On cores of weakly oligomorphic relational structures

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Weakly oligomorphic structures

Definition

A relational structure $\mathbf{A}$ is called *weakly oligomorphic* if for every arity there are finitely many relations that can be defined by sets of positive existential formulæ.

Proposition

A countable structure $\mathbf{A}$ *is weakly oligomorphic if and only if* $\mathrm{End}(\mathbf{A})$ *is oligomorphic (i.e. there are just finitely many invariant relations of* $\mathrm{End}(\mathbf{A})$ *of any arity).*

A local homomorphism of a structure **A** is a homomorphism from a finite substructure of **A** to **A**.

Definition (Cameron, Nešetřil 2002)

A structure **A** is **homomorphism-homogeneous** if every local homomorphism of **A** extends to an endomorphism of **A**.

HAP

Homo-almagamation property of a class $\mathcal{C}$ of finite relational structures

If $\mathbf{A}, \mathbf{B_1}, \mathbf{B_2} \in \mathcal{C}$, $f_1 : \mathbf{A} \mapsto \mathbf{B_1}$ is a homomorphism, and $f_2 : \mathbf{A} \hookrightarrow \mathbf{B_2}$ is an embedding, then there are $\mathbf{C} \in \mathcal{C}$, an embedding $g_1 : \mathbf{B_1} \hookrightarrow \mathbf{C}$, and a homomorphism $g_2 : \mathbf{B_2} \mapsto \mathcal{C}$ such that the following diagram commutes:

$$\begin{tikzcd}
\mathbf{A} \arrow[r, "f_1"] \arrow[d, hook, "f_2"'] & \mathbf{B_1} \arrow[d, hook, "g_1"] \\
\mathbf{B_2} \arrow[r, "g_2"] & \mathbf{C}
\end{tikzcd}$$

i.e. $g_1 \circ f_1 = g_2 \circ f_2$.

Characterization of the ages of countable homomorphism-homogeneous relational structures

Theorem

(a) *The age of any homomorphism-homogeneous structure has property HAP.*

(b) *If a class $\mathcal{C}$ of finite relational structures*

1. *is closed under isomorphism,*
2. *has only countably many isomorphism types, and*
3. *has properties HP, JEP and HAP,*

then there is a countable homomorphism-homogeneous relational structure whose age is $\mathcal{C}$.

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then there is a countable homomorphism-homogeneous relational structure whose age is $\mathcal{C}$.

A class of finite relational structures over the same signature, that is closed under isomorphism and that has properties HP, JEP, and HAP is called **hom-almagamation class**.

Definition

A structure $\mathbf{A}$ is **weakly homomorphism-homogeneous** if whenever $\mathbf{B} < \mathbf{C}$ are finite substructures of $\mathbf{A}$, then every homomorphism $f : \mathbf{B} \to \mathbf{A}$ extends to $\mathbf{C}$.

Weakly homomorphism-homogeneous structures

Definition

A structure **A** is **weakly homomorphism-homogeneous** if whenever $\mathbf{B} < \mathbf{C}$ are finite substructures of **A**, then every homomorphism $f : \mathbf{B} \to \mathbf{A}$ extends to **C**.

Remark

A countable structure is weakly homomorphism-homogeneous iff it is homomorphism-homogeneous.

Quasi-order on structures

Definition

Let $\mathbf{H}$ and $\mathbf{H}'$ be two relational structures. We write $\mathbf{H} \preceq_h \mathbf{H}'$ if

- $\mathrm{Age}(\mathbf{H}) \supseteq \mathrm{Age}(\mathbf{H}')$, and
- for all finite $\mathbf{A} \leq \mathbf{B} \leq \mathbf{H}$ we have that every homomorphism from $\mathbf{A}$ to $\mathbf{H}'$ extends to a homomorphism from $\mathbf{B}$ to $\mathbf{H}'$.

