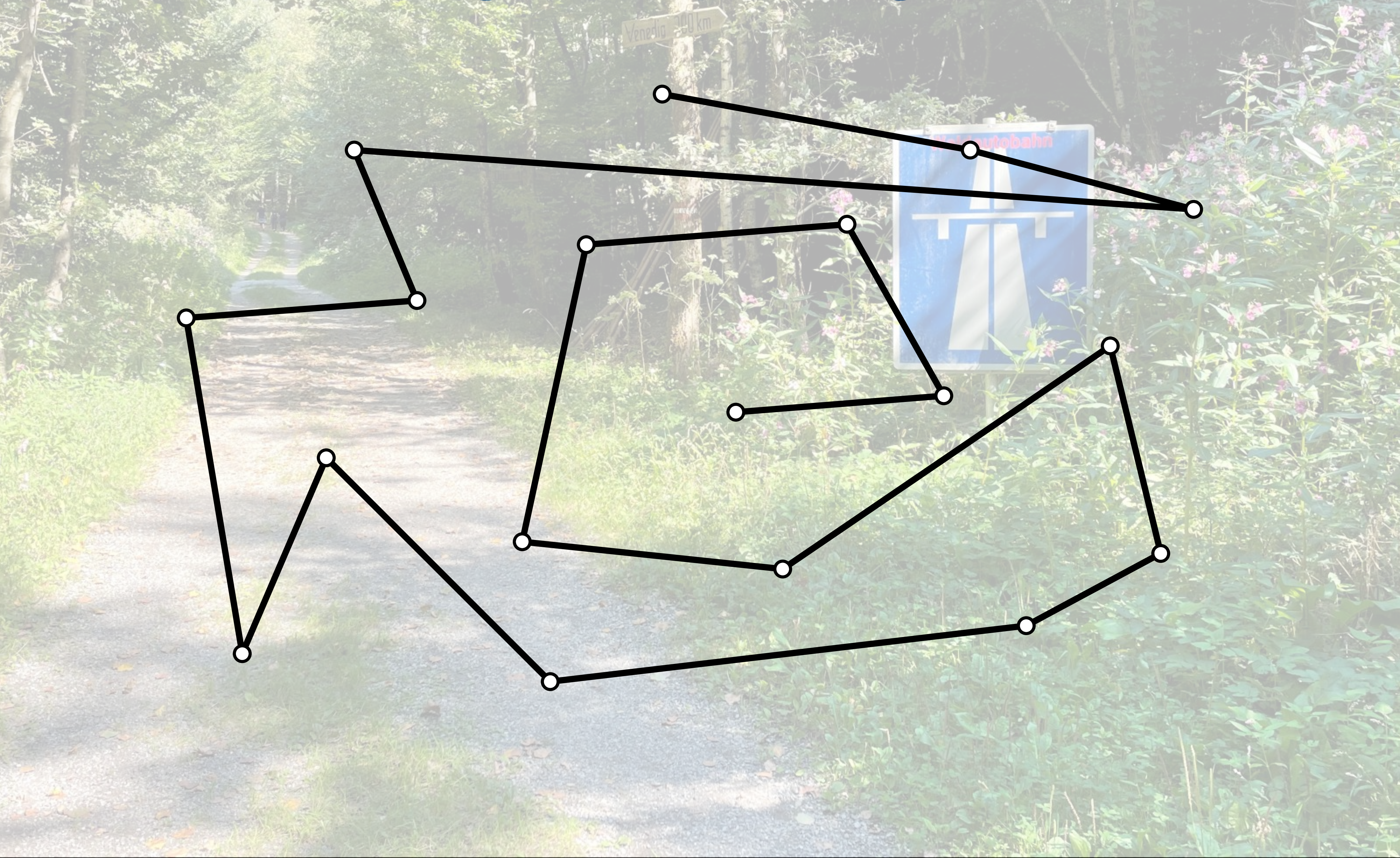
Open Problems for Trees

Open Problem 2: For which flip types does the Happy Edge Property hold for crossing-free spanning trees?

Open Problem 3: What is the complexity of computing the flip distance for crossing-free spanning trees for general point sets / point sets in convex position?

Open Problem 4: What is the tight bound for the diameter of the flip graph for crossing-free spanning trees for general point sets / point sets in convex position?

Crossing-free Spanning Paths



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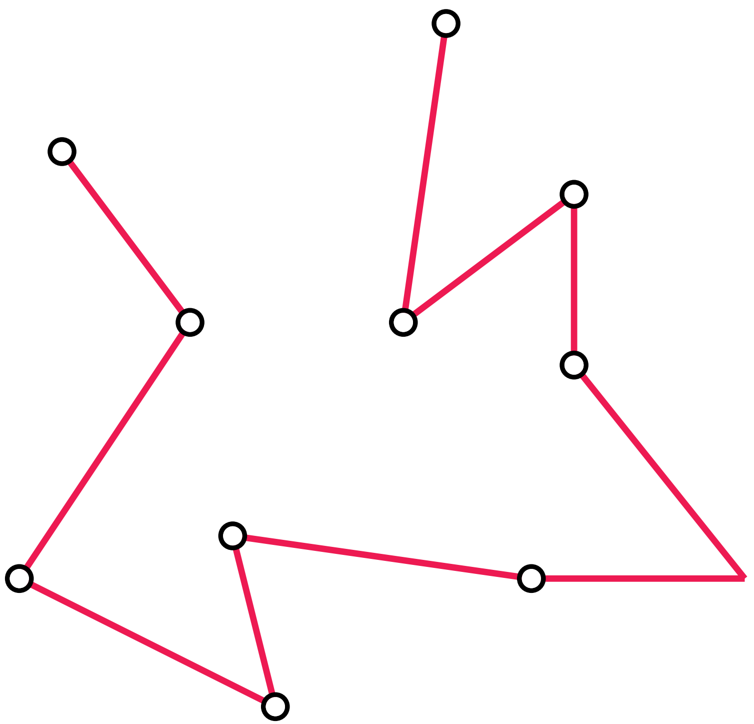

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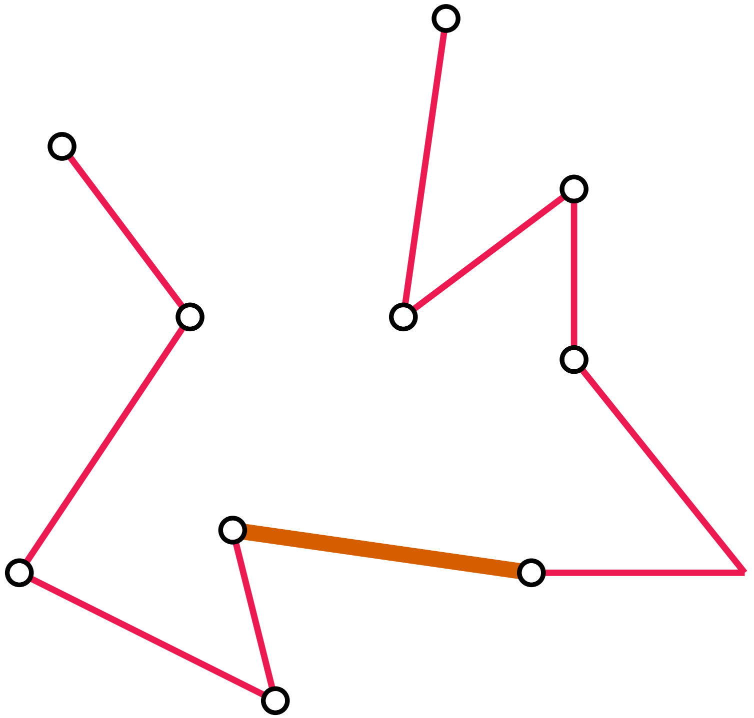

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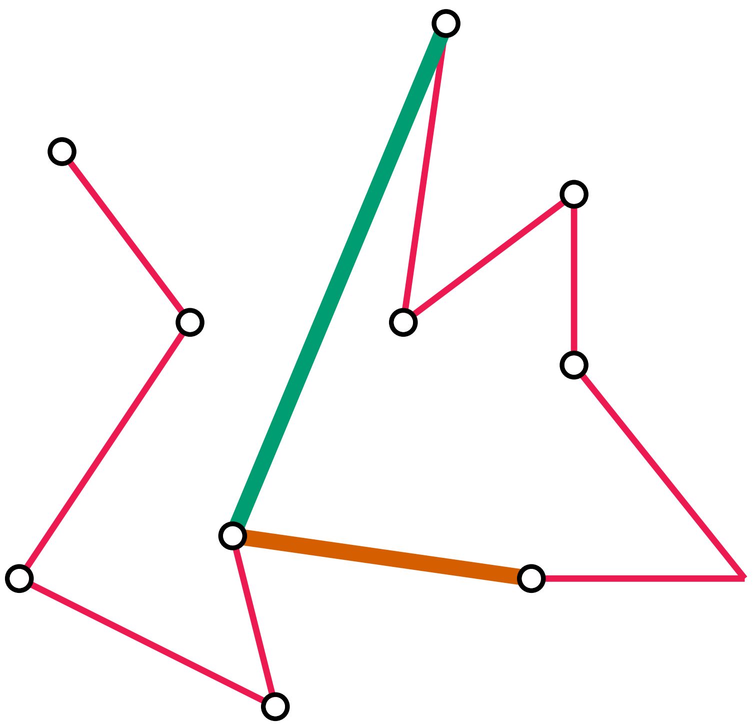

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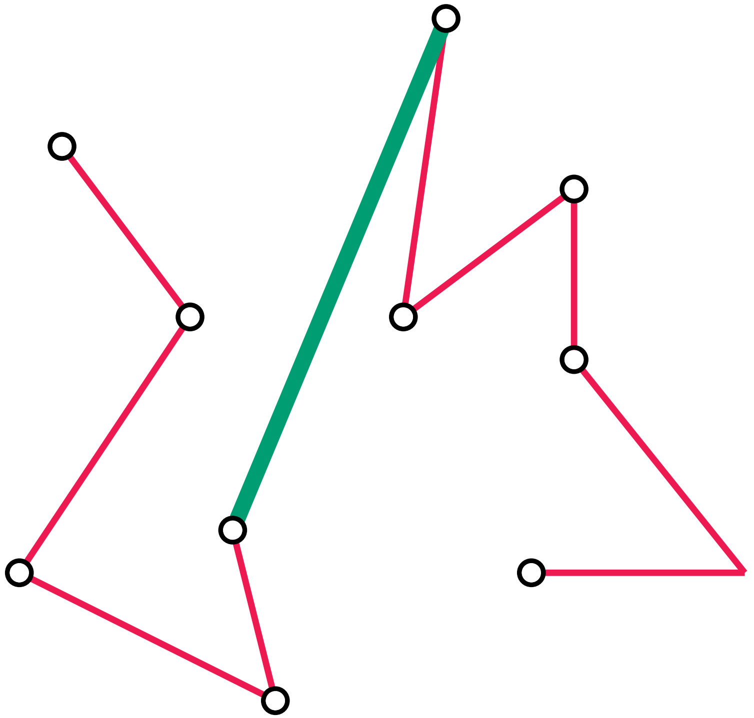

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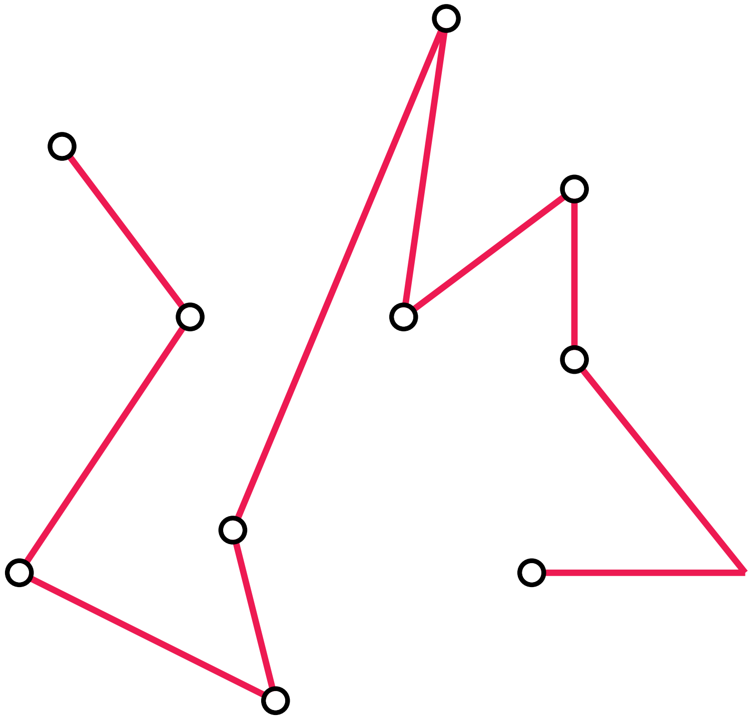

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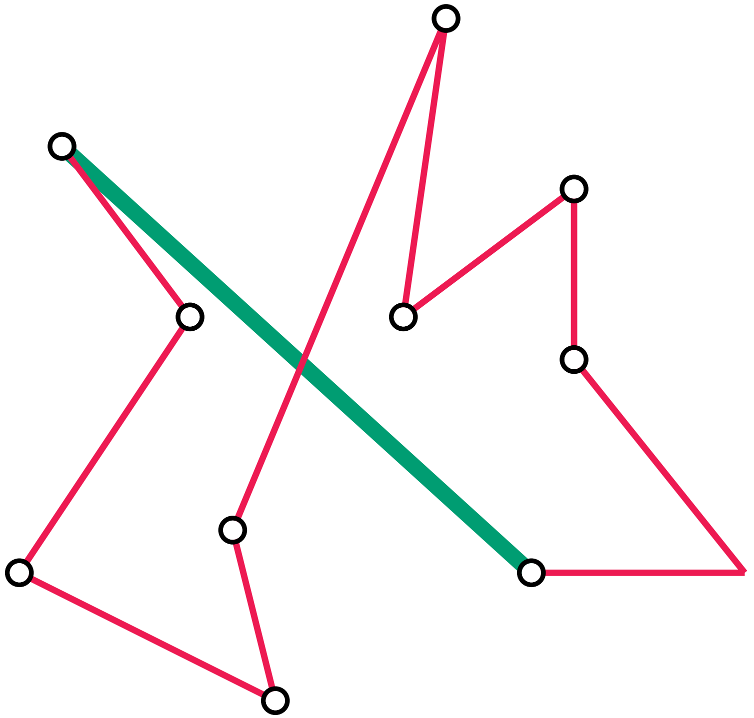

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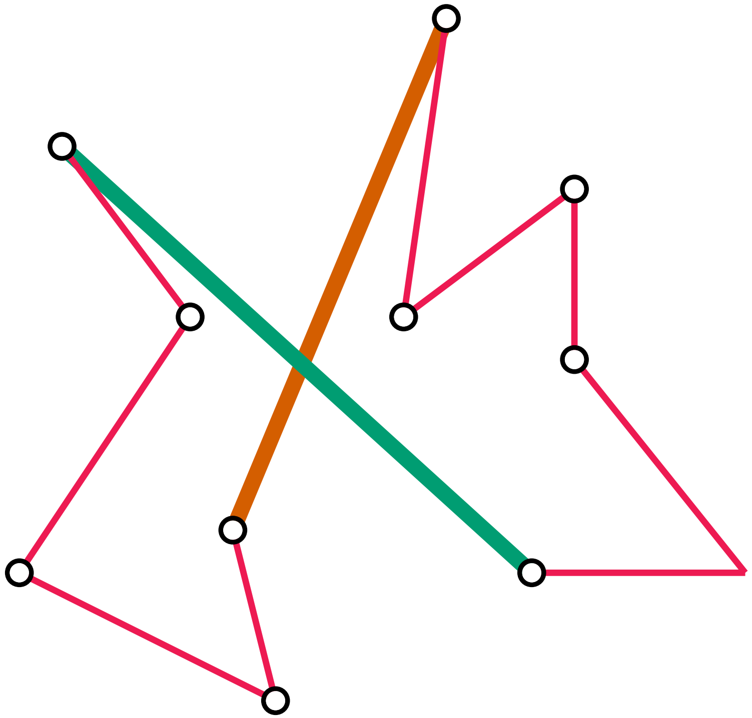

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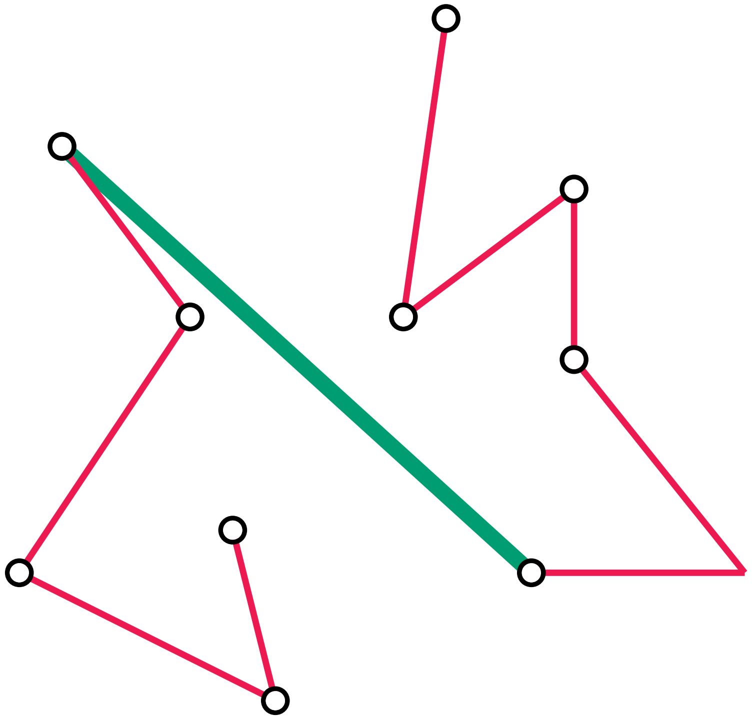

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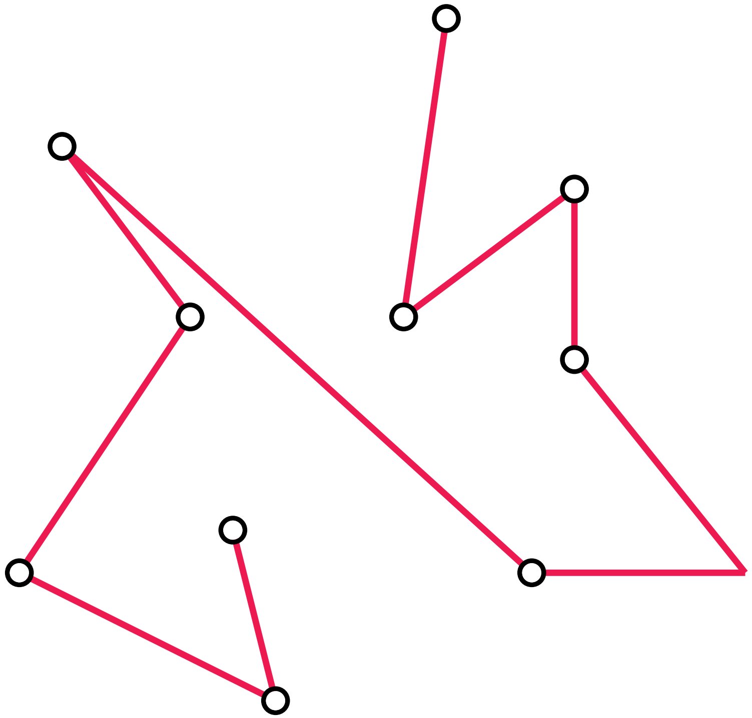

