Actions of countable groups on homogeneous structures

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I. Background

Polish spaces and groups

Definition

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Example

The permutation group of the integers, denoted by $\mathcal{S}_\infty$.
If $\sigma, \tau \in \mathcal{S}_\infty$, let $d(\sigma,\tau) = \inf\{2^{-n} \colon \sigma_{|n} = \tau_{|n}\}$.
This is a left-invariant separable (ultra)metric.
It is not complete; however the following metric is:

$$d'(\sigma,\tau) = d(\sigma,\tau) + d(\sigma^{-1},\tau^{-1}).$$

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- Or particular interest to us: automorphism groups of Fraïssé limits, notably Urysohn spaces whose distance takes values in $\{0, \ldots, n\}$ (denoted $\mathbb{U}_n$), $\mathbb{N}$ ($\mathbb{U}_\mathbb{N}$) or $\mathbb{Q}$ ($\mathbb{U}_\mathbb{Q}$).

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- The unitary group $\mathcal{U}(\ell_2)$.
- The group $\mathrm{Aut}(\mu)$ of measure-preserving bijections of a standard atomless probability space (X, μ).
- The isometry group $\mathrm{Iso}(\mathbb{U})$ of the Urysohn space.

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Question

What does a typical element of $\mathrm{Hom}(\Gamma, G)$ look like? Which properties are *generic* in $\mathrm{Hom}(\Gamma, G)$?

II. Conjugacy classes

Definition

G naturally acts on $\mathrm{Hom}(\Gamma, G)$ by conjugacy:

$$g \cdot \pi(\gamma) = g\pi(\gamma)g^{-1} \ .$$

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- This implies the analoguous result for $\mathrm{Iso}(\mathbb{U})$.

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Definition

G is said to have *ample generics* if there exist comeager conjugacy classes in $\mathrm{Hom}(\mathbb{F}_n, G)$ for all n. This property has very strong consequences on the structure of G.

Ample generics and the extension property.

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Definition

Let $\mathcal{K}$ be a Fraïssé class. Say that $\mathcal{K}$ has the *extension property* if for any $A \in \mathcal{K}$ there exists $B \in \mathcal{K}$ in which A embeds in such a way that all *partial* automorphisms of A extend to *global* automorphisms of B.

Theorem (Herwig–Lascar 2000)

Let $\mathcal{L}$ be a finite relational language, $\mathcal{T}$ be a finite family of $\mathcal{L}$-structures, A be a finite $\mathcal{T}$-free structure and P a set of partial automorphisms of A. Assume that there exists a $\mathcal{T}$-free structure M in which A embeds in such a way that all elements of P extend to global automorphisms of M. Then there exists a *finite* $\mathcal{T}$-free structure B in which A embeds in such a way that all elements of P extend to global automorphisms of B.

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The algebraic heart of the proof is a theorem of Ribbes and Zaleskiĭ about free groups. This result was used by Solecki to show that many "natural" Fraïssé classes of metric spaces have the extension property.

Property (RZ)

Definition

Γ has the *Ribbes–Zaleskiĭ property* if, whenever $\Gamma_1, \ldots, \Gamma_n$ are finitely generated subgroups of Γ, their product $\Gamma_1 \cdots \Gamma_n$ is closed in the profinite topology.

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Theorem (Rosendal 2011)

Let Γ be a finitely generated group with property (RZ). Then there is a generic element in $\mathrm{Hom}(\Gamma, \mathrm{Iso}(\mathbb{U}_{\mathbb{Q}}))$.

Contrasting situations in the discrete and continuous settings

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- Conjugacy classes in $\mathrm{Hom}(\Gamma, \mathrm{Aut}(\mu))$ are meager for any infinite amenable Γ (Glasner–Weiss 2005); open in general.

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- Conjugacy classes in $\mathrm{Hom}(\Gamma, \mathrm{Iso}(\mathbb{U}))$ are meager for any abelian Γ containing an infinite cyclic subgroup (M.–Tsankov 2011); open in general.

