Classifying homomorphism-homogeneous structures

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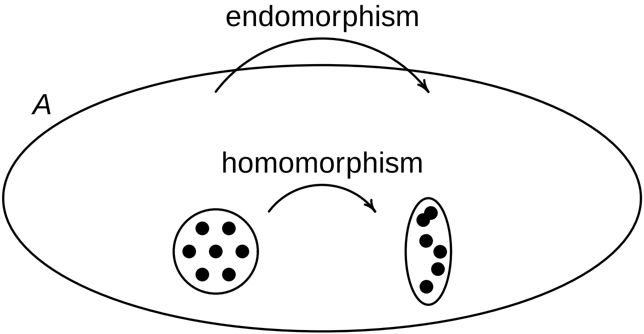
Overview

Next . . .

Homomorphism-homogeneity



Cameron, P. J., Nešetřil, J., *Homomorphism-homogeneous relational structures*, Combinatorics, Probability and Computing 15, 91–103 (2006)

Homomorphism-homogeneity

The General Classification Problem.
Classify homomorphism-homogeneous structures.

Next . . .

Graphs

Objects: finite graphs $(X, \sim)$, no loops
Subobjects: iduced subgraphs
Morphisms: $x \sim y \Rightarrow f(x) \sim f(y)$

Theorem. *[Cameron, Nešetřil 2006]*
A finite graph G is homomorphism-homogeneous if and only if $G \cong k \cdot K_m$ for some positive integers k and m.

Irreflexive structures

Objects: finite structures (X, ρ) where ρ is binary, irreflexive
Subobjects: iduced substructures
Morphisms: $x \; \rho \; y \Rightarrow f(x) \; \rho \; f(y)$

Theorem. *[DM, Nenadov, Škorić 2010]*
A finite irreflexive binary relational structure (X, ρ) is homomorphism-homogeneous if and only if it is one of the following:

1. $k \cdot K_m$ for some positive integers k and m,
2. $k \cdot C_3$ for some positive integer k, where C_3 denotes the oriented 3-cycle.

Posets

Objects: posets
Subobjects: subposets
Morphisms: $x \leqslant y \Rightarrow f(x) \leqslant f(y)$

Theorem. *[DM 2007]*
A poset $(X, \leqslant)$ is homomorphism-homogeneous if and only if it is one of the following:

1. every connected component of X is a chain,
2. X is a tree or a dual tree,
3. X splits into a tree and a dual tree,
4. X is locally bounded and dense in the following sense: whenever $a, b, c, d \in X$ satisfy $\{a, b\} \leqslant \{c, d\}$, there exists an $m \in X$ such that $\{a, b\} \leqslant m \leqslant \{c, d\}$
(the Riesz Interpolation Property)

Posets

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Theorem. *[DM 2007]*
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1. every connected component of X is a chain,
2. X is a tree or a dual tree,
3. X splits into a tree and a dual tree,
4. X is a lattice.

Tournaments with loops

Objects: finite tournaments, vertices may have loops
Subobjects: induced subtournaments
Morphisms: $x \to y \Rightarrow f(x) \to f(y)$

Theorem. *[Ilić, DM, Rajković 2008]*
A finite tournament with loops is homomorphism-homogeneous if and only if it is one of the following:

1. C_3 or C_3°, where C_3° denotes C_3 with all loops,
2. acyclic tournaments with precisely one loopless vertex,
3. acyclic tournaments with two consecutive loopless vertices where both the initial and the final vertex have a loop,
4. acyclic tournaments dense in the following sense:
   - there exist $0, 1 \in V(T)$ such that $0 \rightrightarrows x \rightrightarrows 1$ for all $x \in V(T)$, and
   - for all $x, y \in V(T)$ such that $x \to y$ there is a $z \in V(T)$ such that $z \to z$ and $x \to z \to y$.