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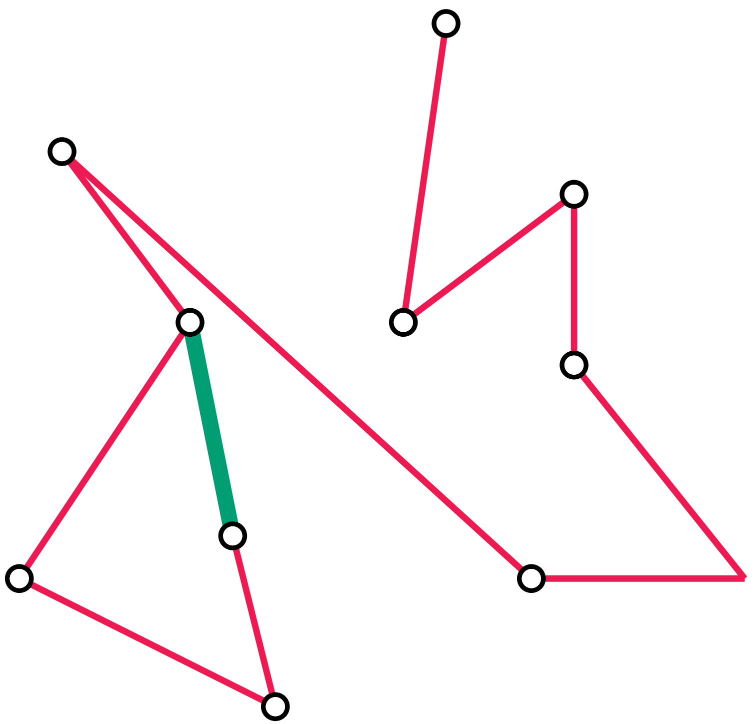

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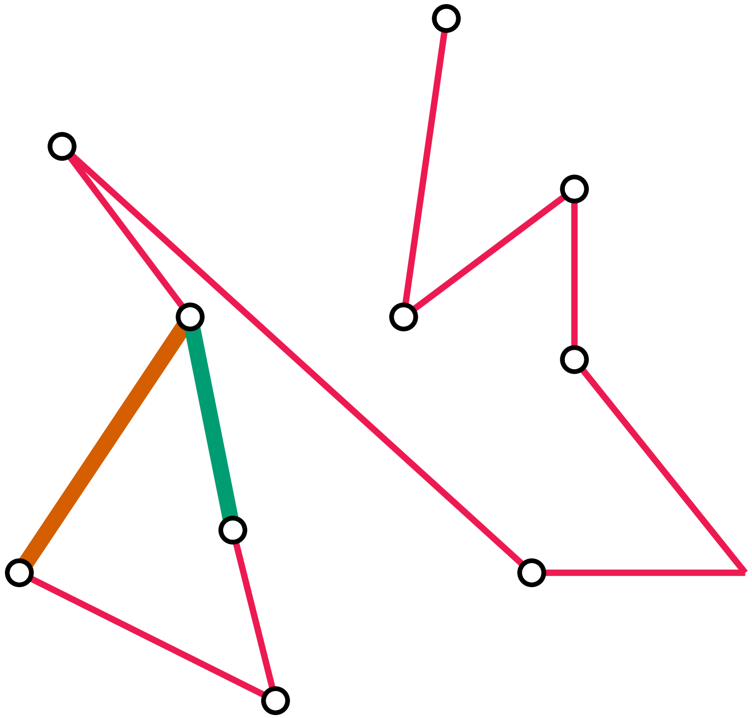

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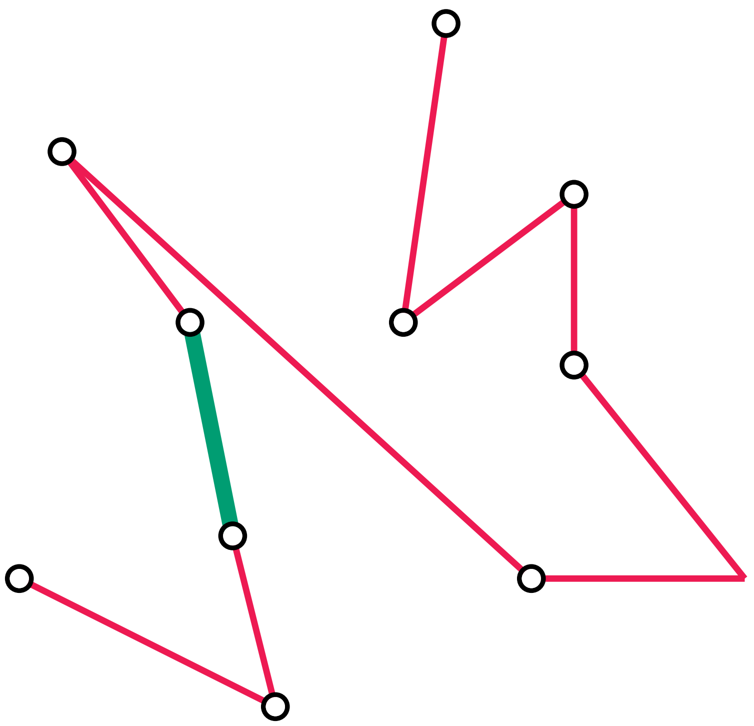

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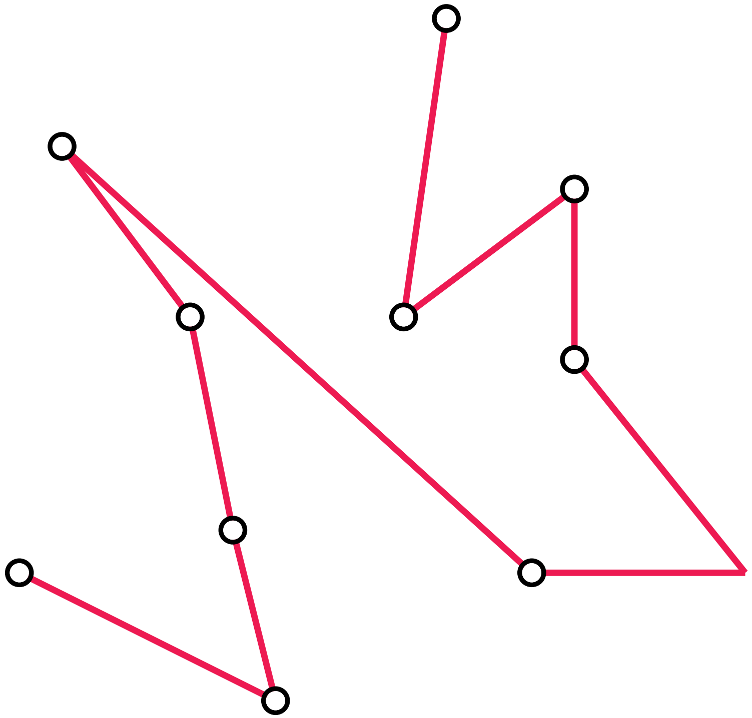

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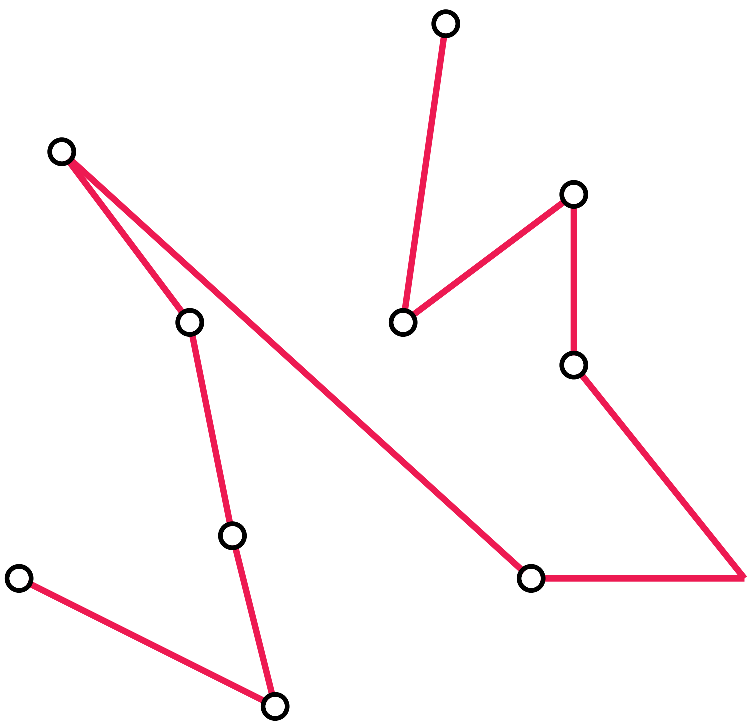
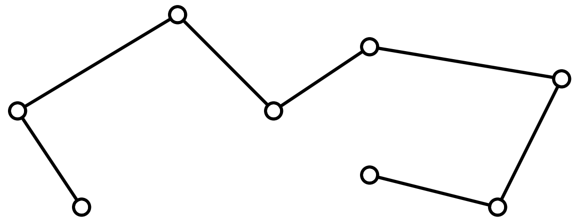


Type 1: e and f share a vertex



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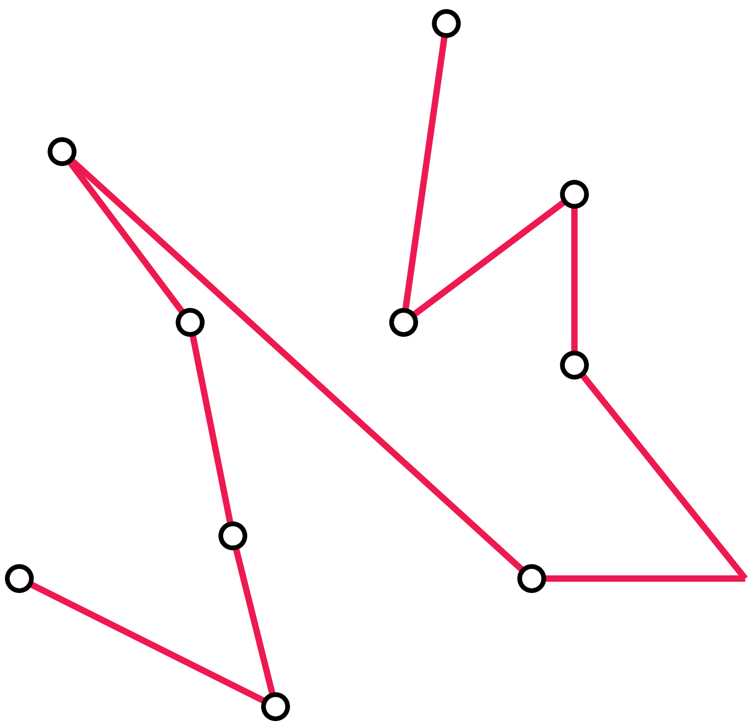

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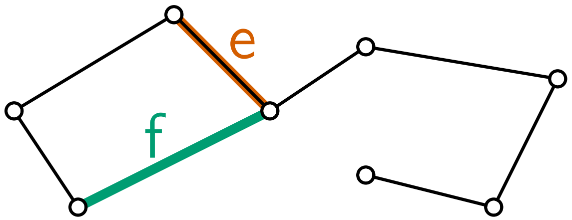

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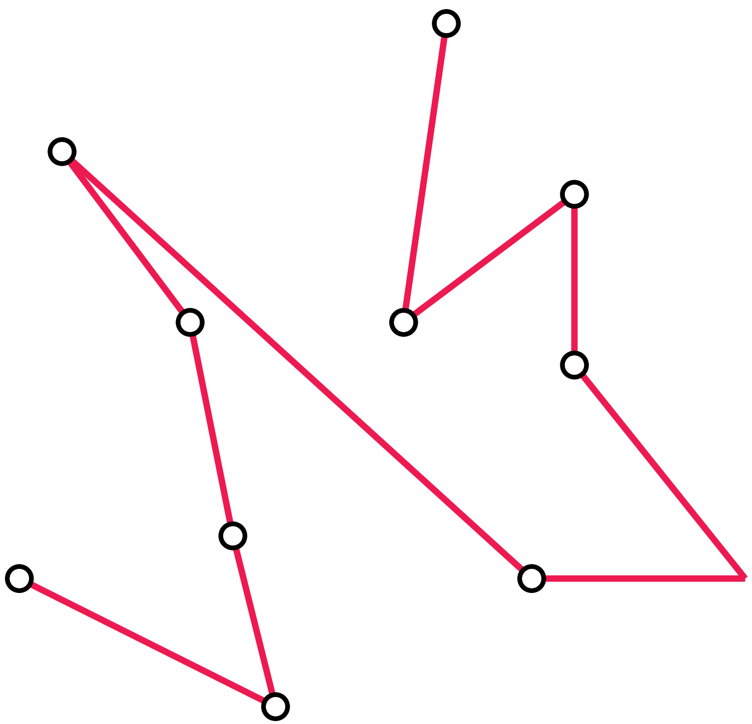

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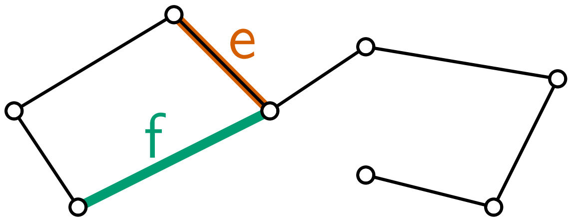
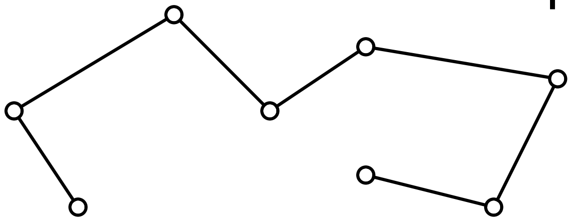


Type 2: path can be closed to a plane cycle



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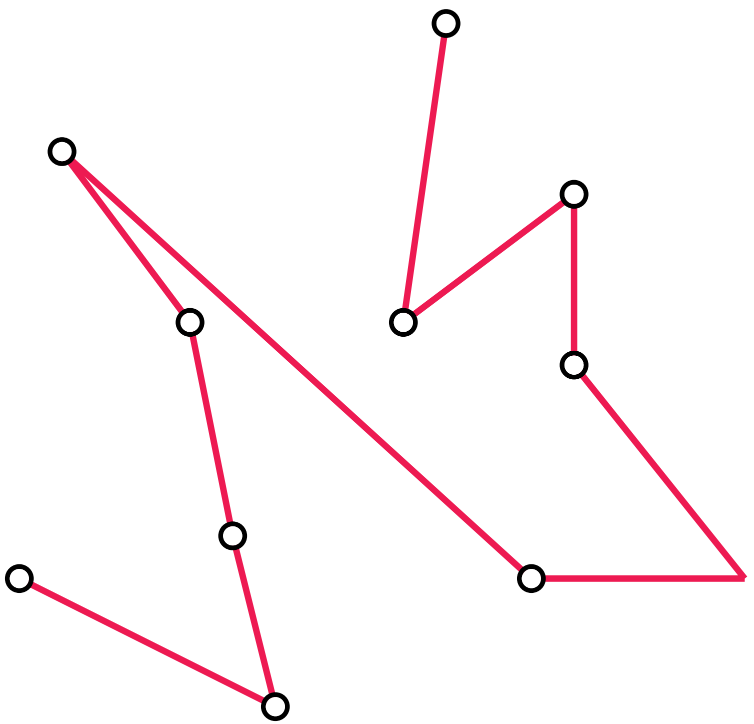

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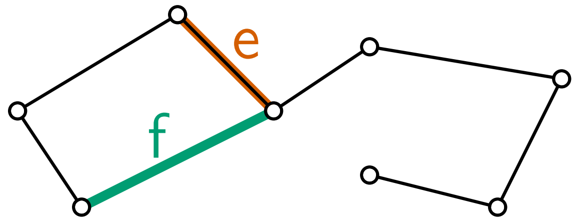

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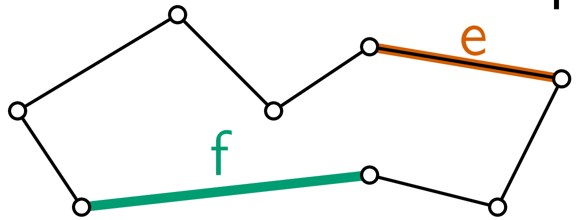

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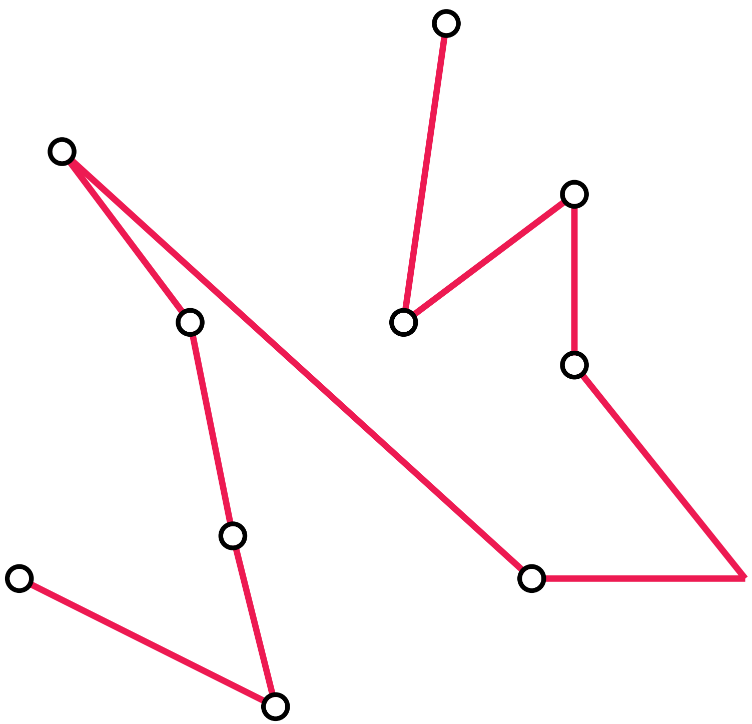

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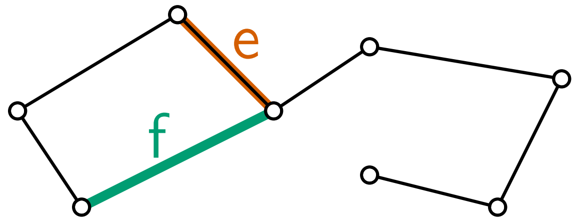

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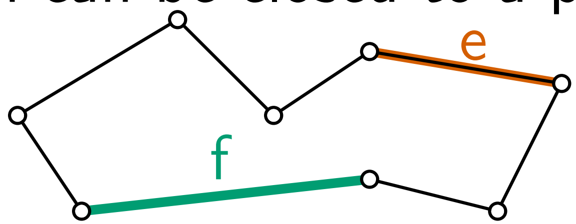
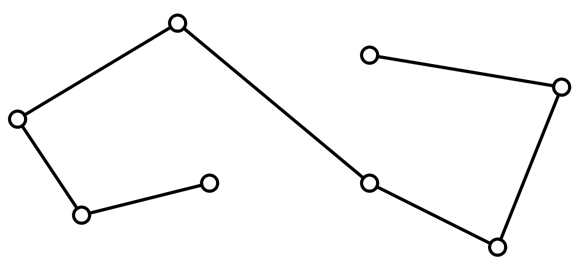


Type 3: cycle contains one crossing



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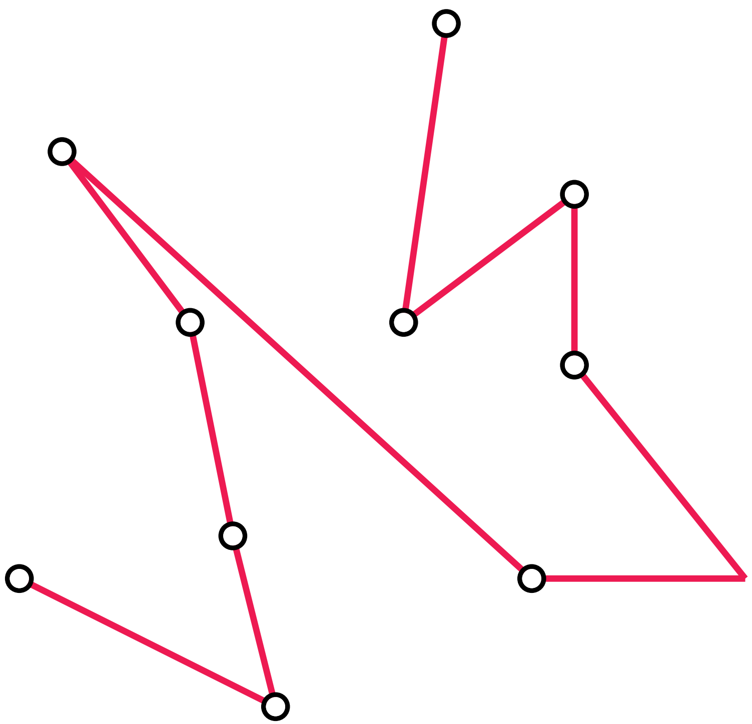

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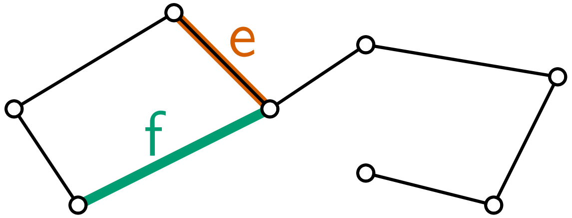

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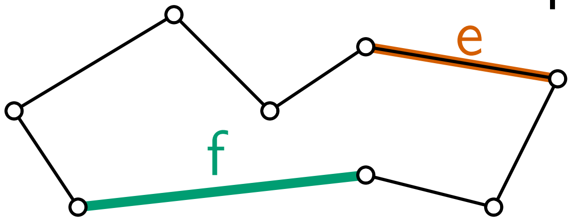

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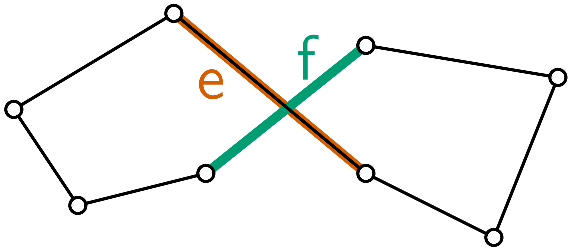

Crossing-free Spanning Paths

Open Problem 5: Is the flip graph for crossing-free spanning paths connected? For which flip-types?