III. The group generated by an action.

Topological properties

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Observation

Whenever $G \leq \mathcal{S}_\infty$ and Γ is finitely generated, the set of $\pi \in \mathrm{Hom}(\Gamma, G)$ such that $\overline{\pi(\Gamma)}$ is compact is G_δ. This is no longer true in the continuous setting (even for $\Gamma = \mathbb{Z}$).

Generating compact subgroups in automorphism groups of discrete structures

Observation (Herwig)

If $G \leq S_\infty$ is the automorphism group of the Fraïssé limit of some class $\mathcal{K}$, the fact that a generic element in G^n generates a relatively compact subgroup for all n is equivalent to the extension property for $\mathcal{K}$.

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Theorem (Rosendal 2011)

Fix $n \in \mathbb{N}$. Then

$$\{\pi \in \mathrm{Hom}(\Gamma, \mathrm{Iso}(\mathbb{U}_n)) \colon \overline{\pi(\Gamma)} \text{ is compact}\}$$

is dense in $\mathrm{Hom}(\Gamma, \mathrm{Iso}(\mathbb{U}_n))$ if, and only if, any product of n finitely generated subgroups of Γ is closed in the profinite topology.

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Theorem (M.–Tsankov)

Let Γ be a countable group, and G be a Polish group. Then

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Assume that Γ is a countable unbounded abelian group.

- A generic element of $\mathrm{Hom}(\Gamma, \mathrm{Aut}(\mu))$ or $\mathrm{Hom}(\Gamma, \mathrm{Iso}(\mathbb{U}))$ generates an extremely amenable subgroup.

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- A generic element of $\mathrm{Hom}(\Gamma, \mathcal{U}(\ell_2))$ generates a closed subgroup isomorphic to $L^0(\mathbb{T})$.

Definition

Let $\Gamma = \mathbb{F}_\infty \times \mathbb{F}_\infty$; G has the *Kirchberg property* if

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Theorem (Kirchberg 1993)

Connes' embedding conjecture is true iff $\mathcal{U}(\ell_2)$ has the Kirchberg property.

IV. Coherence properties

Does the restriction map preserve category?

Question

Assume that $\Delta \leq \Gamma$ are countable groups, and G is a Polish group. How do the generic properties in $\mathrm{Hom}(\Delta, G)$ relate to the generic properties in $\mathrm{Hom}(\Gamma, G)$?

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Let $f: X \to Y$ be a continuous map. Say that f is *category-preserving* if $f^{-1}(O)$ is comeager in X whenever O is comeager in Y.

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Question (revisited)

Assume that $\Delta \leq \Gamma$ are countable groups, and G is a Polish group. When is the restriction map $\mathrm{Res} \colon \mathrm{Hom}(\Gamma, G) \to \mathrm{Hom}(\Delta, G)$ category-preserving?

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Note that the Kuratowski–Ulam theorem implies that the restriction map is always category-preserving when $\Delta = \mathbb{F}_n \leq \mathbb{F}_m = \Gamma$.

Restrictions of measure-preserving actions I

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Theorem (M.)

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Restrictions of measure-preserving actions II

Observation (M.)

There exist a polycyclic group Γ and an infinite cyclic subgroup $\Delta \leq \Gamma$ such that a generic measure-preserving Δ action does not extend to a measure-preserving Γ-action.

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Previous examples of this phenomenon (where Γ was more complicated) were already known. The proof of the above observation depends on a result of King stating that the generic $\overline{\pi(\mathbb{Z})}$ is maximal abelian in $\mathrm{Aut}(\mu)$.

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O. Ageev has recently announced a negative answer; I do not know his proof.

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A generic element of $\mathrm{Aut}(\mu)$ admits roots of all orders.

Extensions of isometric actions

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Does a generic element of $\mathrm{Iso}(\mathbb{U})$ admit roots of all orders?