Digraphs with loops

Objects: finite digraphs, vertices may have loops
Subobjects: induced subdigraphs
Morphisms: $x \to y \Rightarrow f(x) \to f(y)$

Theorem. *[DM (submitted)]*
Let D be a finite digraph with loops which is disconnected or uniform ($=$ all loops, or no loops). Then D is homomorphism-homogeneous if and only if it is one of the following:

1. $L + k \cdot \mathbf{1}$ for some integer $k \geqslant 0$ and some finite homomorphism-homogeneous partially ordered set L;
2. $n \cdot C_3 + m \cdot C_3^\circ + k \cdot \mathbf{1}^\circ$ for some $n, m, k \geqslant 0$;
3. $n \cdot C_3^\circ + m \cdot \mathbf{1}^\circ + k \cdot \mathbf{1}$ for some $n, m, k \geqslant 0$;
4. $n \cdot C_3^\circ + m \cdot \mathbf{1}^\circ + k \cdot A_2^\circ(1)$ for some $n, m, k \geqslant 0$;
5. $n \cdot C_3^\circ + m \cdot \mathbf{1}^\circ + k \cdot A_2^\circ(2)$ for some $n, m, k \geqslant 0$;
6. every connected component of D is a dense tournament;

Digraphs with loops

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Theorem. *[DM (submitted)]*
Let D be a finite digraph with loops which is disconnected or uniform ($=$ all loops, or no loops). Then D is homomorphism-homogeneous if and only if it is one of the following:

7 for every connected component S of D there is a $k \geqslant 1$ such that $D[S] \cong A_k^\circ$ or $D[S] \cong A_k^\circ(1)$;

8 for every connected component S of D there is a $k \geqslant 1$ such that $D[S] \cong A_k^\circ$ or $D[S] \cong A_k^\circ(k)$;

9 for every connected component S of D there exist j and k such that $k \geqslant 1$ and $D[S] \cong A_k^\circ$, or $1 < j < k$ and $D[S] \cong A_k^\circ(j)$, or $1 < j < j+1 < k$ and $D[S] \cong A_k^\circ(j, j+1)$.

Graphs with loops

Adding loops makes the classification problem more interesting!

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Unfortunately, adding loops makes it too much fun . . .

Objects: finite graphs $(X, \sim)$, loops allowed
Subobjects: iduced subgraphs
Morphisms: $x \sim y \Rightarrow f(x) \sim f(y)$

Theorem. *[Rusinov, Schweitzer 2010]*
Deciding whether a finite graph with loops is homomorphism-homogeneous is coNP-complete.

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Another interpretation: there is no “reasonable” classification of finite homomorphism-homogeneous graphs with loops allowed.

Next . . .

1 Introduction

2 Relational structures

3 Algebras

4 Geometries and metric spaces

5 Various kinds of homogeneity

6 Classifiability v. nonclassifiability

Groups

Objects: finite groups
Subobjects: subgroups
Morphisms: homomorphisms of groups

Bertholf D., Walls D.: *Finite quasi-injective groups.*
Glasgow Math. J. 20(1979), 29–33

(NB: finite quasi-injective $=$ finite homomorphism-homog.)

Objects: lattices as algebras $(L, \wedge, \vee)$
Subobjects: sublattices as subalgebras
Morphisms: homomorphisms of lattices as algebras

Theorem. *[Dolinka, DM 2011]*

(a) A lattice L is homomorphism-homogeneous if and only if it is either a chain or every interval of L is a boolean lattice.

(b) A finite lattice L is homomorphism-homogeneous if and only if it is either a chain or a direct power of $0 < 1$.

Semilattices

Objects: semilattices as algebras $(S, \wedge)$
Subobjects: subsemilattices as subalgebras
Morphisms: homomorphisms of semilattices as algebras

Theorem. *[Dolinka, DM 2011]*

(a) A finite homomorphism-homogeneous semilattice is either a tree or the $\wedge$-semilattice reduct of a lattice.

(b) Every tree is a homomorphism-homogeneous semilattice.

(c) The $\wedge$-semilattice reduct of a distributive lattice is homomorphism-homogeneous.

(d) $(M_3, \wedge)$ and $(N_5, \wedge)$ are homomorphism-homogeneous.

Universal algebras

Objects: algebras $(A, \mathcal{F})$
Subobjects: subalgebras
Morphisms: homomorphisms

Theorem. *[Jungábel, DM (to appear)]*
A monounary algebra $\mathcal{A}$ is homomorphism-homogeneous if and only if $\mathcal{A}$ belongs to one of the following classes:

1. every branch in $\mathcal{A}$ is infinite;
2. every connected component in $\mathcal{A}$ is regular, and for any two connected components $S_1, S_2 \subseteq A$, if $\mathrm{cn}(S_1)|\mathrm{cn}(S_2)$ then $\mathrm{ht}(S_1) \geqslant \mathrm{ht}(S_2)$ or $\mathrm{ht}(S_1) = 0$.