Crossing-free Spanning Paths

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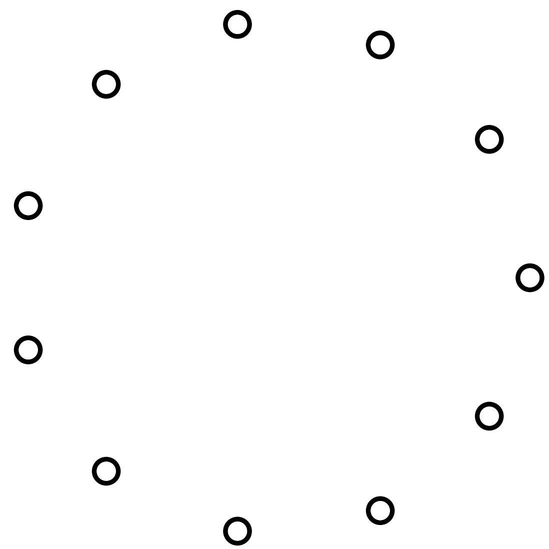

Known so far:

- n points in convex position: connected with diameter $2n-6$ $(n \geq 5)$, tight [Akl, Islam, Meijer, 2007 / Chang, Wu, 2009].
- Connected for $n \leq 11$ points in general position, even for Type 1 flips [shown via order types].

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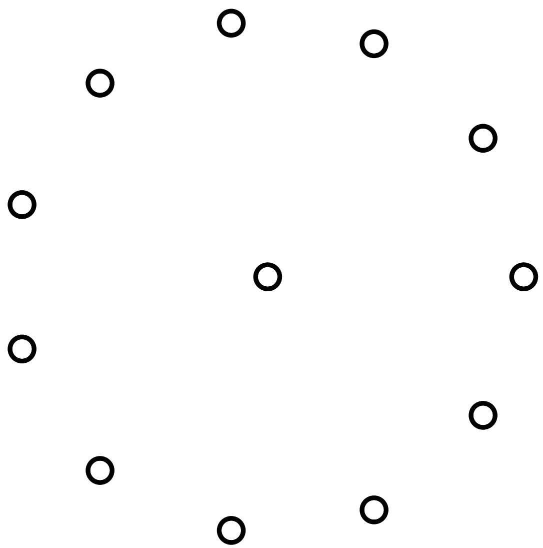

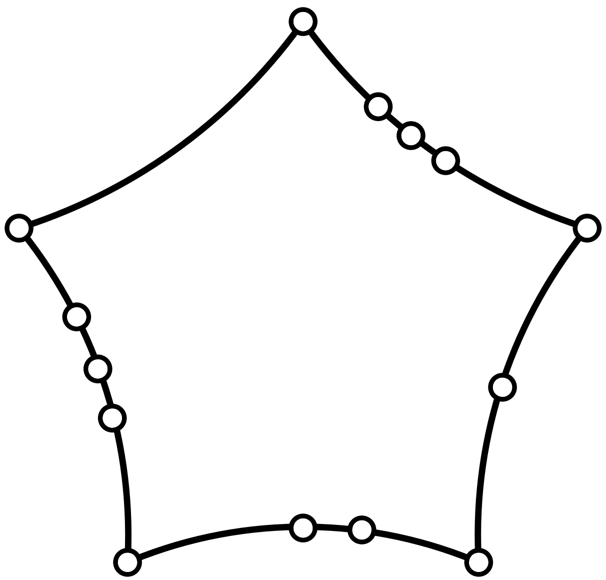

- Connected for wheel sets (diameter $2n-4$), generalized double circles (diameter $O(n^2)$), include ice cream cones, double chains, double circles)

  [A., Knorr, Löffler, Masárová, Mulzer, Obenaus, Paul, Vogtenhuber, 2023]

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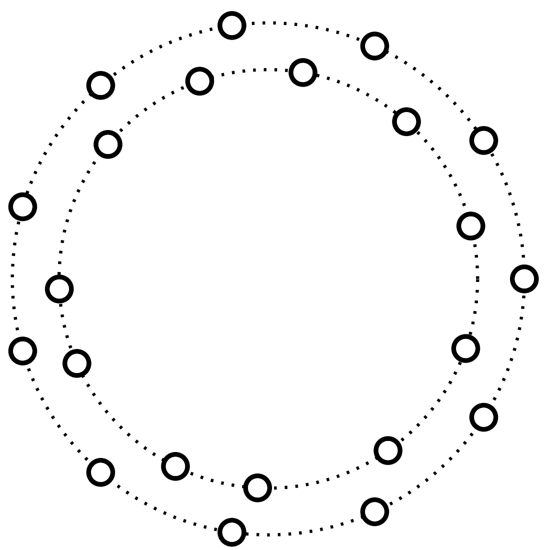
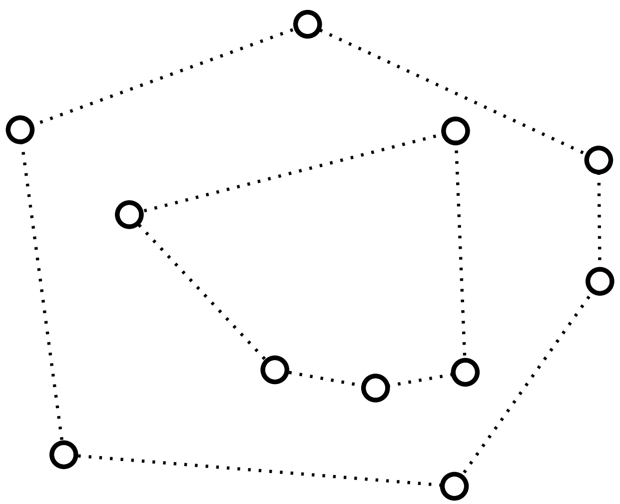




- Connected for Sets with two Convex Layers

  [L. Kleist, P. Kramer, Ch. Rieck, 2024]

Crossing-free Spanning Paths

Open Problem 5: Is the flip graph for crossing-free spanning paths connected? For which flip-types?

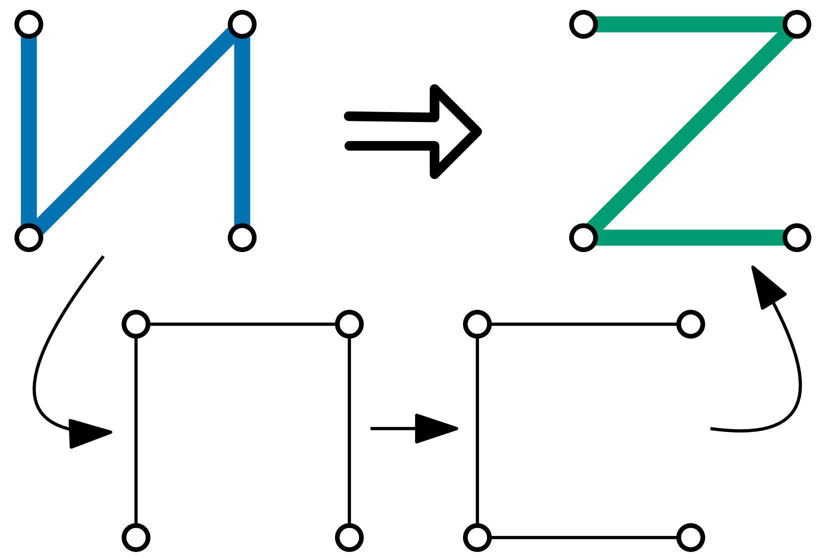
Open Problem 6: In case of connectedness (convex sets etc) how fast can the shortest flip-sequence be found?

Open Problem 7: Assume that starting and target paths are together crossing-free (aka compatible). Can they be embedded into a triangulation, so that flipping from one to the other can be done via paths of this triangulation?

Open Problem 8: Is the flip graph for crossing-free spanning paths connected via compatible paths? Two paths are compatible if their union is crossing-free.

Crossing-free Spanning Paths

The Happy Edge Property does **NOT** hold for crossing-free spanning-path, not even for points in convex position:



The diagonal is an happy edge, but is the only one that can be flipped.

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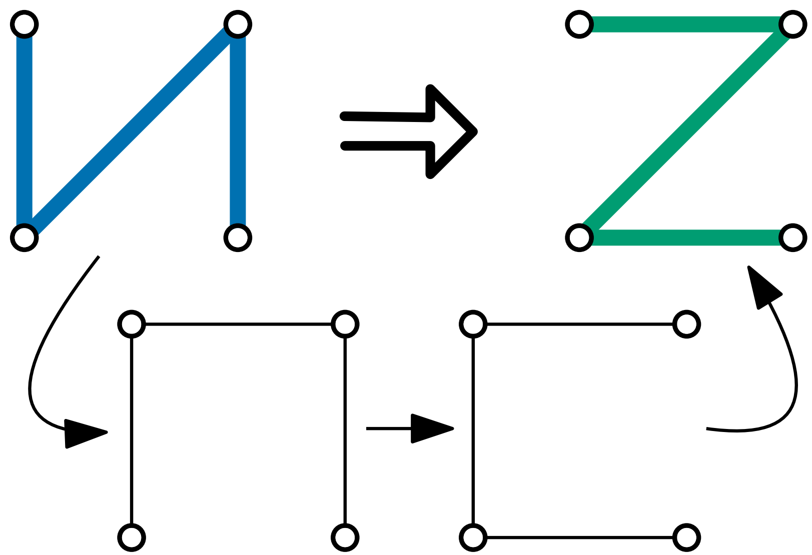

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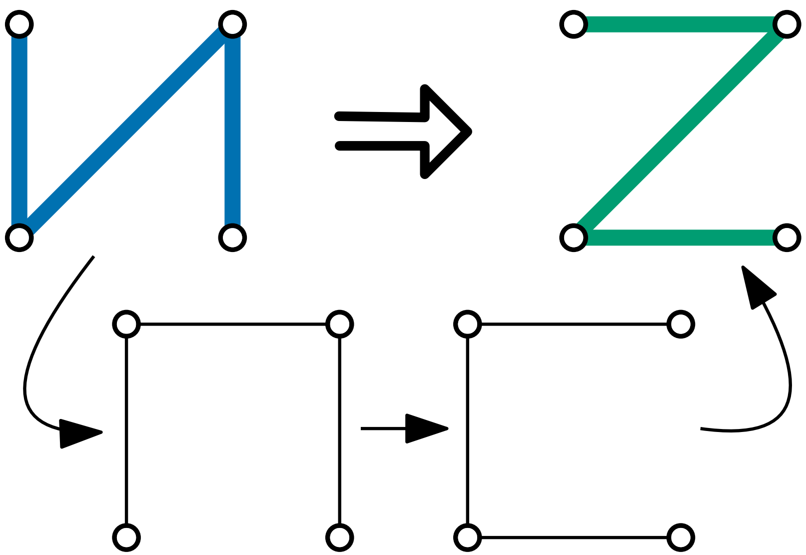

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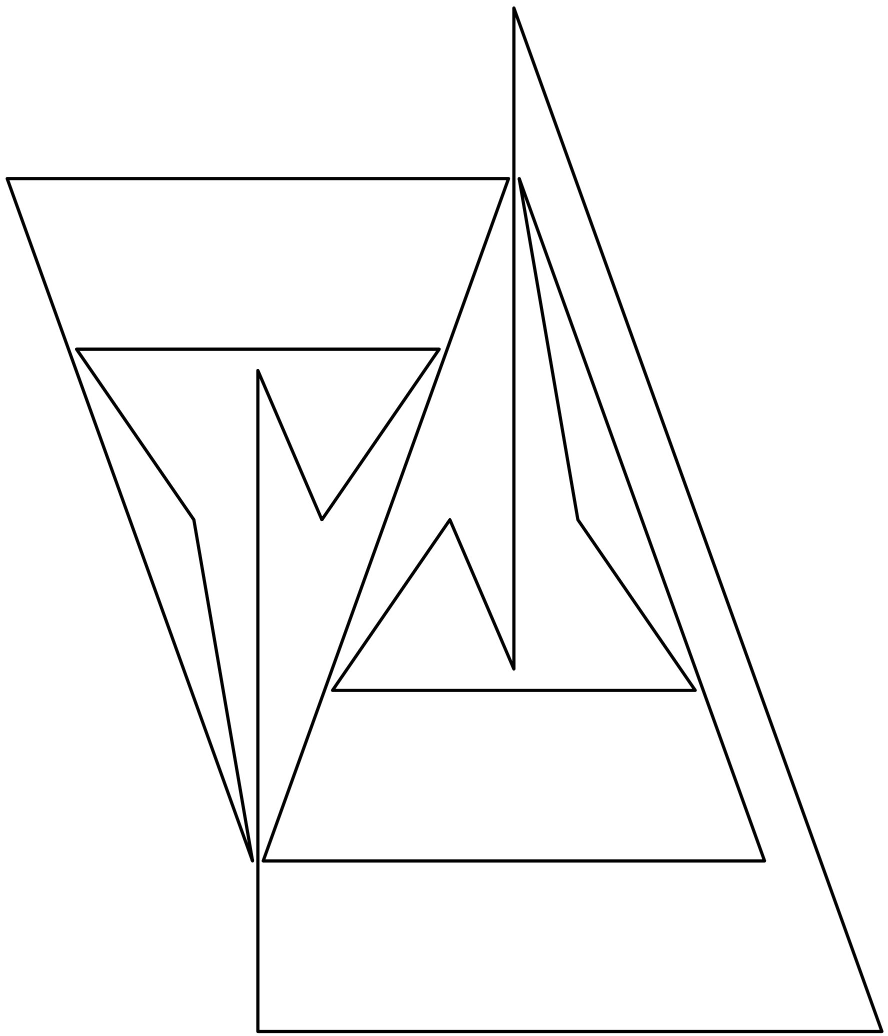
Reconfiguration problems seem to get harder when the happy edge property does not hold, but for **Problem 6**:

Wednesday April 9

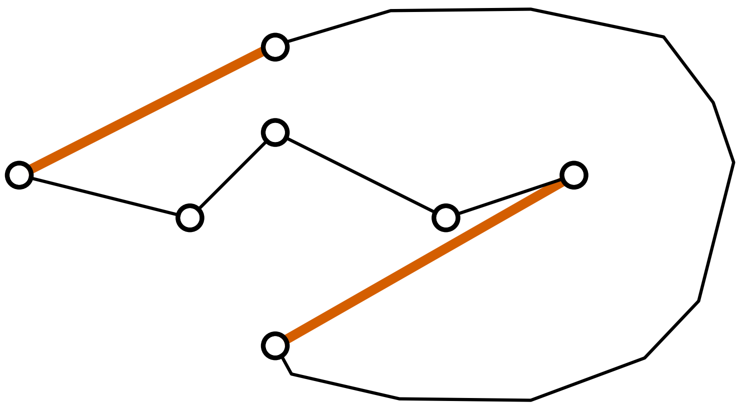
17:15 - 17:30 *Oswin Aichholzer and Joseph Dorfer*
A Linear Time Algorithm for Finding Minimum Flip Sequences between Plane Spanning Paths in Convex Point Sets

Crossing-free Hamiltonian Cycles

What if we **close** the path to a crossing-free spanning cycle?



- aka polygonalizations
- a flip consists of 2 edges



A flip is not always possible:
counterexample with 19 points
[Hernandoa, Houle, Hurtado 2002]

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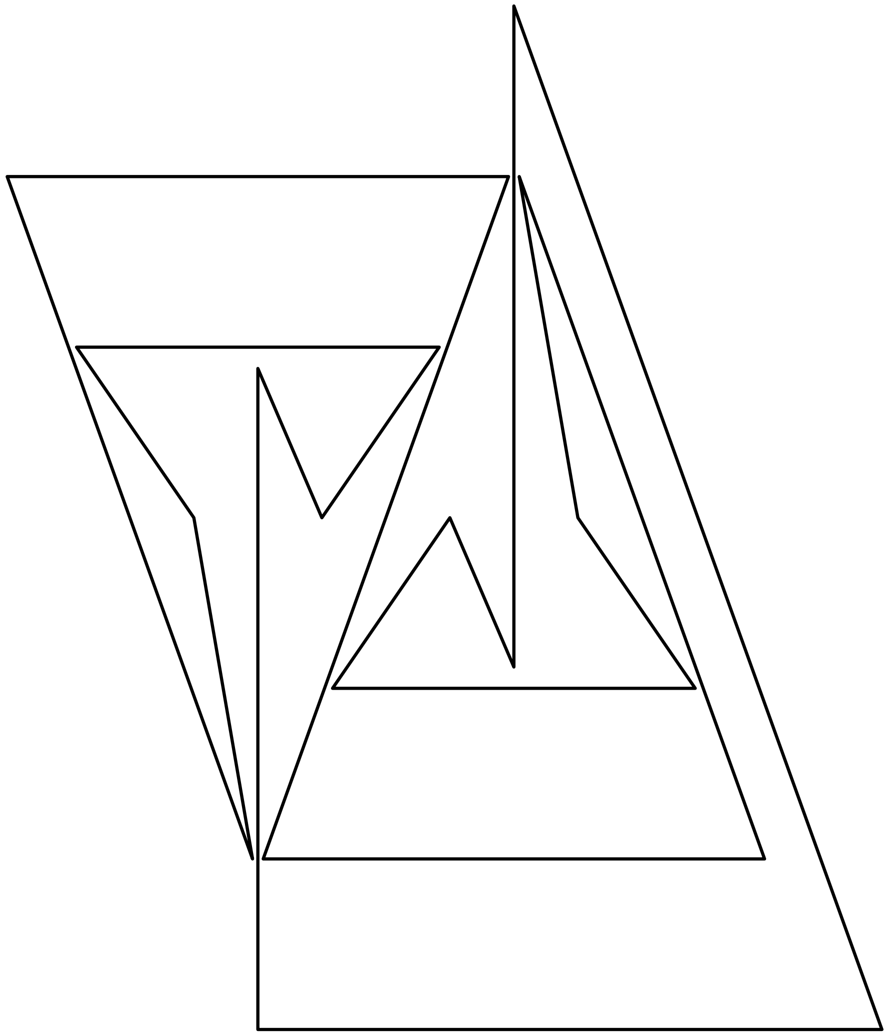

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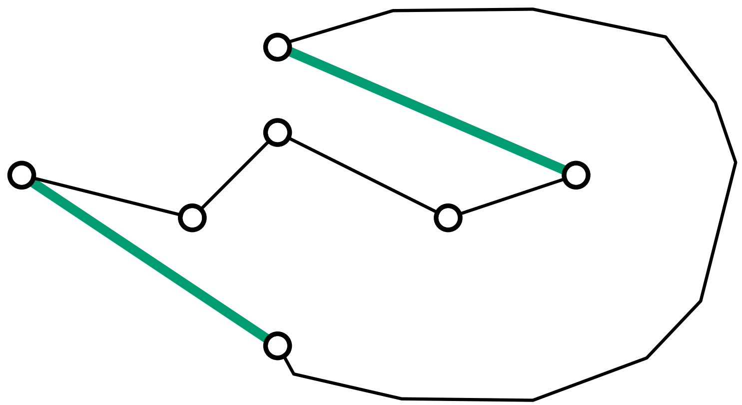

