Expanding Fraïssé classes into Ramsey classes

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July 2012

Ramsey property

Definition

A class $\mathcal{K}$ of finite (first order) structures has the Ramsey property (= is Ramsey) when for any:

- $X \in \mathcal{K}$ *(small structure, to be colored),*
- $Y \in \mathcal{K}$ *(medium structure, to be reconstituted),*
- $k \in \mathbb{N}$ *(number of colors),*

there exists $Z \in \mathcal{K}$ (very large structure) such that:

$$Z \longrightarrow (Y)^X_k .$$

i.e. whenever copies of X in Z are colored with k colors,
there is $\tilde{Y} \cong Y$ where all copies of X have same color.

Examples and non examples of Ramsey classes

The following are Ramsey classes:

- Finite sets (Ramsey, 30).
- Finite Boolean algebras (Graham-Rothschild, 71).
- Finite vector spaces (Graham-Leeb-Rothschild, 72).

The following are NOT Ramsey classes:

- Finite graphs, finite relational structures in a fixed countable language.
- Finite K_n-free graphs.
- Finite posets.
- Finite equivalence relations.

...BUT...

Non-examples of Ramsey classes

...They can be expanded into Ramsey classes:

- Finite graphs, finite relational structures in a fixed countable language: Add arbitrary linear orderings (Nešetřil-Rödl, 77; Abramson-Harrington, 78).
- Finite K_n-free graphs: Arbitrary linear orderings (Nešetřil-Rödl, 83).
- Finite posets: Linear extensions (Nešetřil-Rödl, 84).
- Finite equivalence relations: Convex linear orderings (Rado, 54).

Those results do have a substantial combinatorial content. In some sense, those classes are "close" to be Ramsey.

Question

Can we formalize this notion of being "close to be Ramsey" more precisely?

G-flows

Definition

Let G be a Hausdorff topological group.
A G-flow is a continuous action of G on a compact Hausdorff space X.
Notation: $G \curvearrowright X$.
$G \curvearrowright X$ is minimal when every $x \in X$ has dense orbit in X:

$$\forall x \in X \quad \overline{G \cdot x} = X$$

$G \curvearrowright X$ is universal when:

$$\forall G \curvearrowright Y \text{ minimal}, \quad \exists \pi : X \longrightarrow Y \text{ continuous, onto, and so that}$$
$$\forall g \in G \quad \forall x \in X \quad \pi(g \cdot x) = g \cdot \pi(x).$$

"Every minimal G-flow is a continuous image of $G \curvearrowright X$."

Universal minimal flow

Theorem (Folklore)

Let G be a Hausdorff topological group.
Then there is a unique G-flow that is both minimal and universal.
Notation: $G \curvearrowright M(G)$.

Remark

- *When G is compact, $M(G) = G$ with action on itself by left translation.*
- *When G is not compact:*
  - *$M(G)$ may be not metrizable (E.g. G locally compact)*
  - *$M(G)$ may be a singleton, G is then called extremely amenable (eg: $\mathrm{Aut}(\mathbb{Q}, <)$, Pestov, 98).*
  - *$M(G)$ may be metrizable (eg: $M(S_\infty) = S_\infty \curvearrowright LO(\mathbb{N})$, Glasner-Weiss, 02)*

Kechris-Pestov-Todorcevic theorem

Theorem (Kechris-Pestov-Todorcevic, 05)

Let $\mathcal{K}$ be a Fraïssé class whose elements are rigid (have no non-trivial automorphisms). Let $\mathbb{F}$ be its Fraïssé limit. TFAE:

i) $\mathrm{Aut}(\mathbb{F})$ *is extremely amenable.*

ii) $\mathcal{K}$ *has the Ramsey property.*

Question

Is there a similar theorem for those Fraïssé classes that admit a Ramsey expansion?

A trivial answer

Proposition

Every Fraïssé class $\mathcal{K}$ admits a Ramsey expansion.

Proof.

Consider $\mathbb{F} = \{x_n : n \in \mathbb{N}\}$, the Fraïssé limit of $\mathcal{K}$. Expand it with countably many unary relations $A_n^*, n \in \mathbb{N}$:

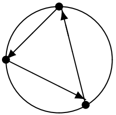
$$A_n^*(x) \Leftrightarrow x = x_n.$$

Then $\mathbb{F}^* := (\mathbb{F}, (A_n^*)_{n \in \mathbb{N}})$ is rigid, and the class of its finite substructures is a Ramsey expansion of $\mathcal{K}$. □

Of course, the above result has empty combinatorial content. We must rephrase the question and ask which classes admit "non-trivial" expansions.

Only linear orderings?