Universal algebras

Objects: algebras $(A, \mathcal{F})$
Subobjects: subalgebras
Morphisms: homomorphisms

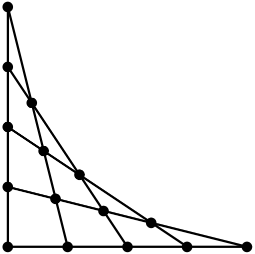
Theorem. *[DM (submitted)]*
Let $\mathcal{K}$ be the class of all finite algebras whose signature contains at least one at least binary operation. Deciding whether an algebra from $\mathcal{K}$ is homomorphism-homogeneous is a coNP-complete problem.

Next . . .

Point-line geometries

Definition. A *point-line geometry* is an ordered pair $(X, \mathcal{L})$ where X is a set of *points*, $\mathcal{L} \subseteq \mathcal{P}(X)$ is a set of *lines* and the following is satisfied:

- every line contains at least two points, and
- every pair of points is contained in at most one line.



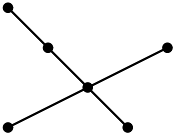
Objects: point-line geometries

Subobjects: induced subgeometries $(Y, \mathcal{L}|_Y)$ where
$\mathcal{L}|_Y = \{\ell \cap Y : \ell \in \mathcal{L}, |\ell \cap Y| \geqslant 2\}$

Morphisms: functions that map collin. points to collin. points
$\forall \ell \in \mathcal{L}\ \exists m \in \mathcal{L}\ (f(\ell) \subseteq m)$

Point-line geometries

A point-line geometry is *proper* if it contains a pair of intersecting *proper* lines (proper line $=$ line with at least 3 points).



Theorem. *[Jungábel, DM (in preparation)]*
Deciding whether a finite connected improper point-line geometry which is not a graph is homomorphism-homogeneous is a coNP-complete problem.

Point-line geometries

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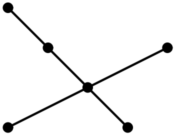


Theorem. *[DM (to appear)]*
A finite connected proper point-line geometry is homomorphism-homogeneous if and only if it is one of the following:

1. a pencil of lines,
2. the Fano plane,
3. a subdivision of the triangular space $T(n)$, $n \geqslant 1$,
4. a particular trivial projective point-line geometry with only two proper lines.

Metric spaces

Objects: metric spaces with rational distances
Subobjects: subspaces
Morphisms: nonexpansive maps
$d(f(x), f(y)) \leqslant d(x, y)$

Fact. Deciding whether a finite metric space with rational distances is homomorphism-homogeneous is a coNP-complete problem.

Theorem. *[Dolinka 2012]*
The rational Urysohn space (the Fraïssé limit of the class of all finite metric spaces with rational distances) is homomorphism-homogeneous.

Traditional metric spaces

Fix a “traditional” normed space $(\mathbb{R}^n, \|\cdot\|_p)$, $n \geqslant 1$, $p \in [1, \infty]$

Morphisms: nonexpansive (1-Lipschitz) maps

$$\|f(x) - f(y)\|_p \leqslant \|x - y\|_p$$

Theorem.
$(\mathbb{R}^n, \|\cdot\|_p)$ is homomorphism-homogeneous if and only if $(\mathbb{R}^n, \|\cdot\|_p)$ has the $(n+1)$-*Kirszbraun Intersection Property*.

Proof. Transfinite induction + Helly’s theorem + Closed balls in $(\mathbb{R}^n, \|\cdot\|_p)$ are convex and compact. $\square$

Traditional metric spaces

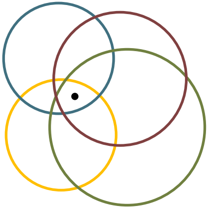
The *m*-Kirszbraun Intersection Property (*m*-KIP).
Let $\overline{B}(x_i, r_i)$, $i \in \{1, \ldots, m\}$, be a collection of m closed balls in a Banach space $(X, \|\cdot\|)$ such that:

$$\bigcap_{i=1}^{m} \overline{B}(x_i, r_i) \neq \varnothing,$$

and let $y_1, \ldots, y_m \in X$ be such that, for all i and j:

$$\|y_i - y_j\| \leqslant \|x_i - x_j\|.$$

Then we also have:



$$\bigcap_{i=1}^{m} \overline{B}(y_i, r_i) \neq \varnothing.$$

Traditional metric spaces

Fact. $(\mathbb{R}, \|\cdot\|_p)$ is homomorphism-homogeneous for all p.

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Theorem. *[Kirszbraun 1934]*
$(\mathbb{R}^n, \|\cdot\|_2)$ has the m-KIP for all $m \geqslant 1$.