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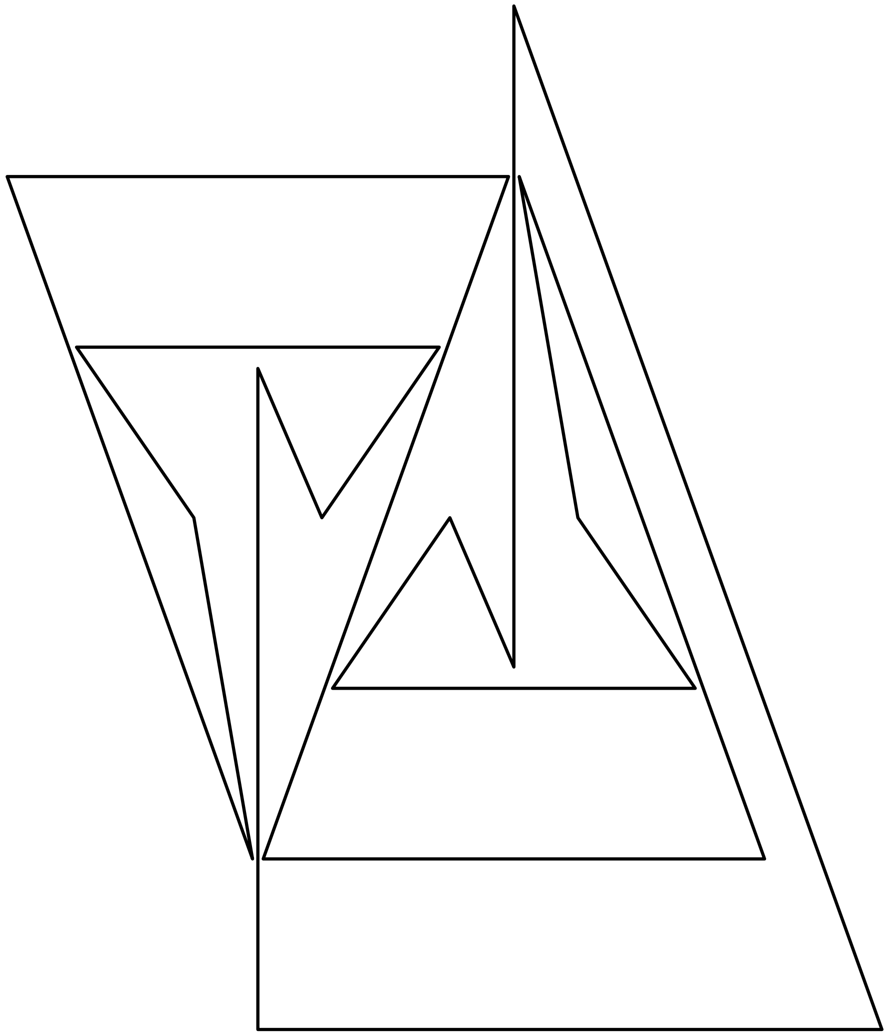

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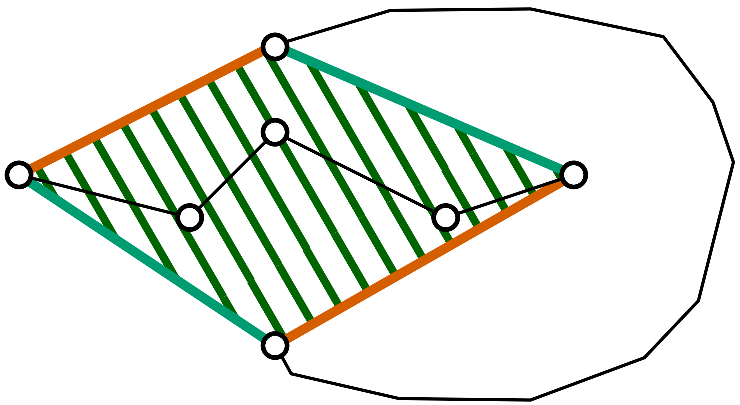

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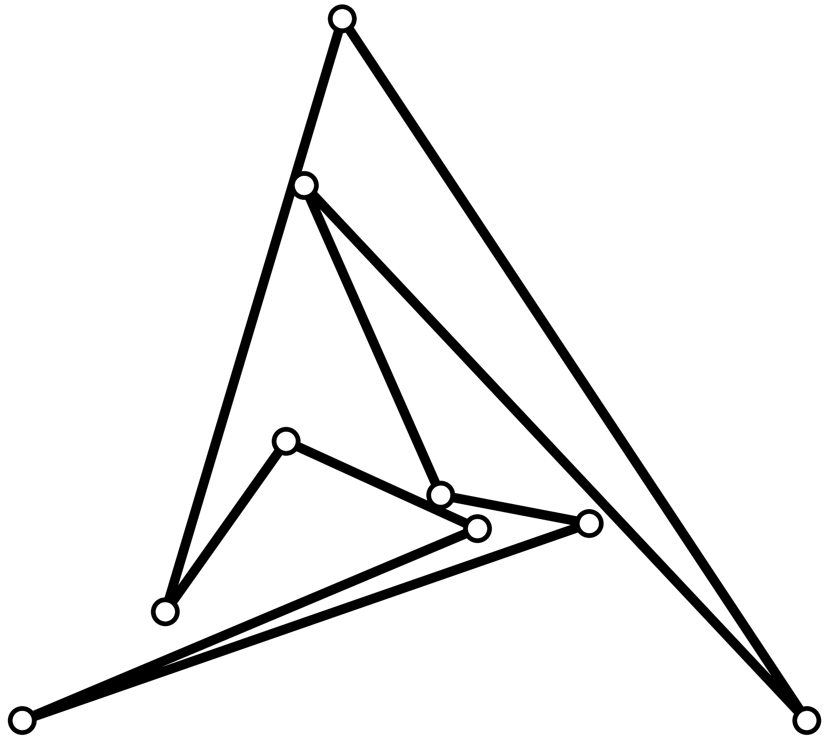

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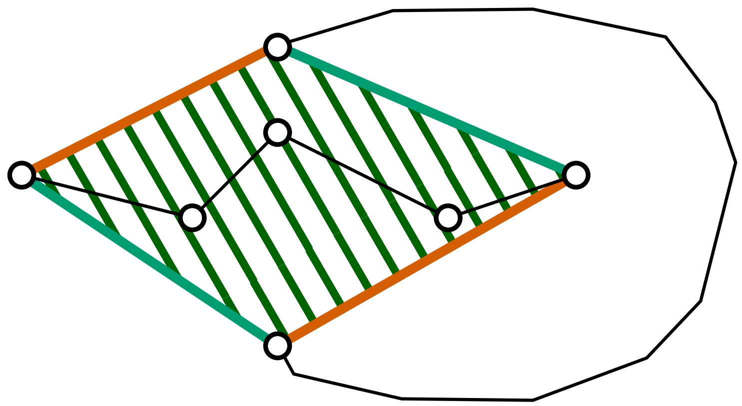
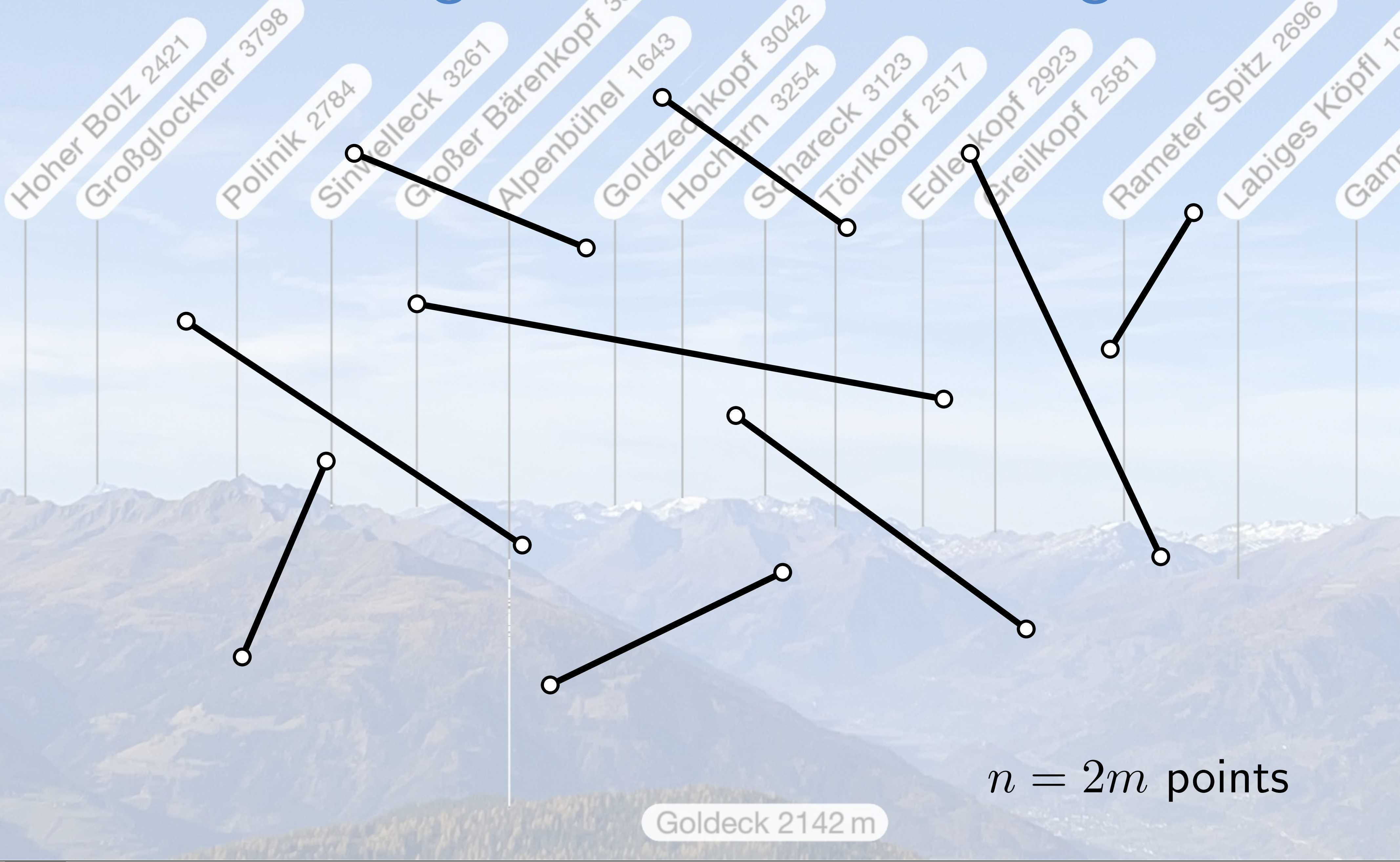


New example (2023) with only 9 vertices, no 2-edge-flip is possible. Minimal w.r.t. the number of points

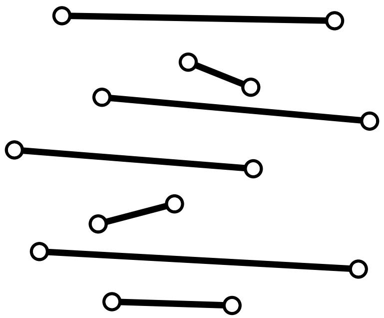
Crossing-free Perfect Matchings



$n = 2m$ points

Crossing-free Perfect Matchings

A natural reconfiguration step is a flip of 2 edges:



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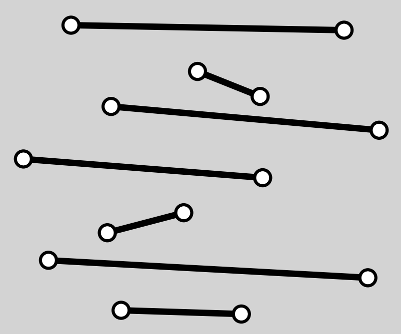

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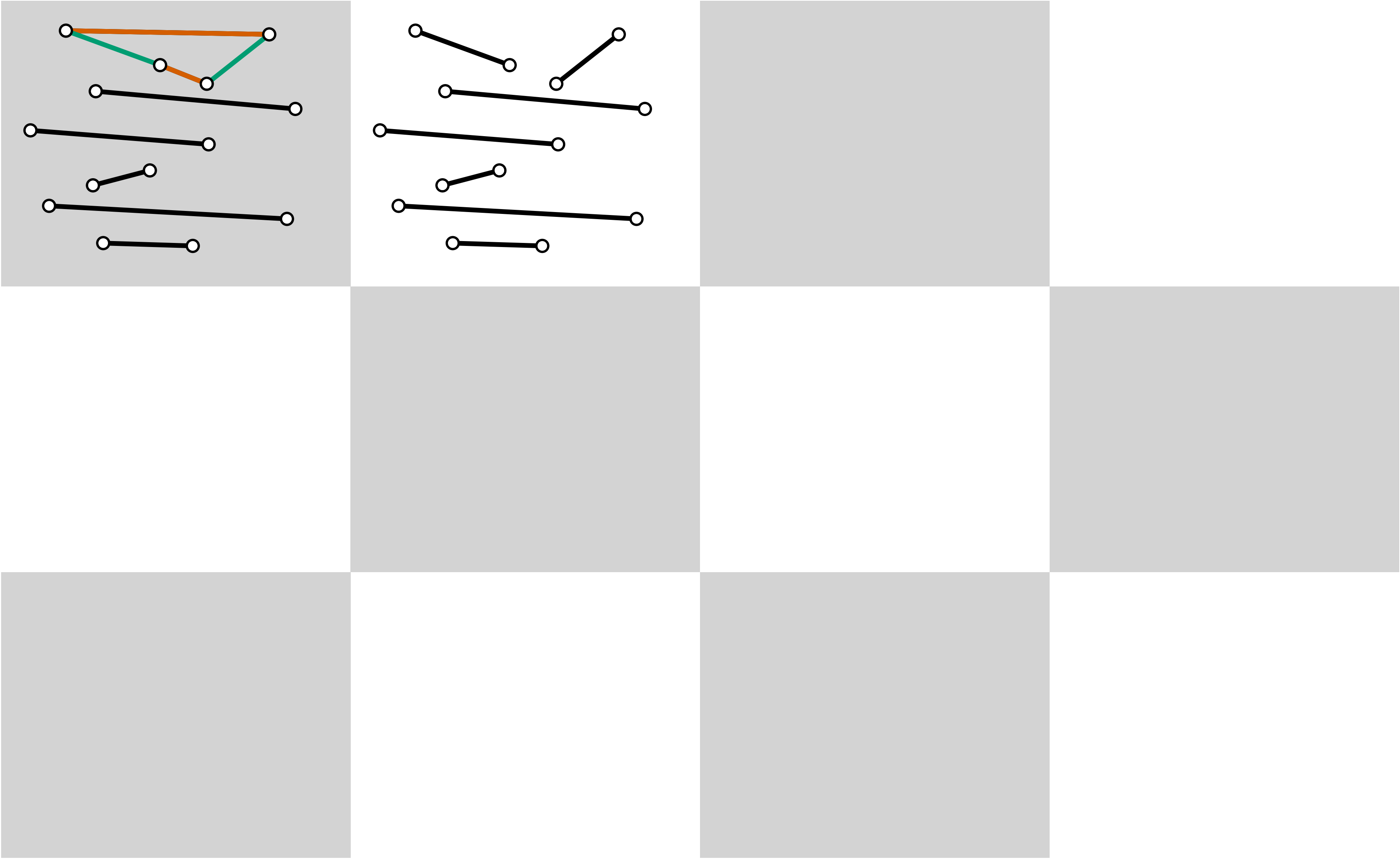

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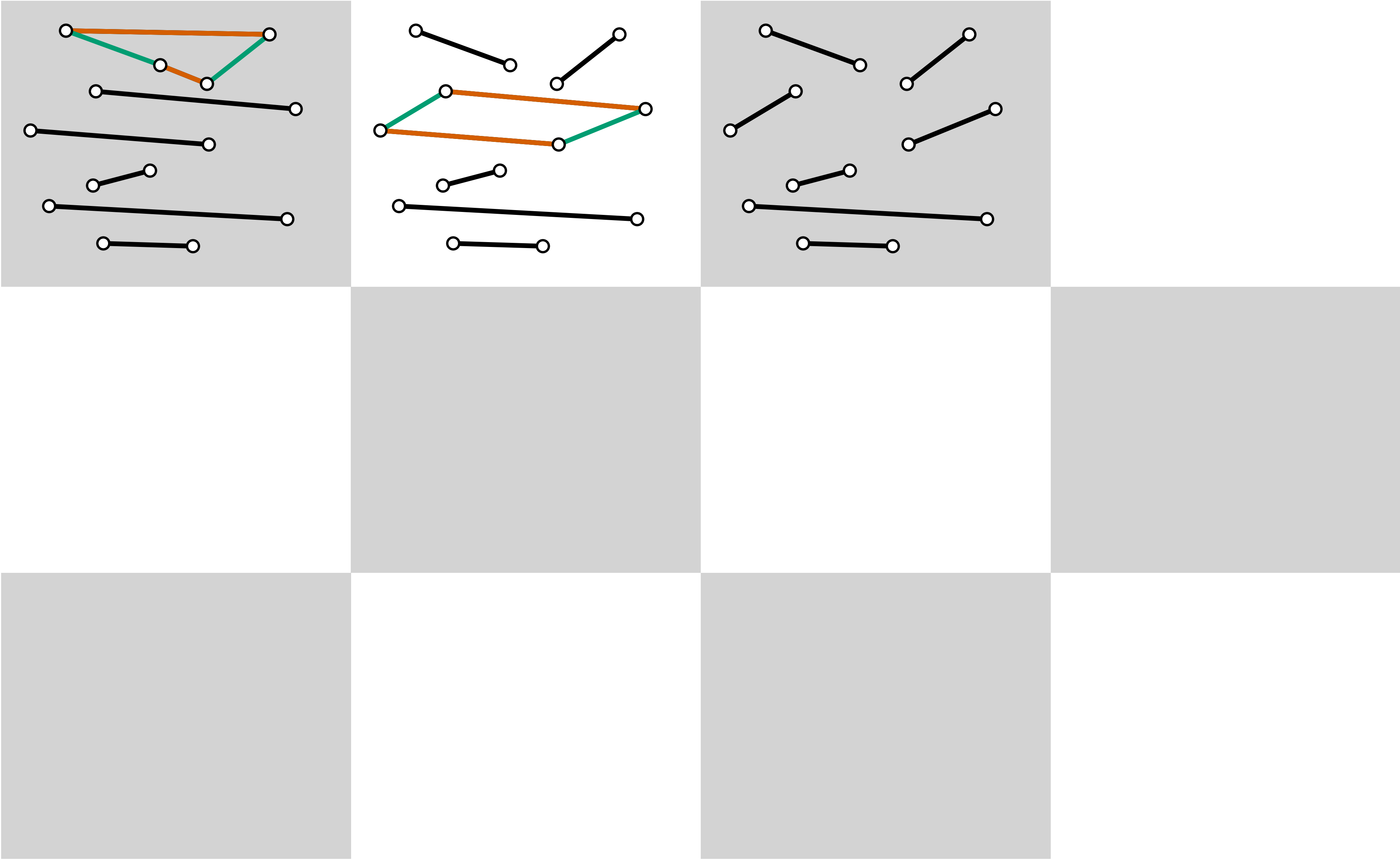