Ages and weak homomorphism-homogeneity

Proposition

Let $\mathbf{H}$ *and* $\mathbf{H}'$ *be two relational structures.*

1. *If* $\mathbf{H} \preceq_h \mathbf{H}'$, *and* $\mathbf{H}$ *is weakly homomorphism-homogeneous, then* $\mathbf{H}'$ *is weakly homomorphism-homogeneous, too.*
2. *If* $\mathbf{H}'$ *is weakly homomorphism-homogeneous, and* $\mathrm{Age}(\mathbf{H}) = \mathrm{Age}(\mathbf{H}')$, *then* $\mathbf{H} \preceq_h \mathbf{H}'$.

Corollary

Let $\mathbf{A}$, *and* $\mathbf{B}$ *be two weakly homomorphism-homogeneous structures with the same age. Then* $\mathbf{A} \preceq_h \mathbf{B}$ *and* $\mathbf{B} \preceq_h \mathbf{A}$. *In particular, any two countable homomorphism-homogeneous relational structures with the same age are homomorphism-equivalent.*

Core

Definition

A relational structure is a *core*, if its all endomorphisms are embeddings.

Definition

Let $\mathcal{C}$ be a class of relational structures over the same signature and let $\mathbf{A} \in \mathcal{C}$. We say that $\mathbf{A}$ is *h-irreducible* if for every $\mathbf{B} \in \mathcal{C}$ and every homomorphism $f : \mathbf{A} \to \mathbf{B}$ holds that f is an embedding.

For a given relational structure $\mathbf{A}$, the class of all finite structures of the same type like $\mathbf{A}$ that are hom-irreducible in the age of $\mathbf{A}$ will be denoted by $\mathcal{C}_{\mathbf{A}}$.

Let $\mathcal{A}, \mathcal{B}$ be classes of rel. structures over a common signature.
$\mathcal{A}$ **projects** onto $\mathcal{B}$ ($\mathcal{A} \to \mathcal{B}$) if

$$(\forall \mathbf{A} \in \mathcal{A})(\exists \mathbf{B} \in \mathcal{B}) \quad \mathbf{A} \to \mathbf{B}$$

- Let $\mathcal{C}$ be a hom-amalgamation class, let $\mathcal{D}$ be the class of all structures from $\mathcal{C}$ that are hom-irreducible in $\mathcal{C}$.
  If $\mathcal{C} \to \mathcal{D}$, then $\mathcal{D}$ is a Fraïssé class.
- If $\mathbf{A}$ is a weakly oligomorphic and weakly homomorphism-homogeneous relational structure, then

$$\mathrm{Age}(\mathbf{A}) \to \mathcal{C}_{\mathbf{A}}$$

Cores and HH structures

Proposition

Let $\mathbf{A}$ *be a countable homomorphism-homogeneous relational structure, such that* $\mathrm{Age}(\mathbf{A}) \to \mathcal{C}_{\mathbf{A}}$*. Then* $\mathbf{A}$ *has a core* $\mathbf{C}$ *with age* $\mathcal{C}_{\mathbf{A}}$*.*

Corollary

Every countable weakly oligomorphic homomorphism-homogeneous relational structure $\mathbf{A}$ *has a core* $\mathbf{C}$ *with age* $\mathcal{C}_{\mathbf{A}}$*.*

Link between HH and H structures

Theorem

Let **A** *be a countable weakly oligomorphic homomorphism-homogeneous relational structure. Then* **A** *contains a substructure* **F** *that is isomorphic to the Fraïssé limit of* $\mathcal{C}_{\mathbf{A}}$. *Moreover,* **F** *and* **A** *are hom-equivalent, and* **F** *is oligomorphic.*

Corollary

Every countable weakly oligomorphic homomorphism-homogeneous relational structure **A** *contains, up to isomorphism, a unique hom-equivalent homomorphism-homogeneous core* **F**. *Moreover,* **F** *is oligomorphic and homogeneous.*

Weak oligomorphy vs. ω-categoricity

Theorem

Let $\mathbf{A}$ *be a countable weakly oligomorphic relational structure. Then* $\mathbf{A}$ *is hom-equivalent to a finite or* ω*-categorical structure* $\mathbf{F}$*. Moreover,* $\mathbf{F}$ *embedds into* $\mathbf{A}$*.*