Traditional metric spaces

Fact. $(\mathbb{R}, \|\cdot\|_p)$ is homomorphism-homogeneous for all p.

Theorem. *[Kirszbraun 1934]*
$(\mathbb{R}^n, \|\cdot\|_2)$ has the m-KIP for all $m \geqslant 1$.

Theorem.
$(\mathbb{R}^n, \|\cdot\|_\infty)$ has the m-KIP for all $m \geqslant 1$.

Proof. Helly's theorem $\square$

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Theorem.
$(\mathbb{R}^2, \|\cdot\|_1)$ has the m-KIP for all $m \geqslant 1$.

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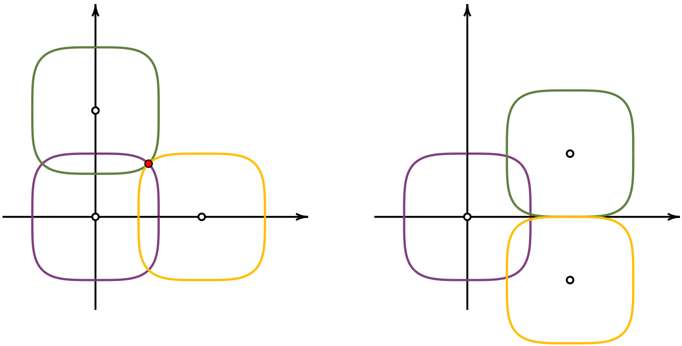
These are the only homomorphism-homogeneous "traditional" metric spaces!

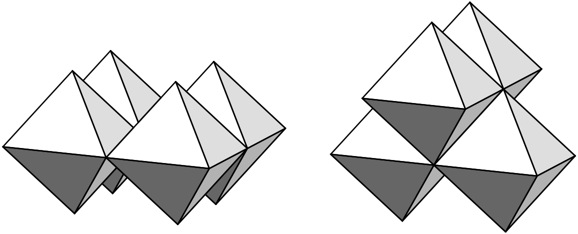
Traditional metric spaces

Example. *[J. T. Schwartz 1969]*
$(\mathbb{R}^n, \|\cdot\|_p)$ *doesn't have 3-KIP for* $p \in (1,2) \cup (2,\infty)$, $n \geqslant 2$*:*
there exist $x_1, x_2, x_3, y_1, y_2, y_3 \in \mathbb{R}^n$ and an $r > 0$ such that
$\|y_i - y_i\|_p \leqslant \|x_i - x_j\|_p$ for all i and j, and $\bigcap_{i=1}^{3} \overline{B}(x_i, r) \neq \varnothing$, but $\bigcap_{i=1}^{3} \overline{B}(y_i, r) = \varnothing$.



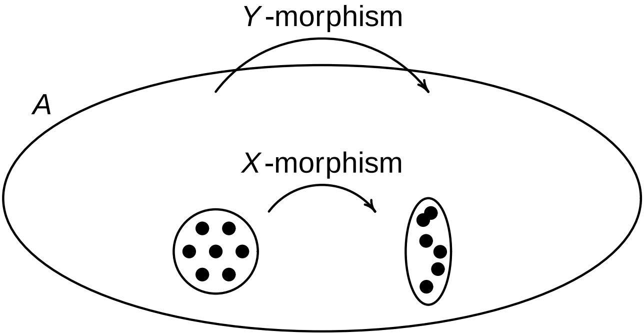
Traditional metric spaces

Example.
$(\mathbb{R}^n, \|\cdot\|_1)$ *doesn't have 4-KIP for $n \geqslant 3$:*
there exist $x_1, \ldots, x_4, y_1, \ldots, y_4 \in \mathbb{R}^n$ and an $r > 0$ such that
$\|y_i - y_i\|_1 \leqslant \|x_i - x_j\|_1$ for all i and j, and $\bigcap_{i=1}^{4} \overline{B}(x_i, r) \neq \varnothing$, but
$\bigcap_{i=1}^{4} \overline{B}(y_i, r) = \varnothing$.



Next . . .