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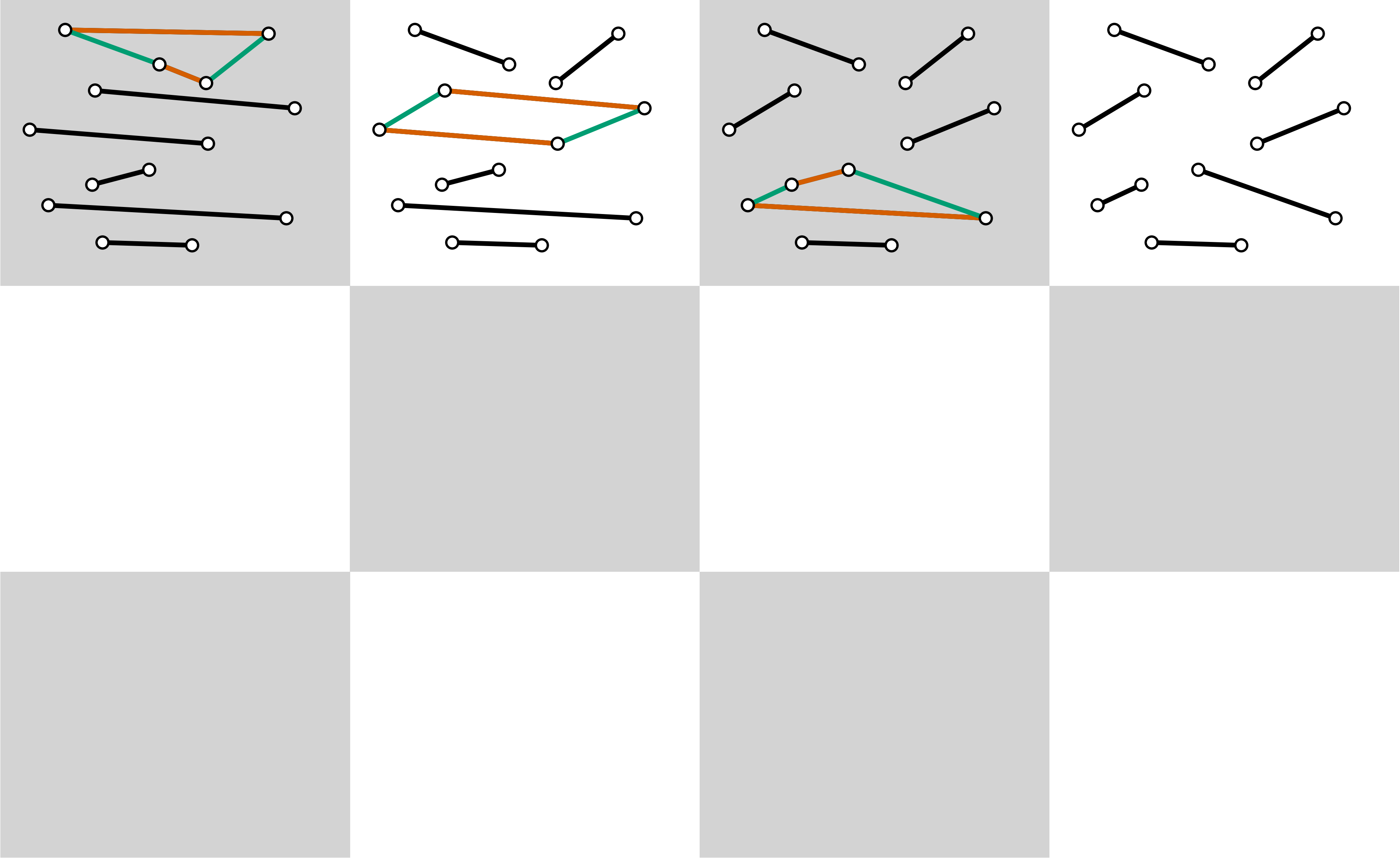

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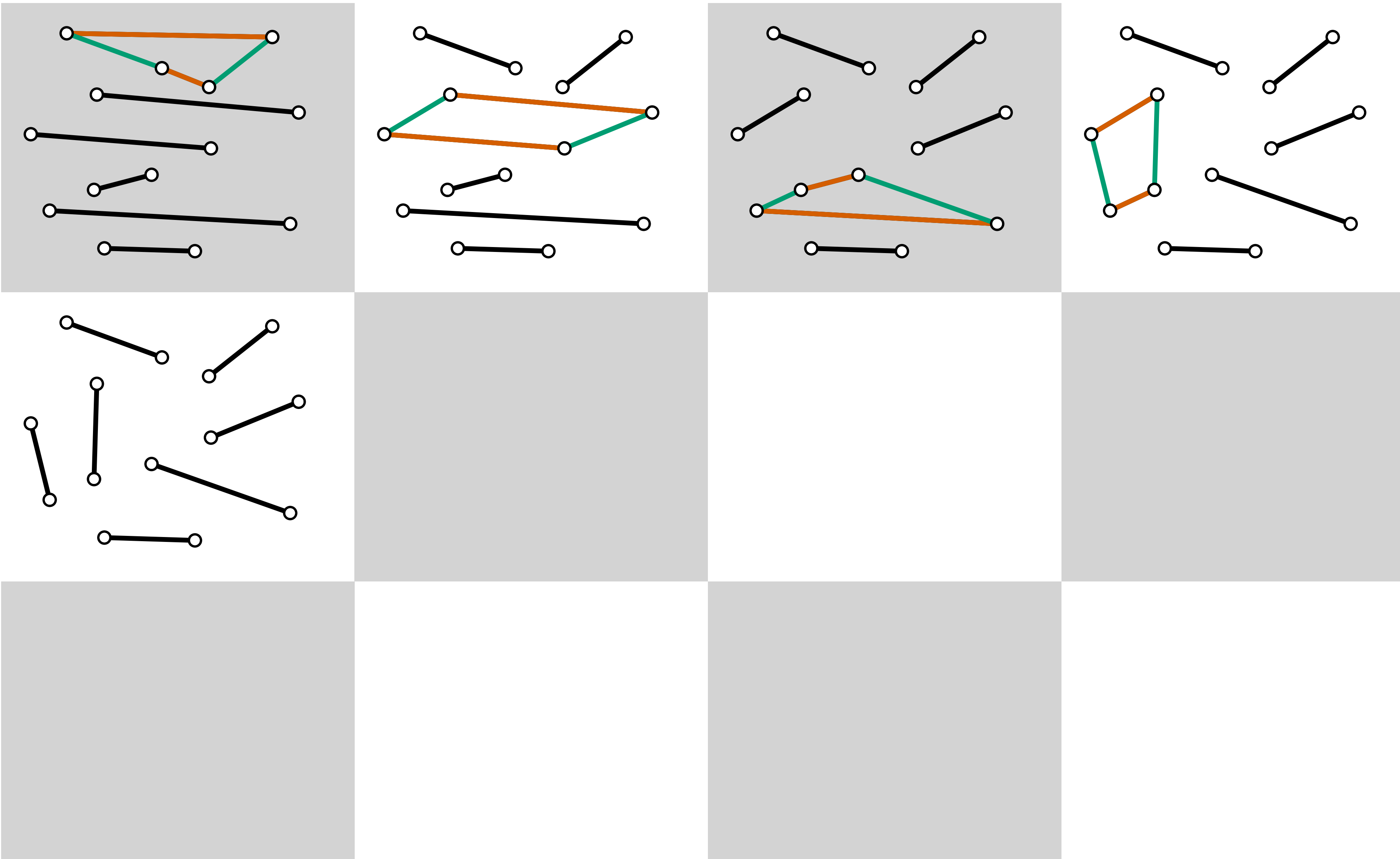

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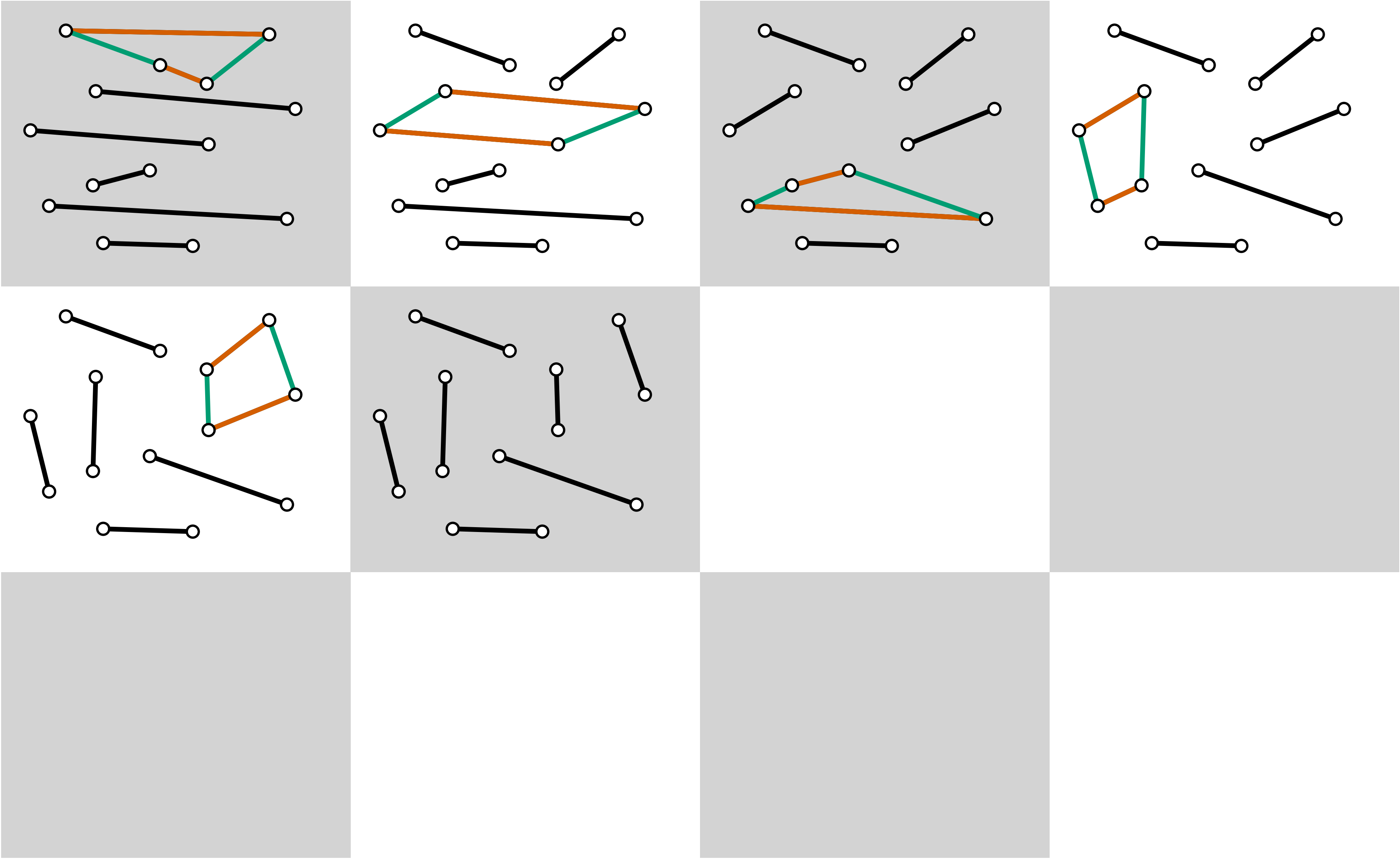

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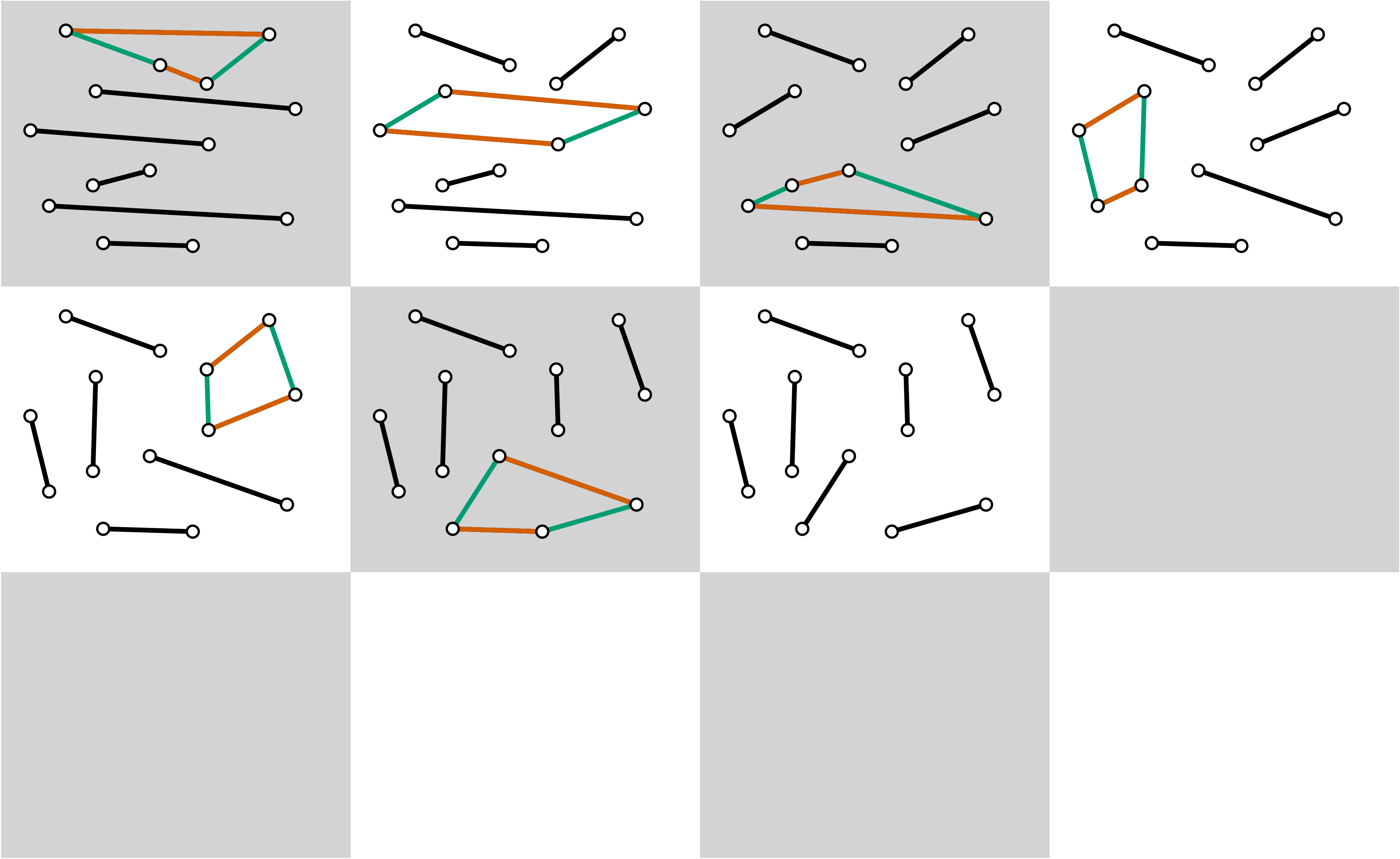

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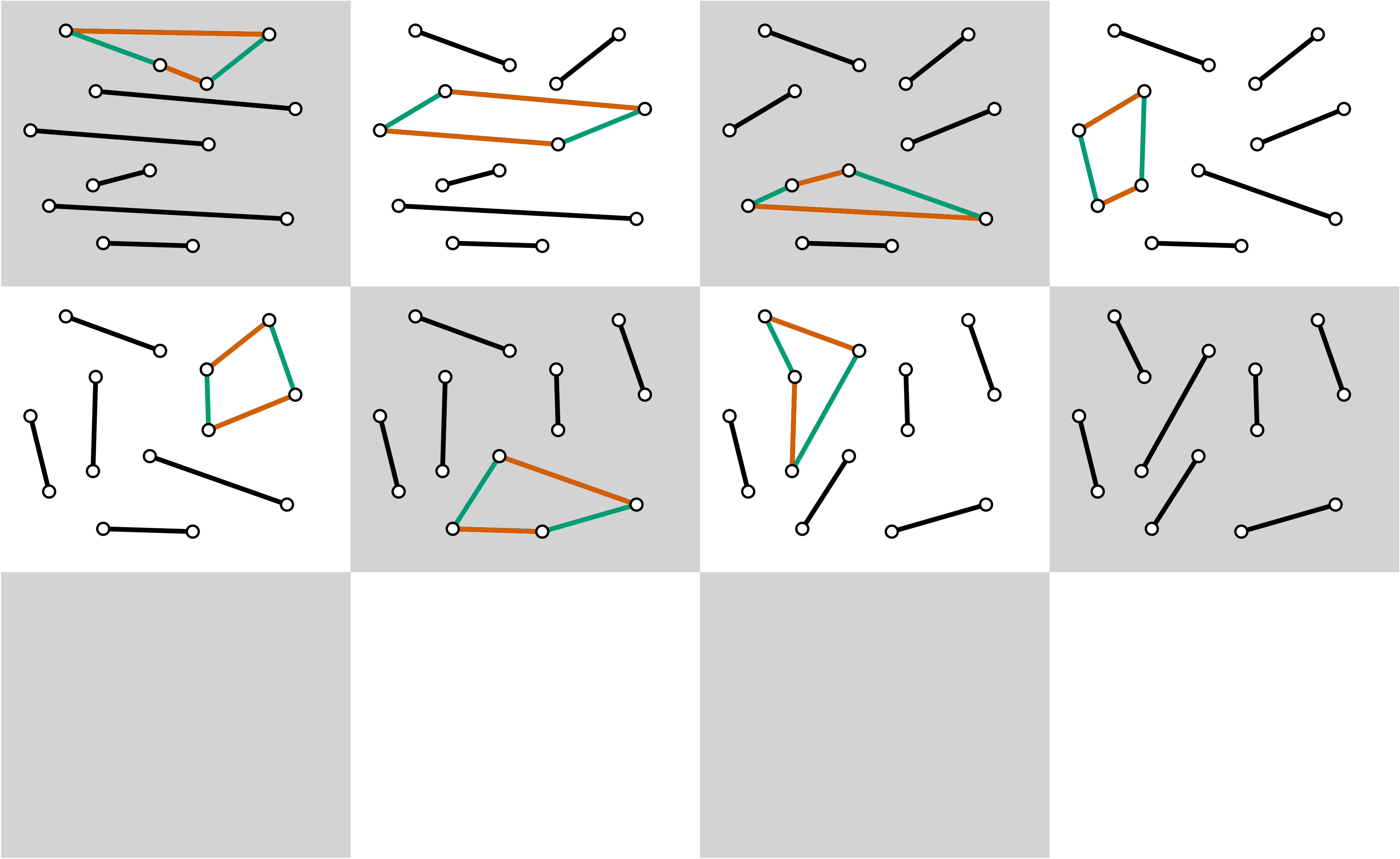

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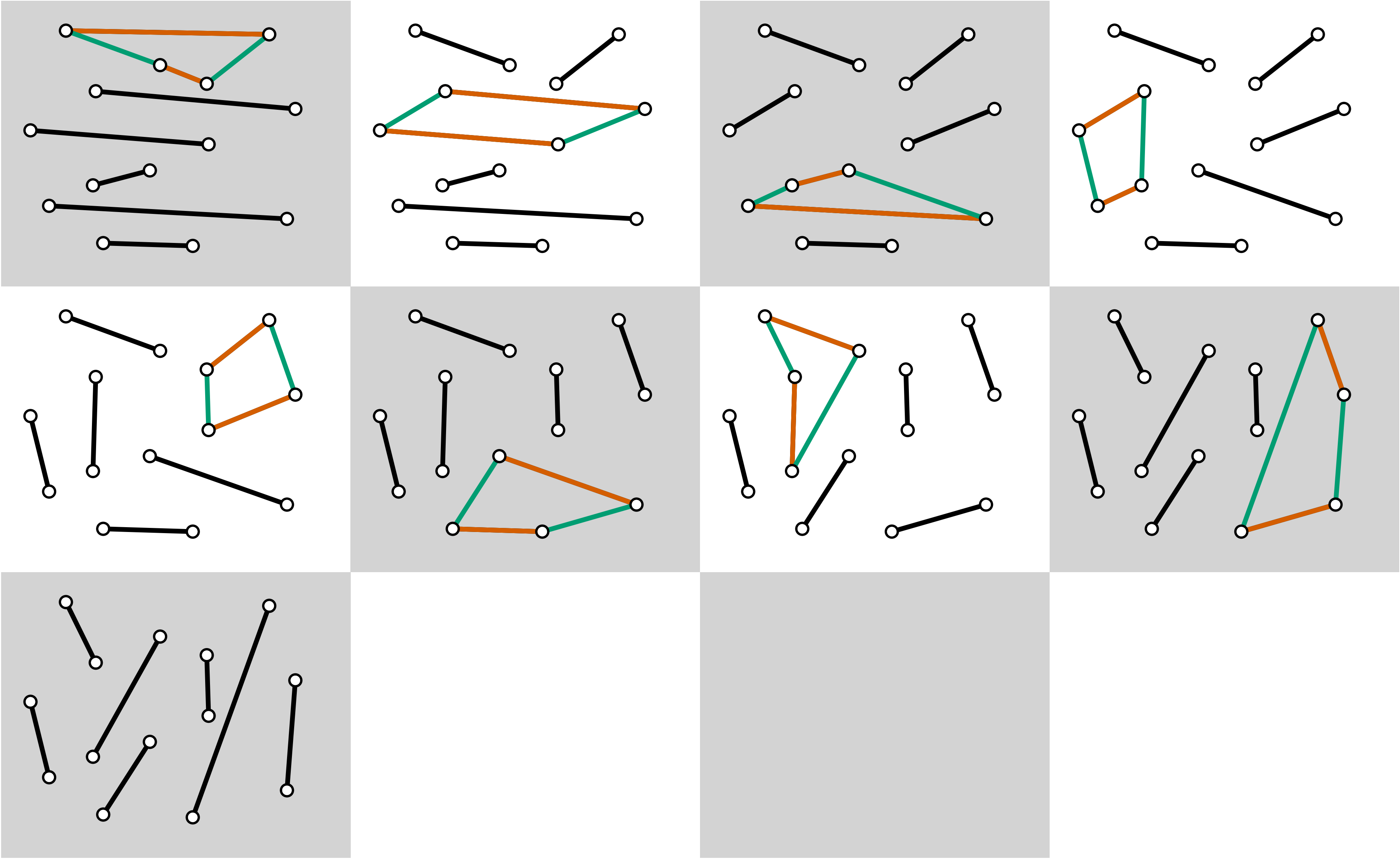