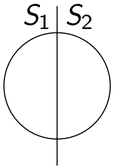
- In view of the aforementioned classical results, expansions by linear orderings should definitely by considered as "non-trivial".
- But we should allow more: Recall that the dense local order $S(2)$ is the tournament defined by:
  Vertices: Rational points of $\mathbb{S}^1$ (no antipodal pair).
  Arcs: $x \to y$ iff (counterclockwise angle from x to y) $< \pi$.



- For a linear ordering $<$ on $S(2)$, the class of finite substructures of $(S(2), <)$ is never Ramsey: there is 2-coloring of the vertices with no monochromatic 3-cycle, namely, left and right part.

The case of $S(2)$

- Ramsey property holds if $S(2)$ is enriched differently:



- Key fact: $(S(2), S_1, S_2) \cong (\mathbb{Q}, Q_1, Q_2, <)$, Q_1, Q_2 dense subsets of $\mathbb{Q}$ (Reversing the arcs between points in different parts).
- The corresponding class of finite substructures is Ramsey, and not for trivial reasons.

Precompact expansions

Definition

Let $\mathcal{K}$ be a class of finite structures in some some language L, $\mathcal{K}^*$ an expansion of $\mathcal{K}$ in a language $L^* \supset L$. Then $\mathcal{K}^*$ is a precompact expansion of $\mathcal{K}$ when every element of $\mathcal{K}$ only has finitely many expansions in $\mathcal{K}^*$.

Theorem

Let $\mathcal{K}$ be a Fraïssé class. Call $\mathbb{F}$ the corresponding Fraïssé limit and set $G = \mathrm{Aut}(\mathbb{F})$. TFAE:

1. *$\mathcal{K}$ admits a Fraïssé, precompact expansion $\mathcal{K}^*$ that is Ramsey and has rigid elements.*
2. *$M(G)$ is metrizable and has a generic orbit.*
3. *G admits a closed, extremely amenable subgroup G^* such that G/G^* is precompact.*

What the theorem says

- Admitting a precompact Ramsey expansion seems to be a reasonable notion for "being close to Ramsey", and suggests that many other non trivial Ramsey theorems could be found: start from your favorite Fraïssé class, and try to expand it in a precompact way to make it Ramsey!
- Item 3 indicates that looking for a large extremely amenable subgroup is the right thing to do in order to prove that a universal minimal flow is metrizable (this method is due to Pestov, and is so far the most powerful one to compute universal minimal flows in concrete cases).

A few words on the proof

- 1⇒2 and 3⇒1 are essentially due to KPT. 2⇒3 uses other facts.
- 1⇒2: Given $\mathcal{K}^*$, refine it into a precompact Ramsey $\mathcal{K}^{**}$ with the so-called the Expansion Property. Ramsey ensures that the flow $\widehat{G/G^{**}}$ is precompact, Expansion property ensures that it is minimal.
- 2⇒3: Let H be the stabilizer of some point in the generic orbit of $M(G)$.
  i) G/H is precompact. Proved by showing that the Samuel compactification of G/H is a continuous image of $M(G)$, hence metrizable.
  ii) The pair (G, H) is relatively extremely amenable (every continuous G-action on a compact space has an H-fixed point). Due to the fact that H is contained in a stabilizer of a point of $M(G)$.
  iii) There is a closed extremely amenable sugbroup G^* of G containing H.
- 3⇒1: Take $\mathcal{K}^*$ corresponding to G^*.

Which Fraïssé classes have Fraïssé precompact Ramsey expansions?

The following admit Fraïssé precompact Ramsey expansions:

- All Fraïssé classes of finite graphs (based on known results).
- All Fraïssé classes of finite tournaments (idem+Laflamme-NVT-Sauer).
- All Fraïssé classes of finite posets (based on work of Sokić).
- In fact, apparently, all Fraïssé classes of finite directed graphs! (Jasiński-Laflamme-NVT).

Conjecture

Every Fraïssé class with finitely many isomorphism types in each cardinality have a Fraïssé precompact Ramsey expansion. Equivalently, every oligomorphic closed subgroup of S_∞ has a metrizable universal minimal flow with a generic orbit.

About the conjecture

My view on the conjecture:

- Test it on any specific case.
- Test it on any class of structures where a classification result is known (e.g. Fraïssé classes of *n*-tournaments).
- There are known counterexamples when G is not oligomorphic (e.g $\mathrm{Aut}(\mathbb{Z}, <^{\mathbb{Z}}, d^{\mathbb{Z}}) = \mathbb{Z}$)
- Would say that Ramsey classes are not so rare after all, and that there are plenty of interesting combinatorial cases to be discovered.
- Will not say anything about how to expand Fraïssé class into Ramsey classes in practice (so no risk of losing your job if you are working in structural Ramsey theory).
- So far, the most reasonable attempt of proof is from topological dynamics, as the combinatorics still exhibits a variety of seemingly different situations.