Various kinds of homogeneity



Cameron, P. J., Nešetřil, J., *Homomorphism-homogeneous relational structures*, Combinatorics, Probability and Computing 15, 91–103 (2006)

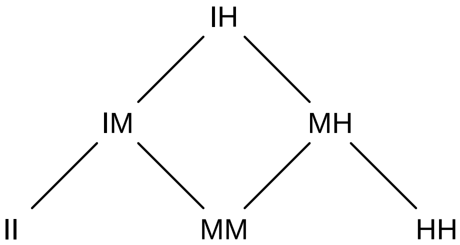
Various kinds of homogeneity

$\downarrow$ *homomorphism-homogeneity*

HH-homogeneity:	homomorphism	$\rightsquigarrow$	homomorphism
MH-homogeneity:	monomorphism	$\rightsquigarrow$	homomorphism
IH-homogeneity:	isomorphism	$\rightsquigarrow$	homomorphism
MM-homogeneity:	monomorphism	$\rightsquigarrow$	monomorphism
IM-homogeneity:	isomorphism	$\rightsquigarrow$	monomorphism
II-homogeneity:	isomorphism	$\rightsquigarrow$	isomorphism

$\uparrow$ *(ultra)homogeneity*

Various kinds of homogeneity



Various kinds of homogeneity

Theorem. *[Cameron, Lockett 2010]*

$$\begin{tikzcd}
\mathrm{IH} = \mathrm{MH} = \mathrm{HH} \arrow[d, dash] \\
\mathrm{IM} = \mathrm{MM} \arrow[d, dash] \\
\mathrm{II}
\end{tikzcd}$$

Countably infinite posets

$$\begin{tikzcd}
\mathrm{IH} = \mathrm{MH} = \mathrm{HH} \arrow[d, dash] \\
\mathrm{IM} = \mathrm{MM} = \mathrm{II}
\end{tikzcd}$$

Finite posets

Various kinds of homogeneity

Theorem. *[Rusinov, Schweitzer 2010]*

$$\begin{tikzcd}
 & \mathrm{IH} \arrow[d, dash] & \\
 & \mathrm{IM} \arrow[dl, dash] \arrow[dr, dash] & \\
\mathrm{II} & & \mathrm{HH} = \mathrm{MH} \arrow[d, dash] \\
 & & \mathrm{MM}
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
\mathrm{IH} \arrow[d, dash] \\
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\mathrm{HH} = \mathrm{MH} \arrow[d, dash] \\
\mathrm{MM}
\end{tikzcd}$$

Countably infinite graphs — Finite graphs

Various kinds of homogeneity

Question. Is MH always equal to HH?

Various kinds of homogeneity

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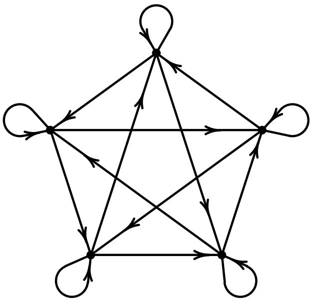
Answer. *[Hartman, Hubička, DM (submitted)]*
NO.

Example 1. Digraphs with loops.



Various kinds of homogeneity

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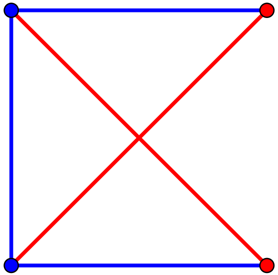
Answer. *[Hartman, Hubička, DM (submitted)]*
NO.

Example 2. Colored graphs.



Next . . .

Classifiability v. nonclassifiability

Where is the borderline between classifiability and nonclassifiability for finite structures?

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Theorem. *[DM, Nenadov, Škorić 2011; Ilić, DM, Rajković 2012]*
$\mathcal{B} =$ all finite structures (X, ρ) where $\rho \subseteq X^2$.
$X' = \{x \in X : x \ \rho \ x\}$, $\rho' = \rho|_{X'}$.

$\mathcal{C} =$ all $(X, \rho) \in \mathcal{B}$ such that (X', ρ') is $\leftrightarrows$-connected.

$\mathcal{D} =$ all $(X, \rho) \in \mathcal{B}$ such that (X', ρ') is $\leftrightarrows$-disconnected.

1. Deciding whether a structure from $\mathcal{D}$ is homomorphism-homogeneous is in P ($\Leftarrow$ we have explicit descriptions).
2. Deciding whether a structure from $\mathcal{C}$ is homomorphism-homogeneous is coNP-complete.

Classifiability v. nonclassifiability

Where is the borderline between classifiability and nonclassifiability for finite structures?

A feeling (Hypothesis?)

For the class of finite relational structures where vertices with “loops” form a “connected” substructure, deciding homomorphism-homogeneity is coNP-complete. Otherwise it is in P.