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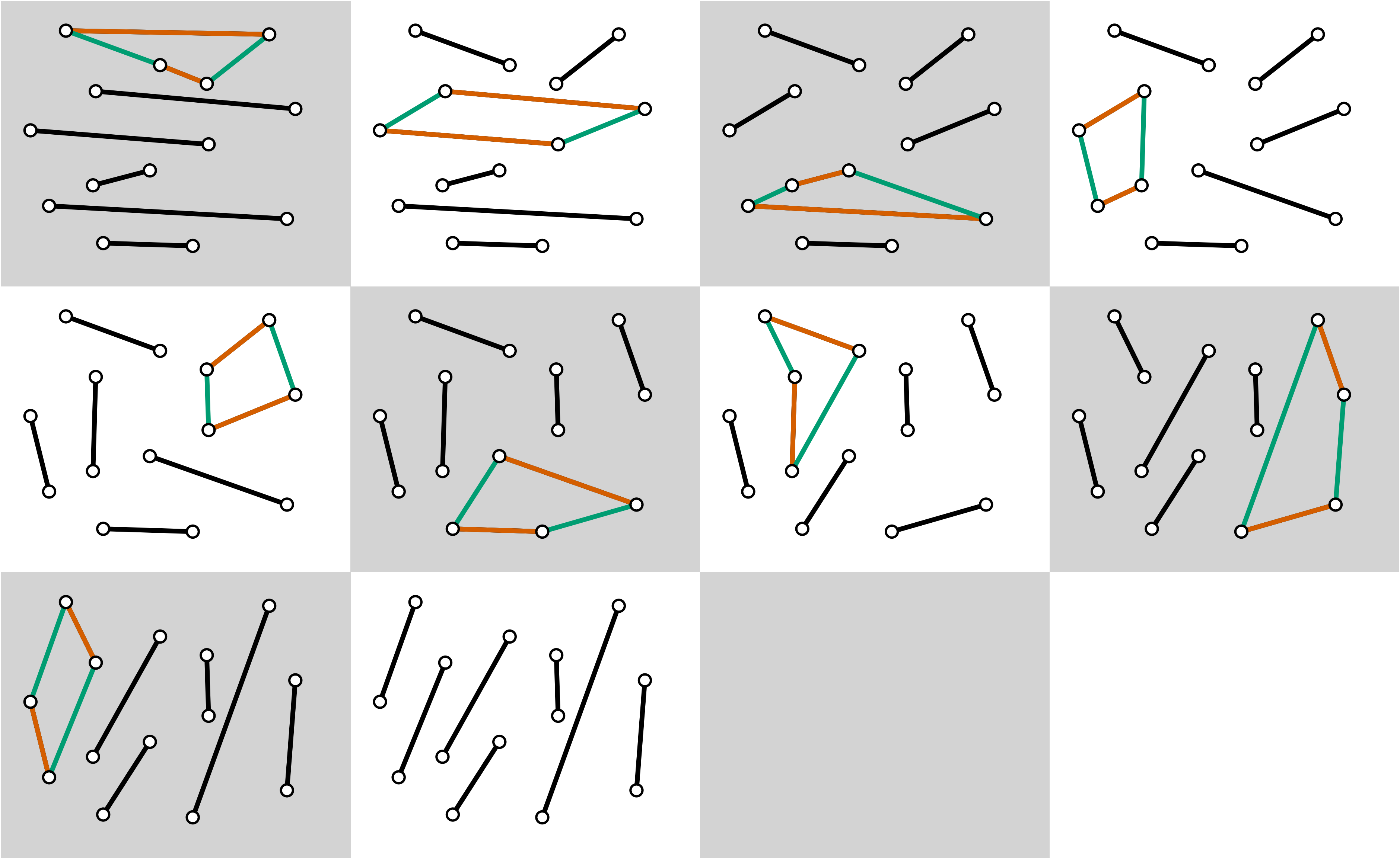

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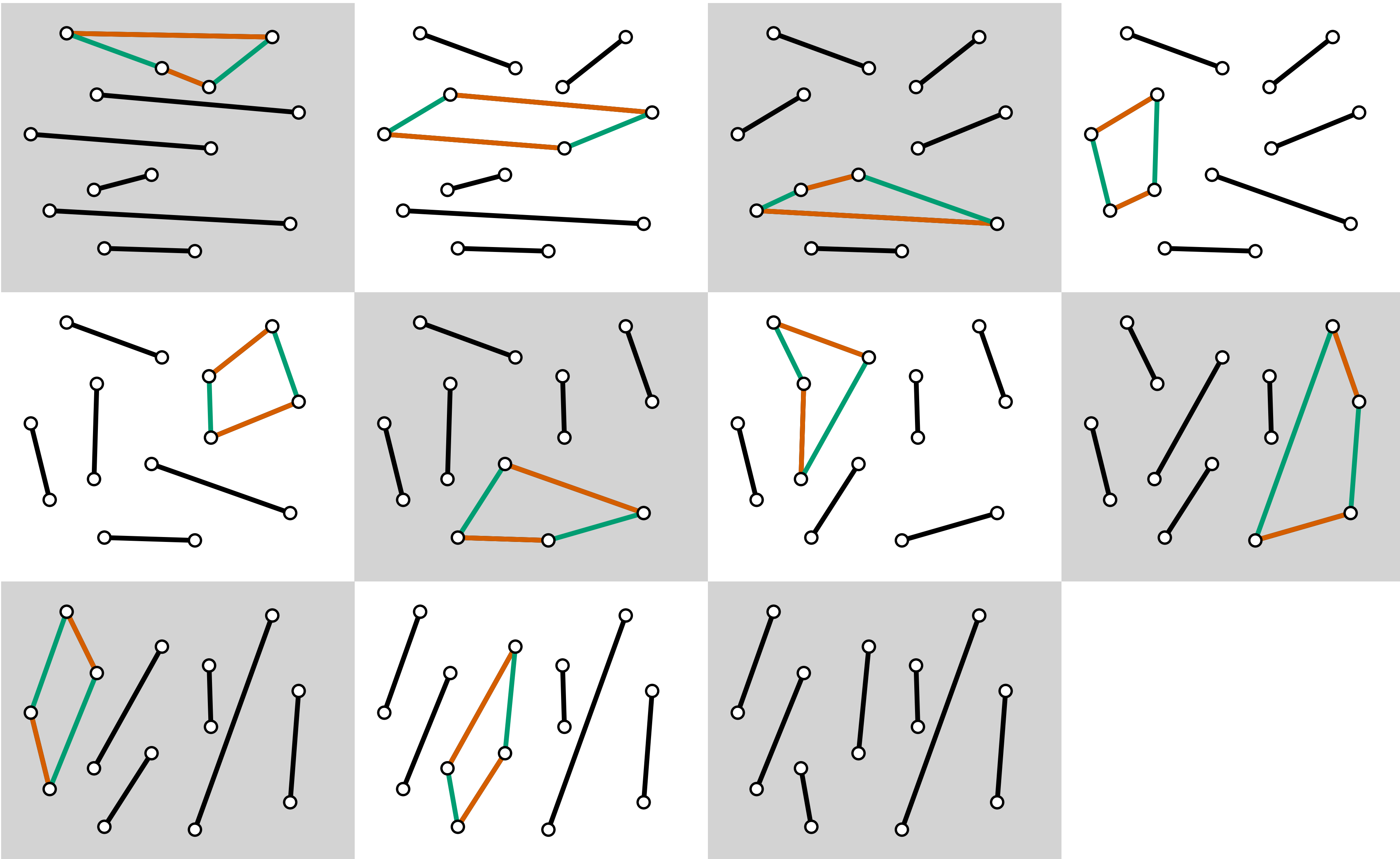

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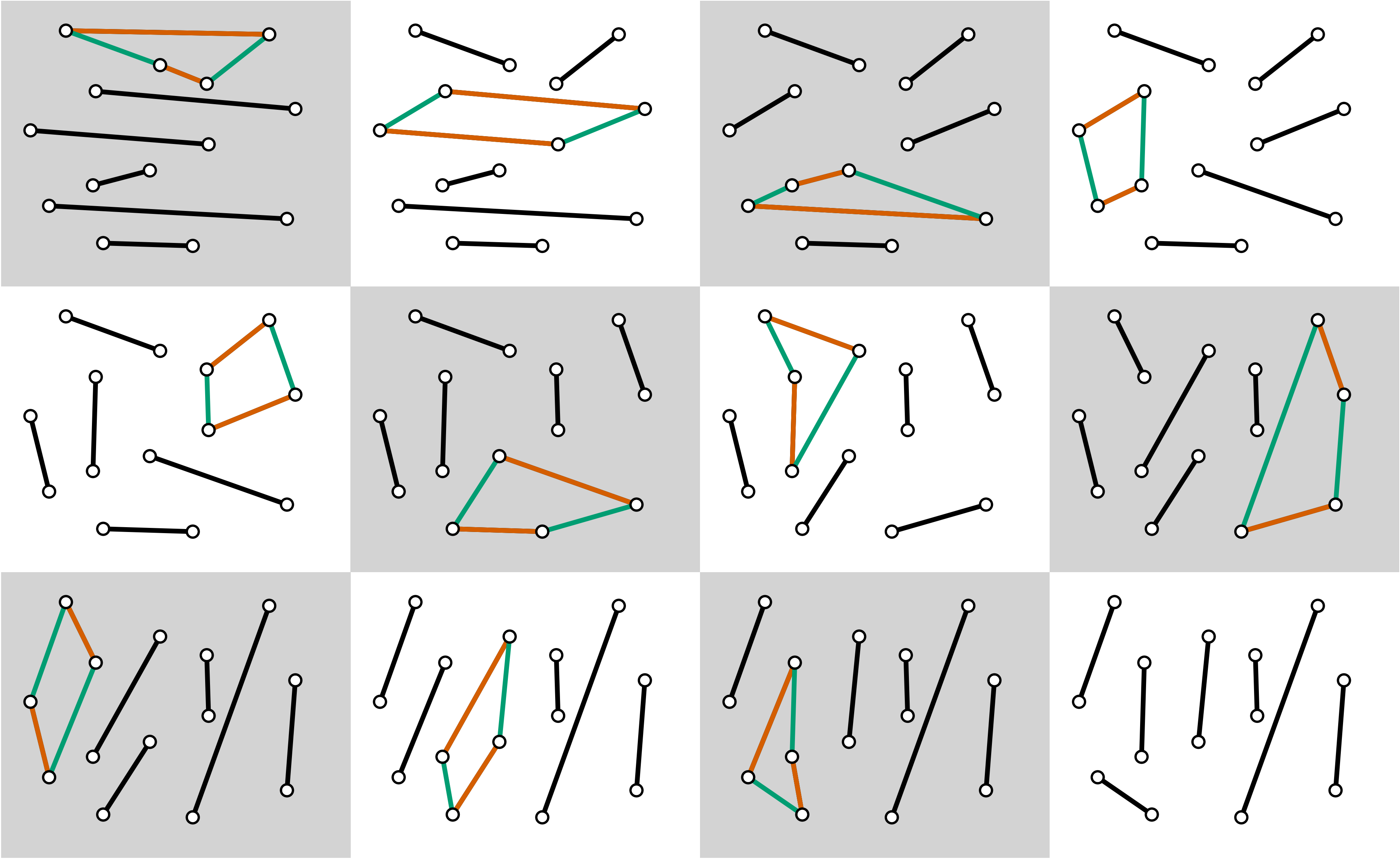

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Open Problem 9: Is the flip graph of crossing-free perfect matchings connected by 2-edge flips?

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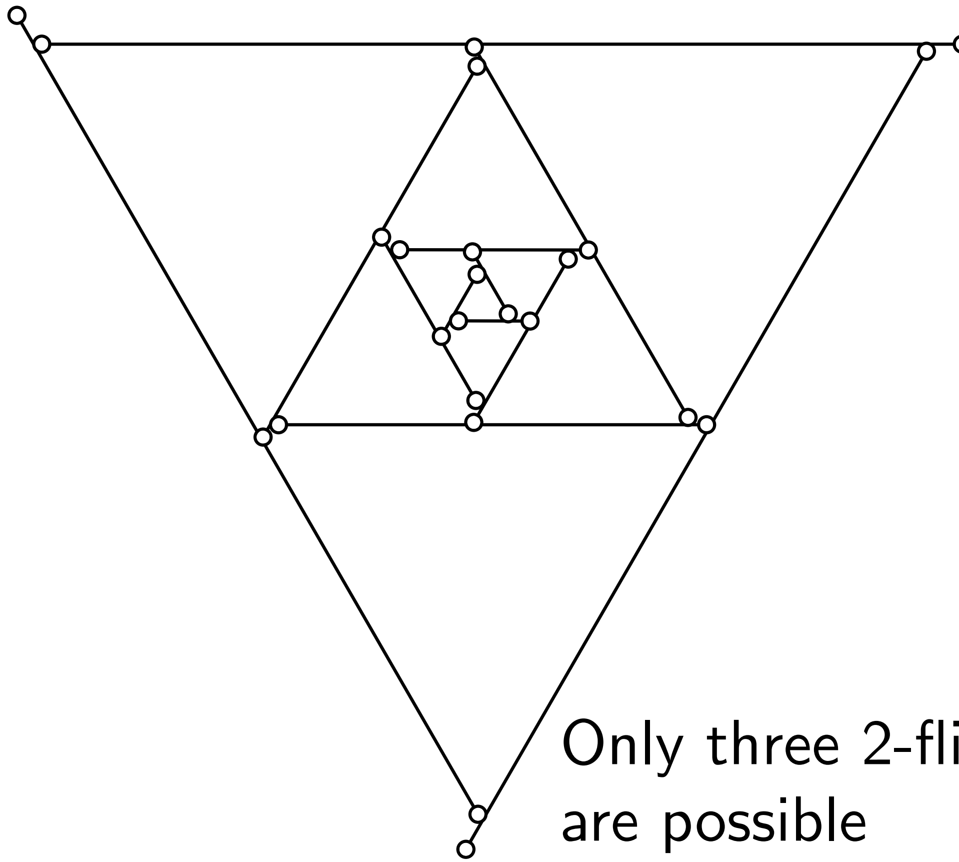
Can we actually show that for any crossing-free perfect matching there always exists at least one 2-flip? For the flip graph that means that there are no singletons, that is, any connected component has at least two elements.



Only three 2-flips are possible

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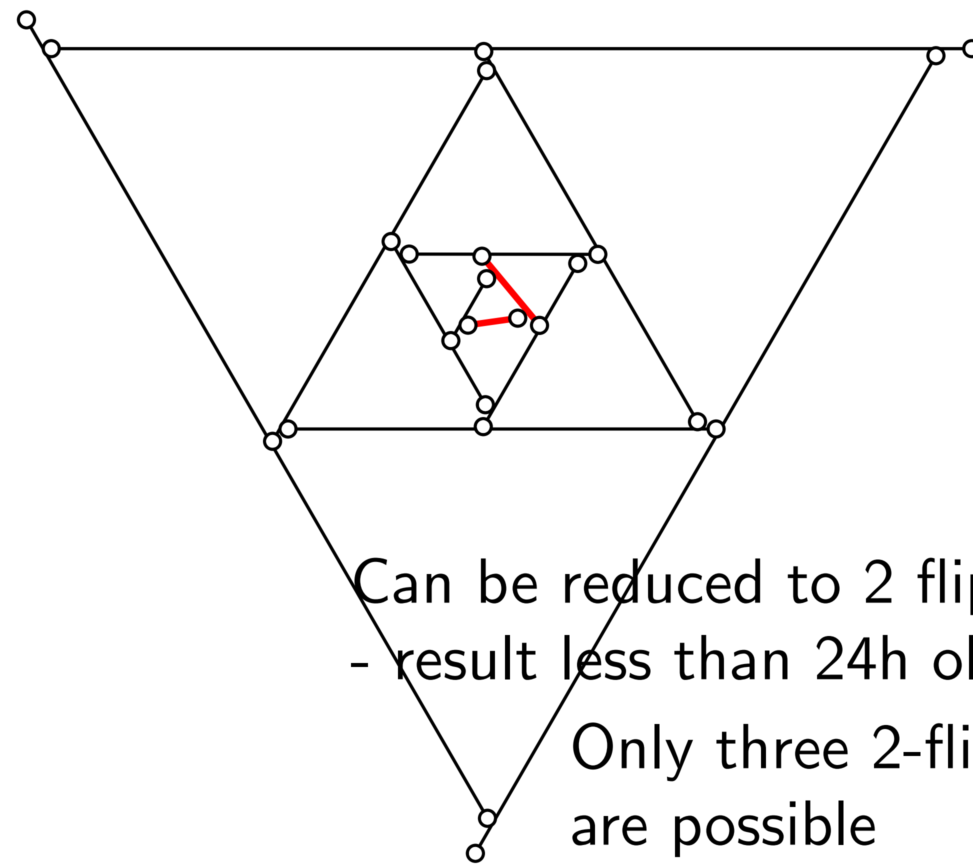

Can be reduced to 2 flips - result less than 24h old

Only three 2-flips are possible

Crossing-free Perfect Matchings

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Remark: Connected with a linear number of 2-edge flips for points in convex position [Hernando, Hurtado, Noy 2002]

Crossing-free Perfect Matchings

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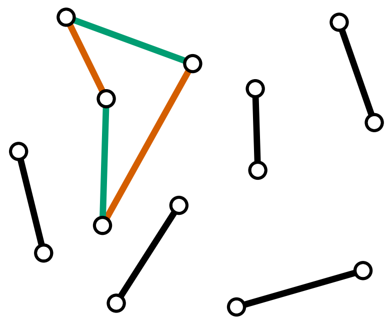
Open Problem 9: Is the flip graph of crossing-free perfect matchings connected by 2-edge flips?

Flipping 2 edges can be seen as "flipping the parity" of a crossing-free 4-cycle



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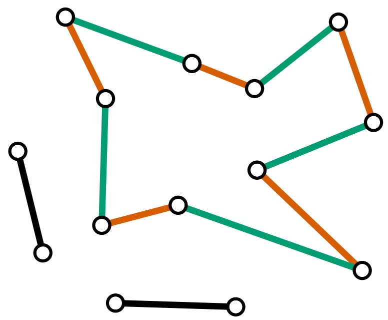
General: Flipping $k \geq 2$ edges can be seen as "flipping the parity" of a crossing-free $2k$-cycle.



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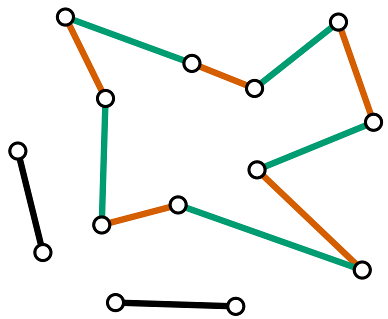

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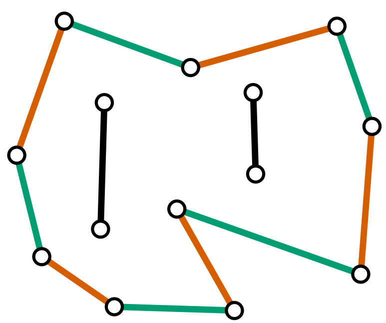
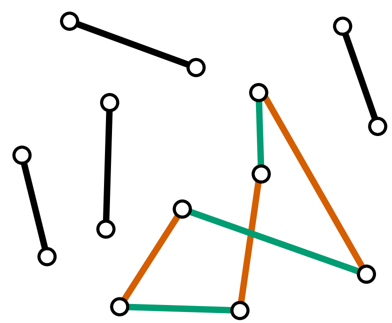


Cycles do not need to be empty in the interior



Cycles might self-intersect, but we consider only crossing-free cycles

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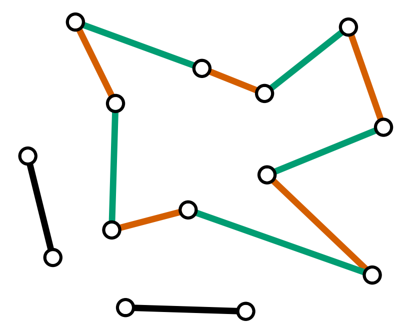

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For unbounded k the flip-graph is connected via flips of crossing-free k-cycles [Houle, Hurtado, Noy, Rivera-Campo, 2005]

Flip distance for crossing-free multi-cycles $O(\log n)$
([A., Bereg, Dumitrescu, Garcia, Huemer, Hurtado, Kano, Marquez, Rappaport, Smorodinsky, Souvaine, Urrutia, Wood, 2008])

Lower bound $\Omega(\log n / \log \log n)$ [Razen, 2008]

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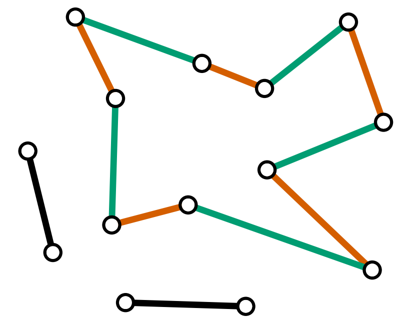

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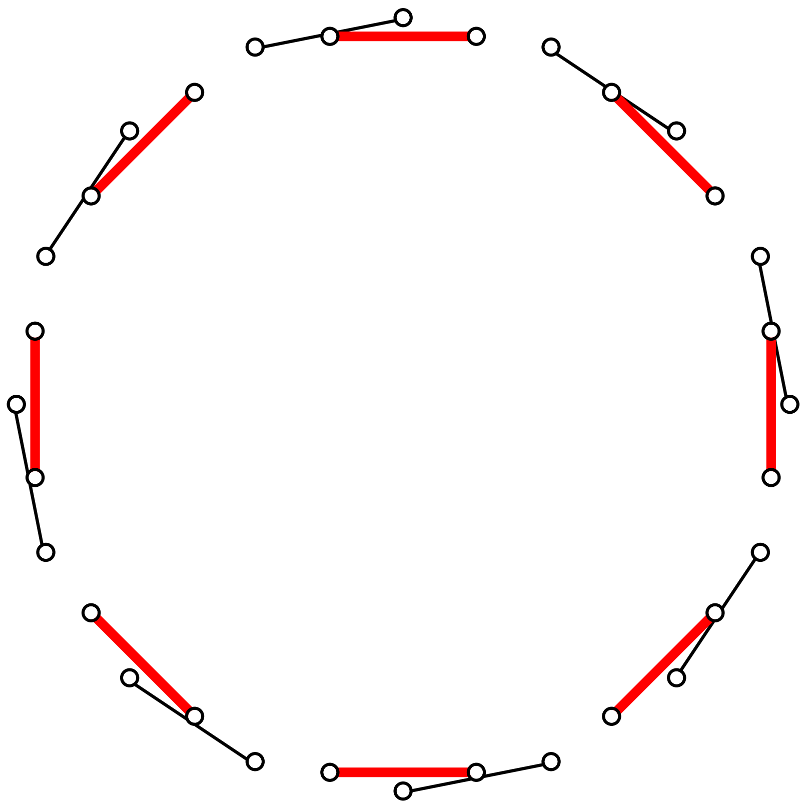
Open Problem 10: Is the flip graph of crossing-free perfect matchings connected via crossing-free $\leq 2k$-cycles, $k = o(n)$?

Crossing-free Perfect Matchings

The Happy Edge Property does not hold for crossing-free perfect matchings with flips of sublinear size:



[M. Schüßler, 2024]

- Red matching edges are happy and should not be flipped
- Respecting happy edges, all black matching edges must be flipped at the same time
- Respecting happy edges, at least half of the edges must be flipped at the same time
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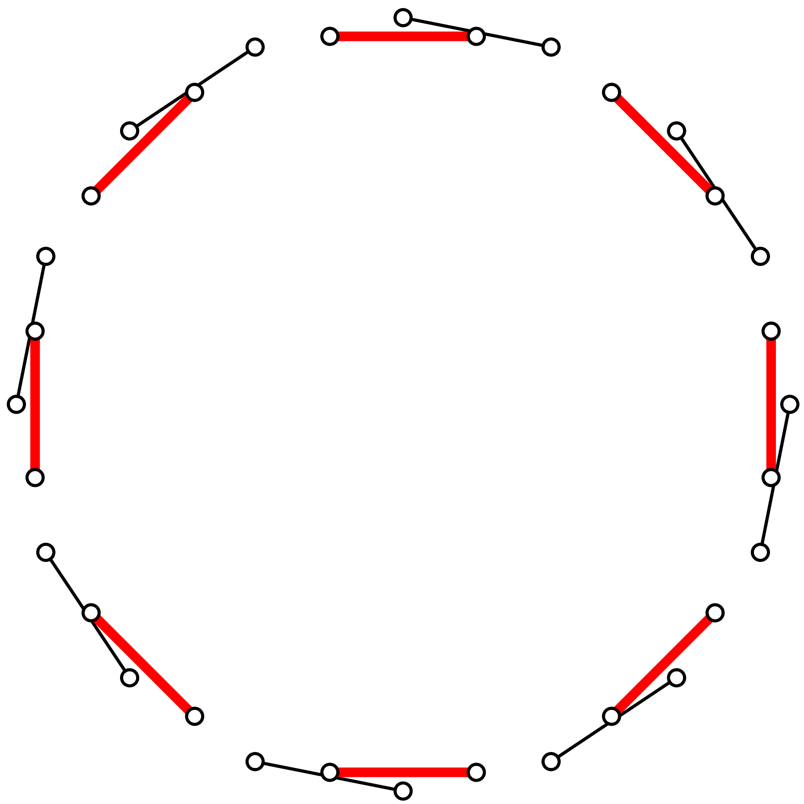

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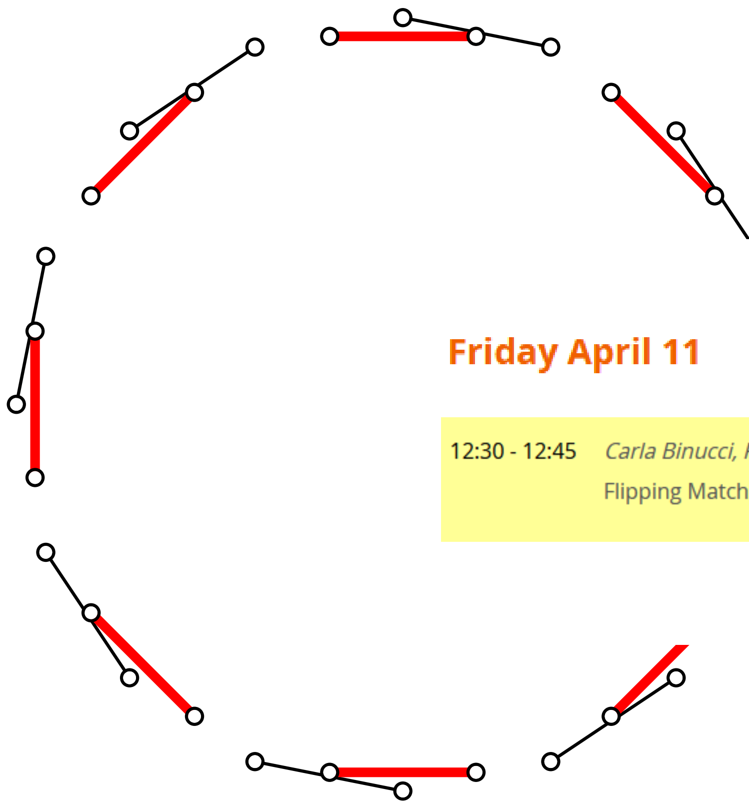

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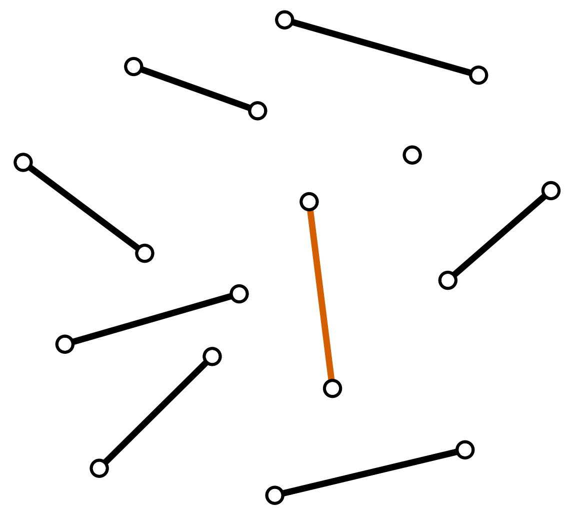
Friday April 11

12:30 - 12:45 *Carla Binucci, Fabrizio Montecchiani, Daniel Perz and Alessandra Tappini*
Flipping Matchings is Hard

Crossing-free Almost Perfect Matchings

Almost perfect matching: $n = 2m + 1$ points, m edges



A flip involves
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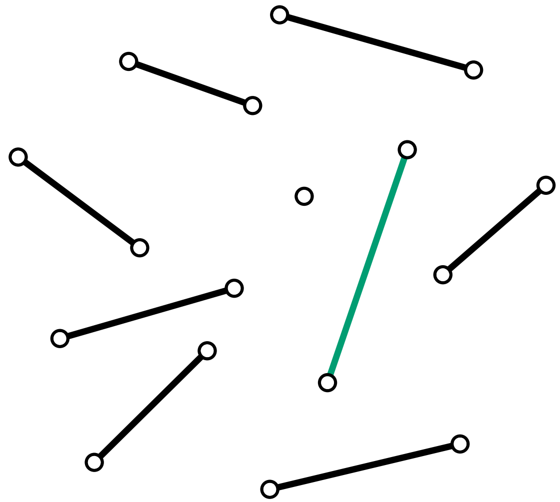

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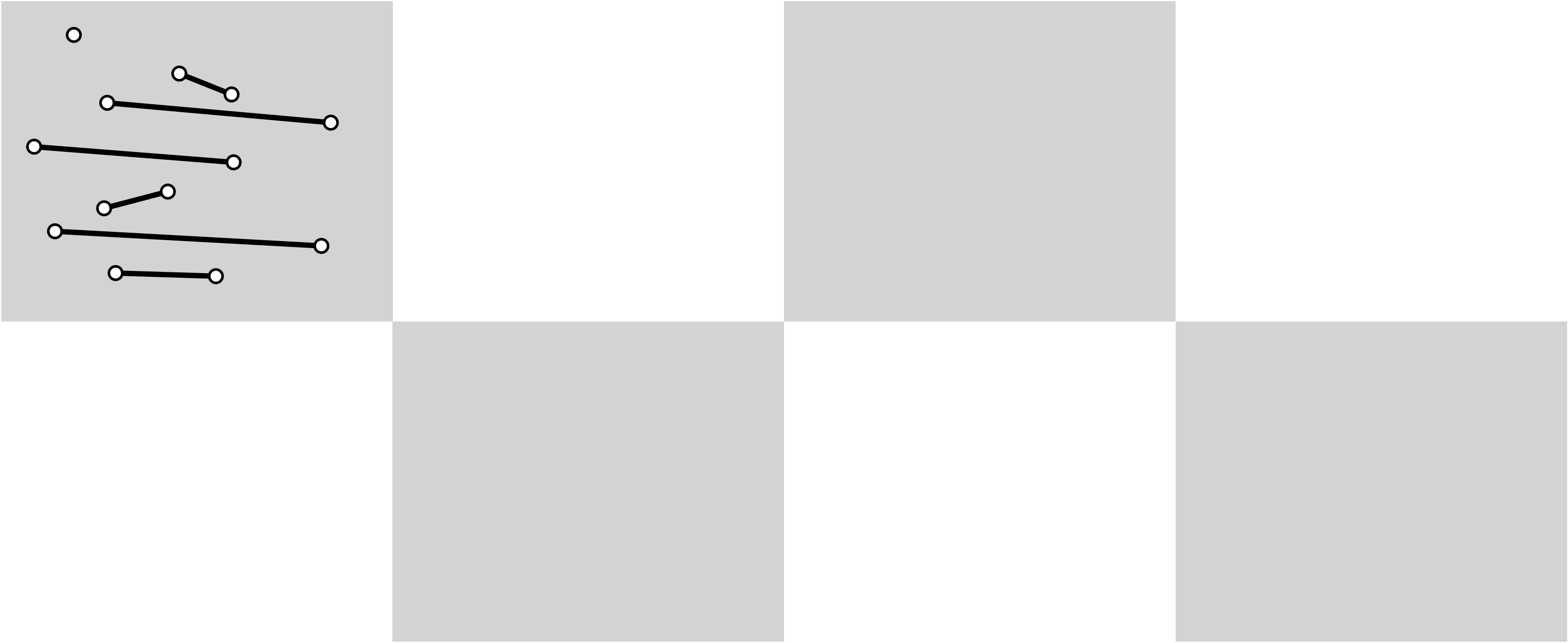

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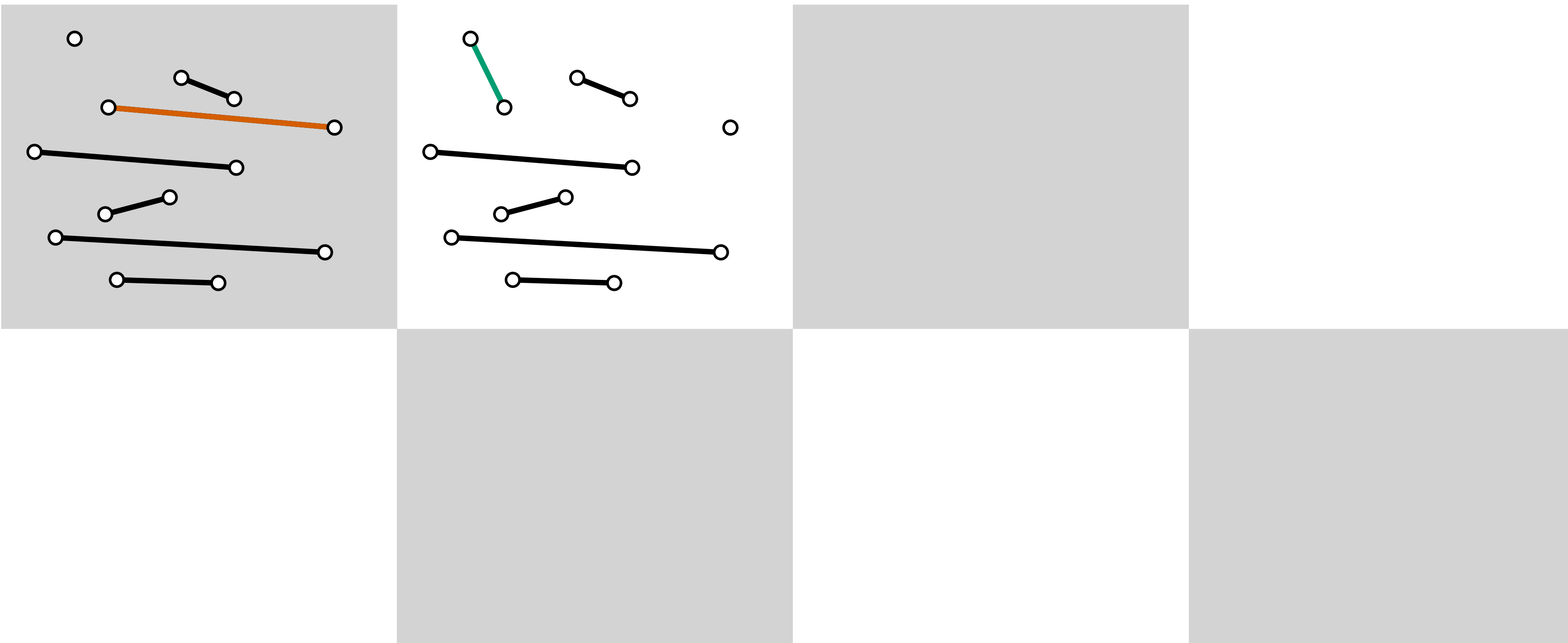

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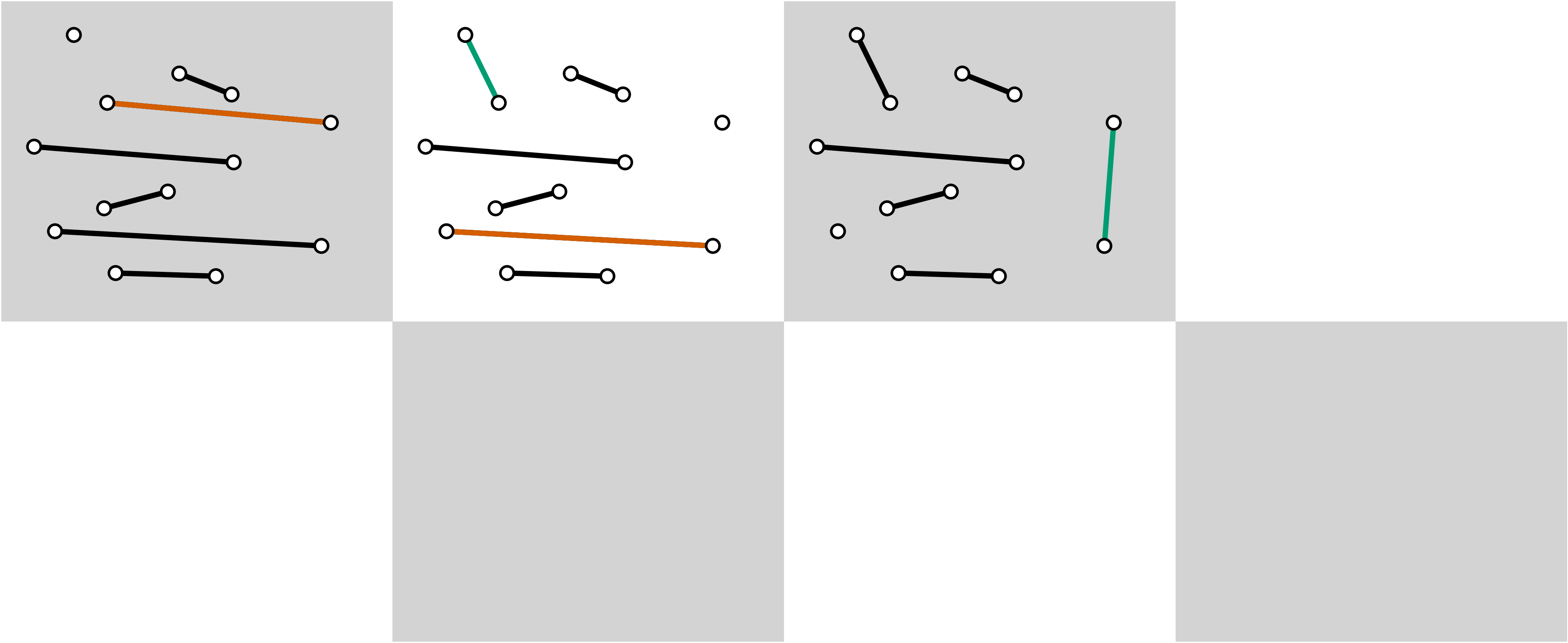

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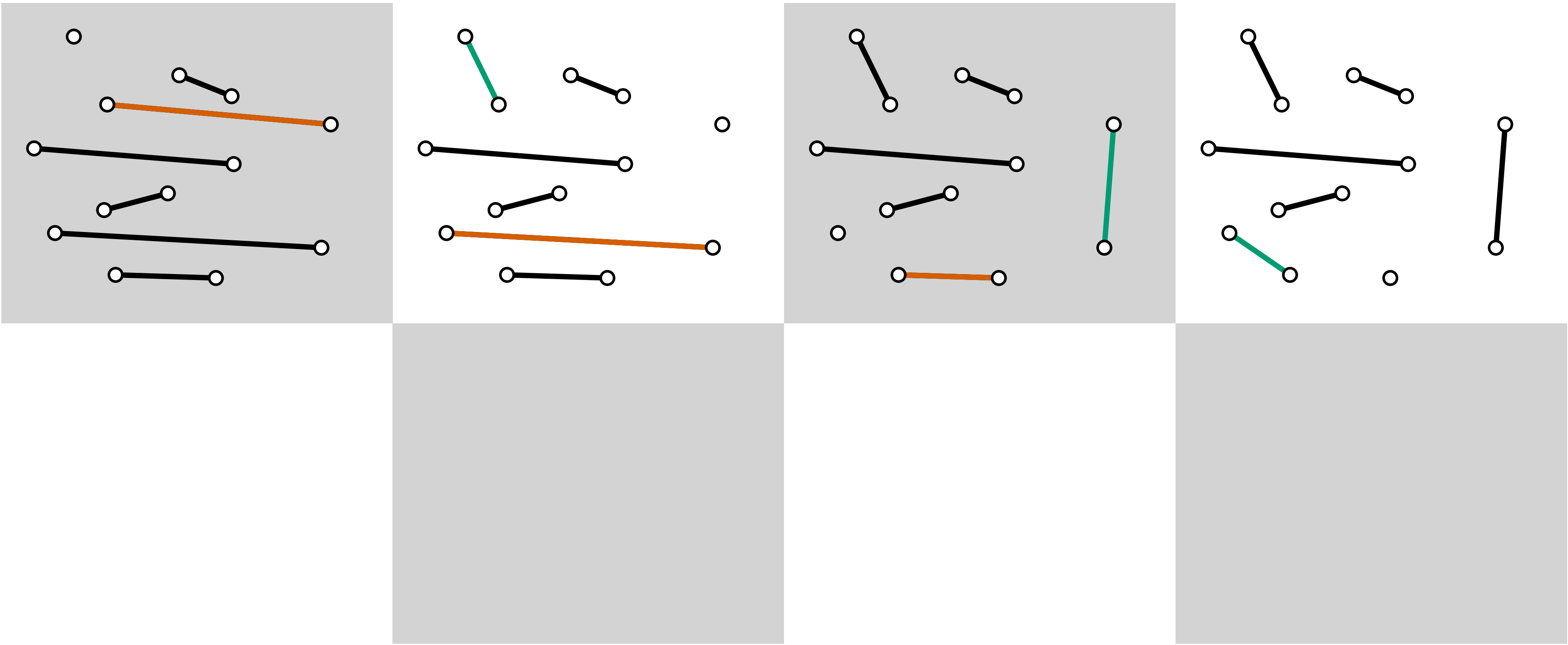

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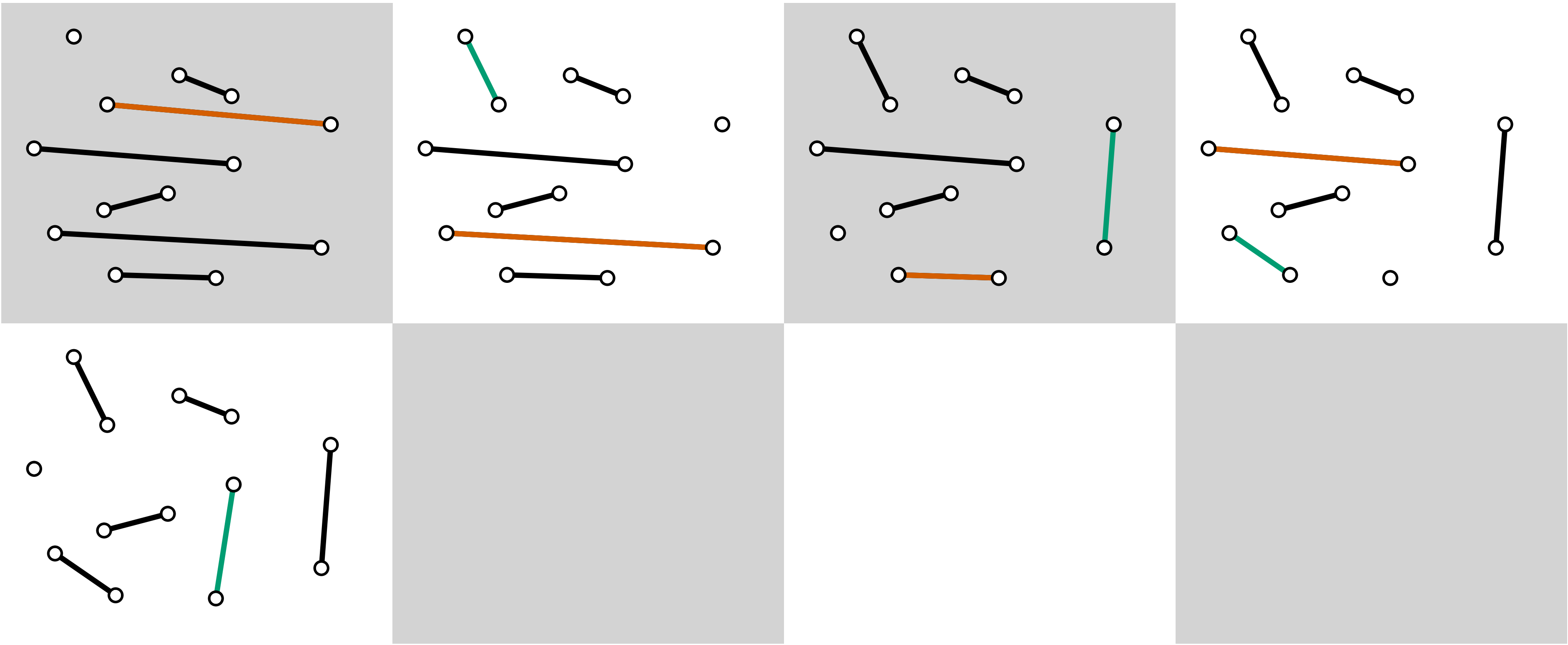

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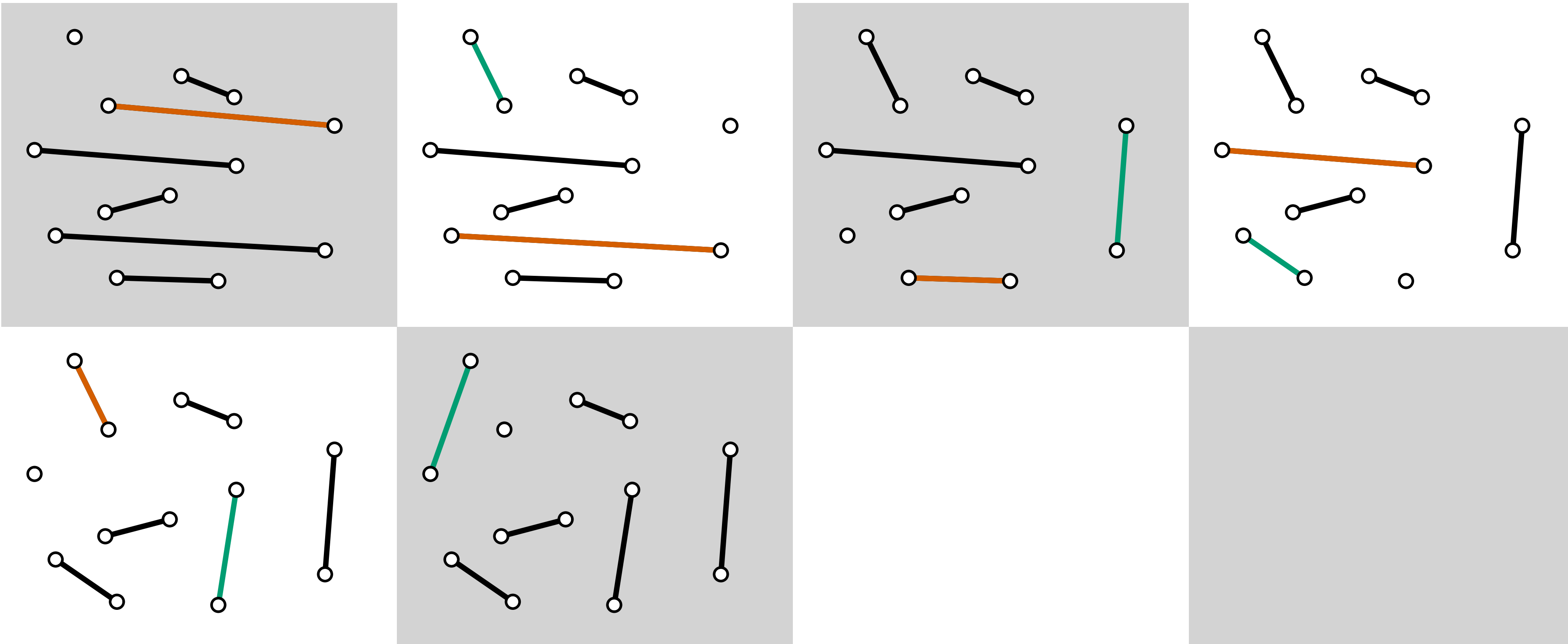
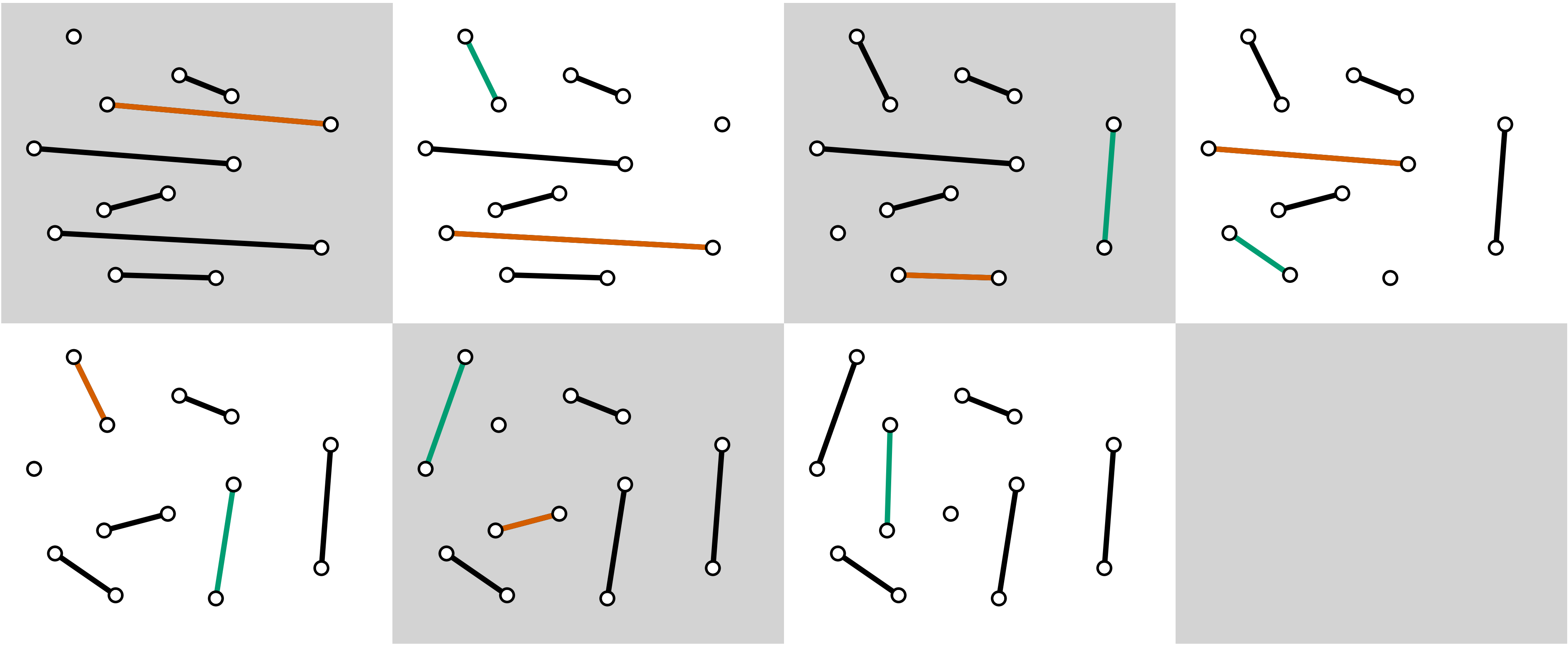


Crossing-free Almost Perfect Matchings

Almost perfect matching: $n = 2m + 1$ points, m edges



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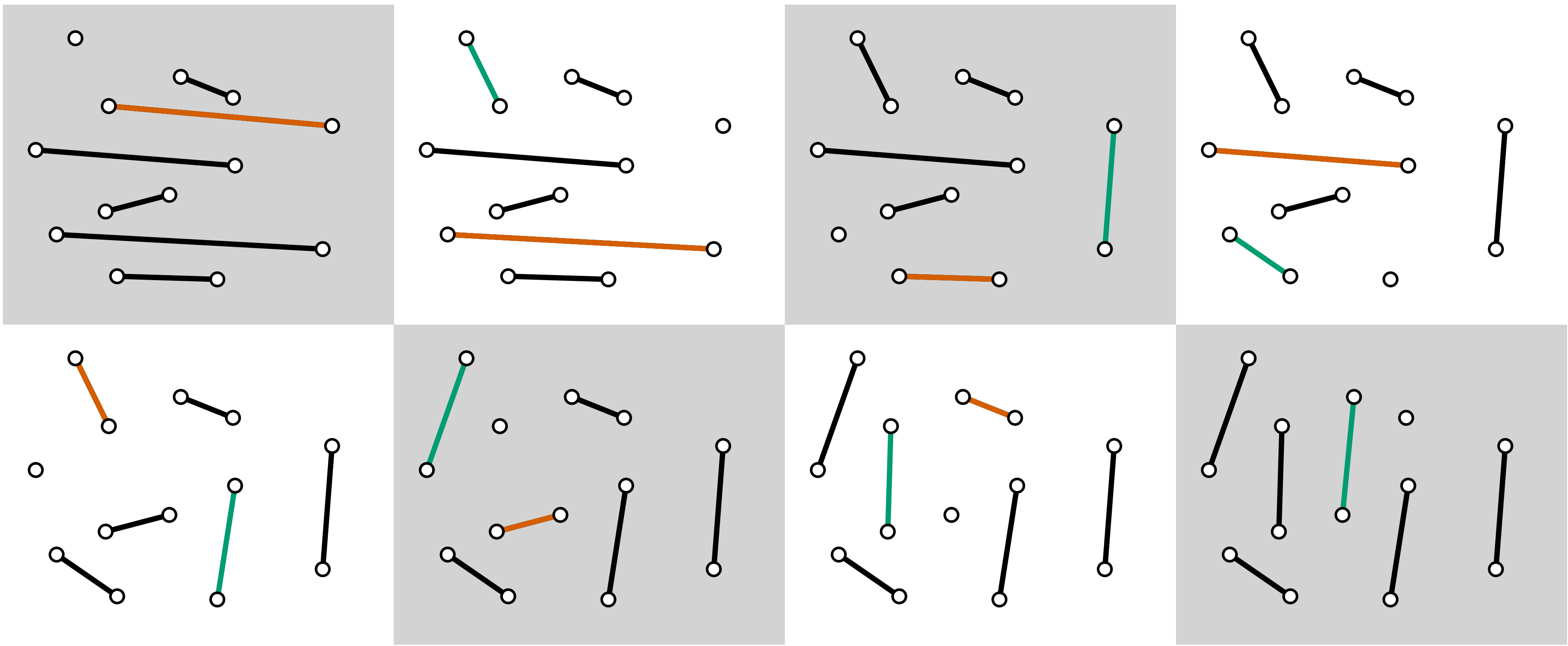

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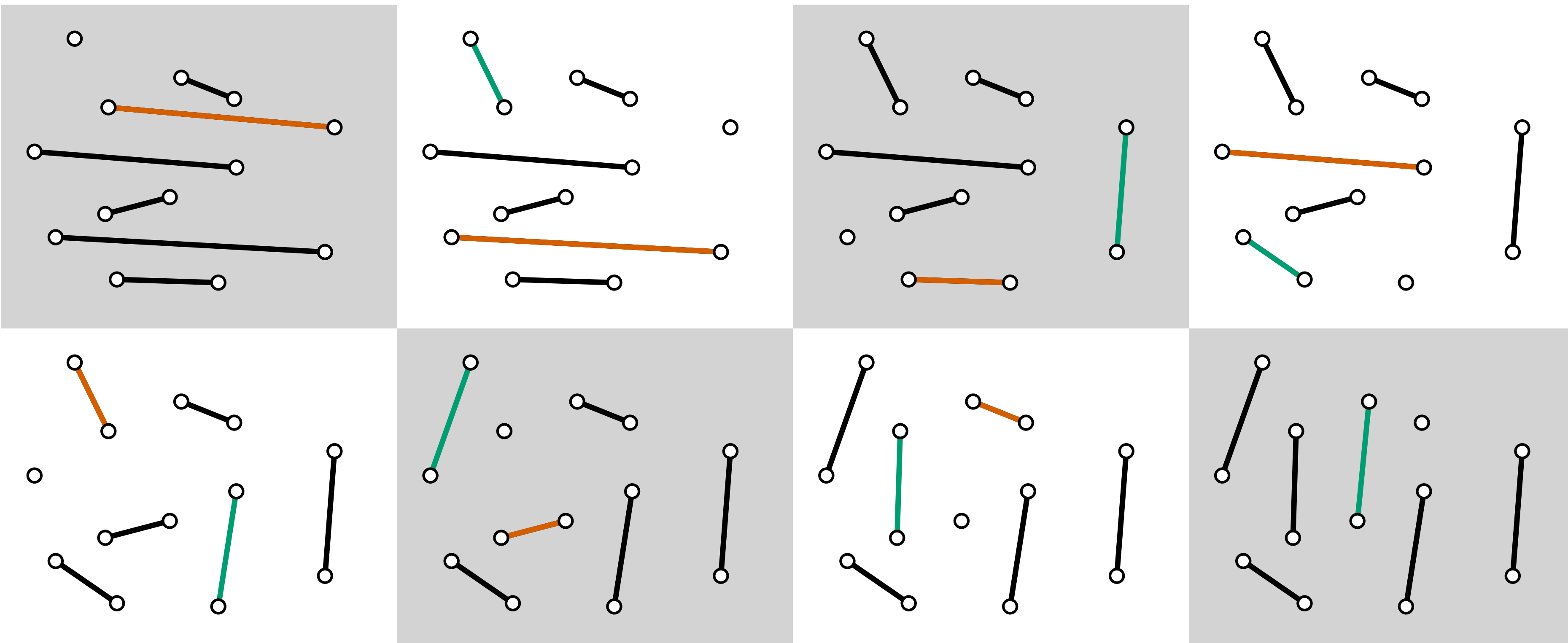

The flip graph of almost perfect matchings is connected with diameter $O(n^2)$ [A., Brötzner, Perz, Schnider 2024]

10 Open Problems

Open Problem 1: What is the complexity of the flip distance of two triangulations of a set of points in convex position (the rotation distance between two binary trees)?
Open Problem 2: Which flip types have the Happy Edge Property for crossing-free spanning trees?
Open Problem 3: What is the complexity of computing the flip distance for crossing-free spanning trees for general point sets / point sets in convex position?
Open Problem 4: What is the tight bound for the diameter of the flip graph for crossing-free spanning trees for general point sets / point sets in convex position?
Open Problem 5: Is the flip graph for crossing-free spanning paths connected? For which flip-types?
Open Problem 6: In case of connectedness (convex sets etc), how fast can the shortest flip-sequence for crossing-free paths path be found?
Open Problem 7: Assume that starting and target paths are together crossing-free (aka compatible). Can they be embedded into a triangulation, so that flipping from one to the other can be done via paths of this triangulation?
Open Problem 8: Is the flip graph for crossing-free spanning paths connected via compatible paths? Two paths are compatible if their union is crossing-free.
Open Problem 9: Is the flip graph of crossing-free perfect matchings connected by 2-edge flips?
Open Problem 10: Is the flip graph of crossing-free perfect matchings connected by flips of crossing-free $\leq 2k$-cycles with $k = o(n)$?

Thank you for your attention!

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Merci beaucoup

Děkuji!

Grazie

धन्यवाद

Tak skal du ha'!

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Muchas gracias por su atención

Danke

Hvala lepa

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Dank U wel

どうも　ありがとう

Köszönöm

Obrigado